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Equivariant Euler characteristics of subspace posets

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The (p-primary) equivariant reduced Euler characteristics of the building for the general linear group over a finite field are determined.

Contents

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1. Introduction

Let G be a finite group, Π a finite G-poset, and $r \geq 1$ a natural number. The *rth equivariant reduced Euler characteristic* of the G-poset Π as defined by Atiyah and Segal [1] and Tamanoi [21] is the normalized sum

$$\widetilde{\chi}_r(\Pi, G) = \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbf{Z}^r, G)} \widetilde{\chi}(C_\Pi(X(\mathbf{Z}^r))) \tag{1.1}$$

of the reduced Euler characteristics of the $X(\mathbf{Z}^r)$-fixed Π-subposets, $C_\Pi(X(\mathbf{Z}^r))$, as X ranges over the set of all homomorphisms of $\mathbf{Z}^r$ to G. For example, when G acts trivially on Π, $\widetilde{\chi}_r(\Pi, G) = \widetilde{\chi}(\Pi)|\mathrm{Hom}(\mathbf{Z}^r, G)|/|G|$ where $|\mathrm{Hom}(\mathbf{Z}^r, G)|/|G| = |\mathrm{Hom}(\mathbf{Z}^{r-1}, G)/G|$ is the number of conjugacy classes of commuting $(r-1)$-tuples of elements of G [9, Lemma 4.13]. In this article we specialize to posets of linear subspaces of finite vector spaces. Let q be a prime power, $n \geq 1$ a natural number, $V_n(\mathbf{F}_q)$ the n-dimensional vector space over $\mathbf{F}_q$, $\mathrm{L}_n(\mathbf{F}_q)$ the $\mathrm{GL}_n(\mathbf{F}_q)$-lattice of subspaces of $V_n(\mathbf{F}_q)$, and $\mathrm{L}_n^*(\mathbf{F}_q) = \mathrm{L}_n(\mathbf{F}_q) - \{0, V_n(\mathbf{F}_q)\}$ the proper part of $\mathrm{L}_n(\mathbf{F}_q)$ consisting of nontrivial and proper subspaces. The general Definition (1.1) takes the following form in this context:

Definition 1.2. The rth, $r \geq 1$, equivariant reduced Euler characteristic of the $\mathrm{GL}_n(\mathbf{F}_q)$-poset $\mathrm{L}_n^*(\mathbf{F}_q)$ is the normalized sum

$$\widetilde{\chi}_r(\mathrm{L}_n^*(\mathbf{F}_q), \mathrm{GL}_n(\mathbf{F}_q)) = \frac{1}{|\mathrm{GL}_n(\mathbf{F}_q)|} \sum_{X \in \mathrm{Hom}(\mathbf{Z}^r, \mathrm{GL}_n(\mathbf{F}_q))} \widetilde{\chi}(C_{\mathrm{L}_n^*(\mathbf{F}_q)}(X(\mathbf{Z}^r)))$$

of the Euler characteristics of the subposets $C_{\mathrm{L}_n^*(\mathbf{F}_q)}(X(\mathbf{Z}^r))$ of $X(\mathbf{Z}^r)$-invariant subspaces as X ranges over all homomorphisms of the free abelian group $\mathbf{Z}^r$ on r generators into the general linear group $\mathrm{GL}_n(\mathbf{F}_q)$.

The generating function for the sequence $\widetilde{\chi}_r(\mathrm{L}_n^*(\mathbf{F}_q), \mathrm{GL}_n(\mathbf{F}_q))$, $n \geq 1$, or rth generating function for short, is the power series

$$F_r(x, q) = 1 + \sum_{n \geq 1} \widetilde{\chi}_r(\mathrm{L}_n^*(\mathbf{F}_q), \mathrm{GL}_n(\mathbf{F}_q)) x^n \in \mathbf{Z}[q][[x]], \qquad r \geq 1 \tag{1.3}$$

with coefficients in integral polynomials in q.

Theorem 1.4. $F_{r+1}(x, q) = \prod_{0 \leq j \leq r} (1 - q^j x)^{(-1)^{r-j}\binom{r}{j}}$ *for all* $r \geq 0$.

The first generating functions $F_r(x, q)$ for $1 \leq r \leq 5$ are

$$1 - x, \quad \frac{1 - qx}{1 - x}, \quad \frac{(1-x)(1-q^2x)}{(1-qx)^2}, \quad \frac{(1-qx)^3(1-q^3x)}{(1-x)(1-q^2x)^3}, \quad \frac{(1-x)(1-q^2x)^6(1-q^4x)}{(1-qx)^4(1-q^3x)^4}$$

When $r = 2$, $F_2(x, q) = 1 + (1-q)\sum_{n\geq 1} x^n$ tells us that $\widetilde{\chi}_2(\mathrm{L}_n^*(\mathbf{F}_q), \mathrm{GL}_n(\mathbf{F}_q)) = 1-q$ for all $n \geq 1$ and all prime powers q, and when $r = 4$,

$$\begin{aligned} F_4(x,q) &= \frac{(1-qx)^3(1-q^3x)}{(1-q^2x)^3(1-x)} \\ &= 1 + (1-q)^3(x + (3q^2+1)x^2 + (6q^4 - q^3 + 3q^2 + 1)x^3 \\ &\quad + (10q^6 - 3q^5 + 6q^4 - q^3 + 3q^2 + 1)x^4 + \cdots) \end{aligned}$$

tells us that

$$\widetilde{\chi}_4(\mathrm{L}_n^*(\mathbf{F}_q), \mathrm{GL}_n(\mathbf{F}_q)) = \begin{cases} (1-q)^3 & n=1 \\ (1-q)^3(3q^2+1) & n=2 \\ (1-q)^3(6q^4 - q^3 + 3q^2 + 1) & n=3 \\ (1-q)^3(10q^6 - 3q^5 + 6q^4 - q^3 + 3q^2 + 1) & n=4 \end{cases}$$

for all prime powers q.

Corollary 1.5. $F_{r+1}(x, q) = \exp\big(-\sum_{n\geq 1}(q^n-1)^r \frac{x^n}{n}\big)$ *for all* $r \geq 0$.

We also discuss the p-primary equivariant reduced Euler characteristics, $\widetilde{\chi}_r(\mathrm{L}_n^*(\mathbf{F}_q), \mathrm{GL}_n(\mathbf{F}_q), p)$, of the $\mathrm{GL}_n(\mathbf{F}_q)$-poset $\mathrm{L}_n^*(\mathbf{F}_q)$ for a given prime p (Definition 4.2). The generating function for the sequence $\widetilde{\chi}_r(\mathrm{L}_n^*(\mathbf{F}_q), \mathrm{GL}_n(\mathbf{F}_q), p)$, $n \geq 1$, or rth p-primary generating function for short, is the power series

$$F_r(x, q, p) = 1 + \sum_{n\geq 1} \widetilde{\chi}_r(\mathrm{L}_n^*(\mathbf{F}_q), \mathrm{GL}_n(\mathbf{F}_q), p)x^n \in \mathbf{Z}[[x]], \qquad r \geq 1 \tag{1.6}$$

with integer coefficients. We have $F_r(x, q, p) = 1 - x$ if $r = 1$, or $r \geq 1$ and q is a power of p. When q is not a power of p, $F_r(x, q, p)$ depends only on the closure $\overline{\langle q \rangle}$ of the cyclic subgroup generated by q in the topological group $\mathbf{Z}_p^\times$ of p-adic units (Lemma 4.9). In any case, the $(r+1)$th p-primary generating function is obtained from the $(r+1)$th generating function of Corollary 1.5 simply by replacing the factor $(q^n-1)^r$ by its p-part, $(q^n-1)_p^r$.

Theorem 1.7. $F_{r+1}(x, q, p) = \exp\big(-\sum_{n\geq 1}(q^n-1)_p^r \frac{x^n}{n}\big)$ *for all* $r \geq 0$.

It is immediate from the elementary Lemma 3.7 that the product expansions of the generating functions of Corollary 1.5 and Theorem 1.7 are

$$F_{r+1}(x, q) = \prod_{k\geq 1}(1-x^k)^{a_{r+1}(k,q)}, \qquad a_{r+1}(k, q) = \frac{1}{k}\sum_{d|k}\mu(k/d)(q^d-1)^r$$

Table 1
Equivariant reduced Euler characteristics of the $\mathrm{GL}_n(\mathbf{F}_2)$-poset $\mathrm{L}_n^*(\mathbf{F}_2)$.

$-\chi_r(n,2)$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
$r=1$	1	0	0	0	0	0	0	0	0	0
$r=2$	1	1	1	1	1	1	1	1	1	1
$r=3$	1	4	12	32	80	192	448	1024	2304	5120
$r=4$	1	13	101	645	3717	20101	104069	521861	2553477	12252805
$r=5$	1	40	760	11056	140848	1657216	18480640	198188800	2062546176	20957358080

Table 2
3-primary equivariant reduced Euler characteristics of the $\mathrm{GL}_n(\mathbf{F}_2)$-poset $\mathrm{L}_n^*(\mathbf{F}_2)$.

$-\chi_r(n,2,3)$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
$r=1$	1	0	0	0	0	0	0	0	0	0
$r=2$	1	1	-1	0	0	1	-1	-1	1	0
$r=3$	1	4	-4	-6	6	16	-16	-49	49	72
$r=4$	1	13	-13	-78	78	403	-403	-2236	2236	10413
$r=5$	1	40	-40	-780	780	10960	-10960	-134590	134590	1500408

$$F_{r+1}(x,q,p)=\prod_{k\geq 1}(1-x^k)^{a_{r+1}(k,q,p)}, \qquad a_{r+1}(k,q,p)=\frac{1}{k}\sum_{d|k}\mu(k/d)(q^d-1)_p^r$$

for all $r\geq 1$.

Using partitions we can express the equivariant reduced Euler characteristics more explicitly. Let us also introduce $\widetilde{\chi}_r^{-1}(\mathrm{L}_n^*(\mathbf{F}_q),\mathrm{GL}_n(\mathbf{F}_q))$ and $\widetilde{\chi}_r^{-1}(\mathrm{L}_n^*(\mathbf{F}_q),\mathrm{GL}_n(\mathbf{F}_q),p)$ for the coefficients of x^n in the reciprocal power series $F_r(x,q)^{-1}$ and $F_r(x,q,p)^{-1}$, respectively. Then (Proposition 3.13, (3.17), Proposition 4.19, (4.20))

$$\widetilde{\chi}_r^{\pm 1}(\mathrm{L}_n^*(\mathbf{F}_q),\mathrm{GL}_n(\mathbf{F}_q))=\frac{1}{n!}\sum_{\lambda\vdash n}(\mp 1)^{|\lambda|}T(\lambda)\prod_{d\in\lambda}(q^d-1)^r,$$

$$\widetilde{\chi}_r^{\pm 1}(\mathrm{L}_n^*(\mathbf{F}_q),\mathrm{GL}_n(\mathbf{F}_q),p)=\frac{1}{n!}\sum_{\lambda\vdash n}(\mp 1)^{|\lambda|}T(\lambda)\prod_{d\in\lambda}(q^d-1)_p^r$$

where $T(\lambda)$, for each partition λ of n, is the number of elements in the symmetric group Σ_n of cycle type λ. The functions $\widetilde{\chi}_2^{-1}(\mathrm{L}_n^*(\mathbf{F}_q),\mathrm{GL}_n(\mathbf{F}_q))$ and $\widetilde{\chi}_2^{-1}(\mathrm{L}_n^*(\mathbf{F}_q),\mathrm{GL}_n(\mathbf{F}_q),p)$ count semi-simple and p-singular semi-simple classes in $\mathrm{GL}_n(\mathbf{F}_q)$, respectively (Corollary 4.22).

Tables 1 and 2 contain examples of concrete values of (p-primary) equivariant reduced Euler characteristics.

Equivariant Euler characteristics have connections to representation theory, combinatorics, and topology. The Knörr–Robinson conjecture [12,22,23] (a reformulation of the (non block-wise) Alperin conjecture) predicts that

$$\widetilde{\chi}_2(\mathcal{S}_G^{p+*},G)+z_p(G)=0$$

where $\mathcal{S}_G^{p+*}$ is the Brown G-poset of nontrivial p-subgroups of G and $z_p(G)$ the number of irreducible complex representations of G of dimension divisible by $|G|_p$. According to Quillen [15, Theorem 3.1], $\widetilde{\chi}_r(\mathrm{L}_n^*(\mathbf{F}_q),\mathrm{GL}_n(\mathbf{F}_q))=\widetilde{\chi}_r(\mathcal{S}_{\mathrm{GL}_n(\mathbf{F}_q)}^{s+*},\mathrm{GL}_n(\mathbf{F}_q))$ where s

is the characteristic of the field $\mathbf{F}_q$. Since $z_s(\mathrm{GL}_n(\mathbf{F}_q)) = q-1$ and the second equivariant reduced Euler characteristic $\widetilde{\chi}_2(\mathrm{L}_n^*(\mathbf{F}_q), \mathrm{GL}_n(\mathbf{F}_q)) = 1-q$, we have verified the Knörr–Robinson conjecture for $\mathrm{GL}_n(\mathbf{F}_q)$ at its defining characteristic. This result is not new, however, as it was proved already by Thévenaz [22], but the approach used here may qualify as a candidate to the combinatorial proof envisioned in [22, Introduction, (1)]. As observed also by Thévenaz [22, Theorem A, B], and investigated further in Section 3.2, the equivariant Euler characteristics of the general linear groups, and presumably also of other families of finite groups of Lie type, lead to combinatorial polynomial identities. The connections to algebraic topology go through the G-space $|\Pi|$, the topological realization of the G-poset Π. It is convenient now to switch to the (unreduced) equivariant Euler characteristics $\chi_r(\Pi, G) = \widetilde{\chi}_r(\Pi, G) + |\operatorname{Hom}(\mathbf{Z}^{r-1}, G)/G|$. The first equivariant Euler characteristic of (Π, G) is the usual Euler characteristic of the quotient space $|\Pi|/G$ (Proposition 2.1). The second equivariant Euler characteristic of (Π, G) is the Euler characteristic of the G-space $|\Pi|$ computed in G-equivariant complex K-theory [1, Theorem 1]. Finally, the rth p-primary equivariant Euler characteristic, $\chi_r(\Pi, G, p) = \widetilde{\chi}_r(\Pi, G, p) + |\operatorname{Hom}(\mathbf{Z}_p^{r-1}, G)/G|$, is the Euler characteristic of the homotopy orbit space $|\Pi|_{hG}$ computed in Morava $K(r)$-theory at p [9] [21, 2-3, 5-1] [14, Remark 7.2].

See [21,14] for (p-primary) equivariant Euler characteristics of Boolean and partition posets.

The following notation will be used in this article:

p	a prime number
$\nu_p(n)$	the p-adic valuation of n
n_p	the p-part of the natural number n ($n_p = p^{\nu_p(n)}$)
$\mathbf{Z}_p$	the ring of p-adic integers
q	a prime power
$\mathbf{F}_q$	the finite field with q elements
$\mathrm{IM}_n(q)$	number of Irreducible Monic polynomials $f \in \mathbf{F}_q[t]$ of degree n with $f(0) \neq 0$
$\mathrm{IM}_n(q,p)$	number of Irreducible Monic polynomials $f \in \mathbf{F}_q[t]$ of degree n and p-power order with $f(0) \neq 0$
$\mathrm{ord}_a(b)$	smallest natural number e such that $b^e \equiv 1 \bmod a$ (a, b are natural numbers with $\gcd(a,b) = 1$)
$\widetilde{\chi}_r(n,q)$	$\widetilde{\chi}_r(\mathrm{L}_n^*(\mathbf{F}_q), \mathrm{GL}_n(\mathbf{F}_q))$ (Definition 1.2)
$\widetilde{\chi}_r(n,q,p)$	$\widetilde{\chi}_r(\mathrm{L}_n^*(\mathbf{F}_q), \mathrm{GL}_n(\mathbf{F}_q), p)$ (Definition 4.2)

2. Equivariant Euler characteristics of subspace posets

The definition (1.1) of the first equivariant Euler characteristic,

$$\widetilde{\chi}_1(\Pi, G) = \frac{1}{|G|} \sum_{g \in G} \widetilde{\chi}(C_\Pi(g))$$

closely resembles the not-Burnside lemma [20, Lemma 7.24.5] or the Lefschetz formula [24, Exercise 4, p. 225]. The topological realization functor takes the G-poset Π to the G-space $|\Pi|$ and the following proposition is nothing surprising so the proof will be omitted.

Proposition 2.1. $\widetilde{\chi}_1(\Pi, G) = \widetilde{\chi}(|\Pi|/G)$.

Let $f_0, f_1 \colon \Pi \to \Pi$ be two poset endomorphisms of the poset Π. We write $f_0 \leq f_1$ if $f_0(x) \leq f_1(x)$ for all $x \in \Pi$ and $f_0 \sim f_1$ if f_0 and f_1 belong to the same class under the equivalence relation generated by this relation. The equivalence relation $f_0 \sim_G f_1$ between G-poset endomorphisms of the G-poset Π is defined similarly. The poset Π is *poset contractible* if there exists a point $x_0 \in \Pi$ such that $1_\Pi \sim x_0$ where 1_Π is the identity map. The G-poset Π is *G-poset-contractible* if there exists a point $x_0 \in C_\Pi(G)$ such that $1_\Pi \sim_G x_0$. The realization of a (G-)poset contractible poset is a (G-)contractible topological space [15, §1.3]. If Π is G-poset-contractible then the subposets $C_\Pi(X)$ are poset contractible for all $X \in \mathrm{Hom}(\mathbf{Z}^r, G)$. Thus we have

$$\begin{aligned}
&\Pi \text{ is poset contractible } \implies \widetilde{\chi}(\Pi) = 0, \\
&\Pi \text{ is } G\text{-poset-contractible } \implies \forall r \geq 1\colon \widetilde{\chi}_r(\Pi, G) = 0
\end{aligned} \tag{2.2}$$

For instance, the Brown poset $\mathcal{S}_G^{p+*}$ of nontrivial p-subgroups of G is G-poset-contractible and $\widetilde{\chi}_r(\mathcal{S}_G^{p+*}, G) = 0$ for all $r \geq 1$ if G admits a nontrivial normal p-subgroup [15, Proposition 2.4]. We shall see something similar in Lemma 2.6.

Here is a basic recursive relation between equivariant reduced Euler characteristics.

Lemma 2.3. *The rth equivariant Euler characteristic (1.1) of (Π, G) is*

$$\widetilde{\chi}_r(\Pi, G) = \sum_{X \in \mathrm{Hom}(\mathbf{Z}, G)/G} \widetilde{\chi}_{r-1}(C_\Pi(X), C_G(X))$$

where the sum extends over conjugacy classes of elements in G and $r \geq 2$.

Proof. Any homomorphism $X \in \mathrm{Hom}(\mathbf{Z}^r, G)$ corresponds to a unique pair of homomorphisms (X_1, X_2) with $X_1 \in \mathrm{Hom}(\mathbf{Z}, G)$ and $X_2 \in \mathrm{Hom}(\mathbf{Z}^{r-1}, C_G(X_1))$. The subposet of Π fixed by X is the subposet fixed by X_2 in the subposet of Π fixed by X_1, $C_\Pi(X) = C_{C_\Pi(X_1)}(X_2)$. The rth equivariant Euler characteristic (1.1) of (Π, G) is

$$\begin{aligned}
\widetilde{\chi}_r(\Pi, G) &= \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbf{Z}^r, G)} \widetilde{\chi}(C_\Pi(X)) \\
&= \frac{1}{|G|} \sum_{X_1 \in \mathrm{Hom}(\mathbf{Z}, G)} \sum_{X_2 \in \mathrm{Hom}(\mathbf{Z}^{r-1}, C_G(X_1))} \widetilde{\chi}(C_{C_\Pi(X_1)}(X_2)) \\
&= \frac{1}{|G|} \sum_{X_1 \in \mathrm{Hom}(\mathbf{Z}, G)} |C_G(X_1)| \widetilde{\chi}_{r-1}(C_\Pi(X_1), C_G(X_1)) \\
&= \sum_{X_1 \in \mathrm{Hom}(\mathbf{Z}, G)/G} \widetilde{\chi}_{r-1}(C_\Pi(X_1), C_G(X_1))
\end{aligned}$$

where the last sum runs over conjugacy classes of elements in G. $\square$

We also need to know that the rth equivariant reduced Euler characteristic is multiplicative. For any lattice L, we write $L^* = L - \{\widehat{0}, \widehat{1}\}$ for the proper part of L of all non-extreme elements.

Lemma 2.4. *The function is multiplicative in the sense that*

$$\widetilde{\chi}_r((\prod_{i \in I} L_i)^*, \prod_{i \in I} G_i) = \prod_{i \in I} \widetilde{\chi}_r(L_i^*, G_i)$$

for any finite set of G_i-lattices L_i, $i \in I$, and any $r \geq 1$.

Proof. This follows immediately from the similar multiplicative rule, $\widetilde{\chi}((\prod_{i \in I} L_i)^*) = \prod_{i \in I} \widetilde{\chi}(L_i^*)$, valid for usual Euler characteristics. Using this property, and assuming for simplicity that the index set $I = \{1, 2\}$ has just two elements, we get

$$\begin{aligned}
&|G_1 \times G_2| \widetilde{\chi}_r((L_1 \times L_2)^*, G_1 \times G_2) \\
&= \sum_{X \in \mathrm{Hom}(\mathbf{Z}^r, G_1 \times G_2)} \widetilde{\chi}(C_{(L_1 \times L_2)^*}(X(\mathbf{Z}^r)) \\
&= \sum_{X_1 \in \mathrm{Hom}(\mathbf{Z}^r, G_1)} \sum_{X_2 \in \mathrm{Hom}(\mathbf{Z}^r, G_2)} \widetilde{\chi}(C_{(L_1 \times L_2)^*}((X_1 \times X_2)(\mathbf{Z}^r)) \\
&= \sum_{X_1 \in \mathrm{Hom}(\mathbf{Z}^r, G_1)} \sum_{X_2 \in \mathrm{Hom}(\mathbf{Z}^r, G_2)} \widetilde{\chi}((C_{L_1}(X_1(\mathbf{Z}^r)) \times C_{L_2}(X_2(\mathbf{Z}^r)))^*) \\
&= \sum_{X_1 \in \mathrm{Hom}(\mathbf{Z}^r, G_1)} \sum_{X_2 \in \mathrm{Hom}(\mathbf{Z}^r, G_2)} \widetilde{\chi}(C_{L_1^*}(X_1(\mathbf{Z}^r)) \times \widetilde{\chi}(C_{L_2^*}(X_2(\mathbf{Z}^r)) \\
&= \sum_{X_1 \in \mathrm{Hom}(\mathbf{Z}^r, G_1)} \widetilde{\chi}(C_{L_1^*}(X_1(\mathbf{Z}^r)) \sum_{X_2 \in \mathrm{Hom}(\mathbf{Z}^r, G_2)} \widetilde{\chi}(C_{L_1^*}(X_1(\mathbf{Z}^r)) \\
&= |G_1| \widetilde{\chi}_r(L_1, G_1) |G_2| \widetilde{\chi}_r(L_2, G_2)
\end{aligned}$$

for any $r \geq 1$. □

We now turn to the case where the poset is $\Pi = \mathrm{L}_n^*(\mathbf{F}_q)$ and the group is $G = \mathrm{GL}_n(\mathbf{F}_q)$. To simplify notation, we shall often write $\widetilde{\chi}_r(n, q)$ for $\widetilde{\chi}_r(\mathrm{L}_n^*(\mathbf{F}_q), \mathrm{GL}_n(\mathbf{F}_q))$.

Proposition 2.5. *Suppose that $r = 1$ or $n = 1$.*

(1) When $r = 1$, $\widetilde{\chi}_1(n, q) = -\delta_{1,n}$ is -1 for $n = 1$ and 0 for all $n > 1$.
(2) When $n = 1$, $\widetilde{\chi}_r(1, q) = -(q-1)^{r-1}$ for all $r \geq 1$.

Proof. The space $|\mathrm{L}_n^*(\mathbf{F}_q)|$ is the simplicial complex of flags in $V_n(\mathbf{F}_q)$. The $\mathrm{GL}_n(\mathbf{F}_q)$-orbit of a flag is described by the dimensions of the subspaces in the flag. Thus the quotient $|\mathrm{L}_n^*(\mathbf{F}_q)| / \mathrm{GL}_n(\mathbf{F}_q)$ is the simplicial complex of all subsets of $\{1, \ldots, n-1\}$, an

$(n-2)$-simplex, Δ^{n-2}. By Proposition 2.1, $\widetilde{\chi}_1(n,q)$ is the usual reduced Euler characteristic of the quotient, $\widetilde{\chi}(\Delta^{n-2})$, which is -1 when $n=1$ and 0 when $n>1$. (Alternatively, this is a special case of Webb's theorem [25, Proposition 8.2.(i)].)

When $n=1$, $\mathrm{L}_1^*(\mathbf{F}_q)=\emptyset$ is empty and as $\widetilde{\chi}(\emptyset)=-1$, the rth equivariant Euler characteristic is

$$\widetilde{\chi}_r(1,q) = -|\operatorname{Hom}(\mathbf{Z}^r, \mathrm{GL}_1(\mathbf{F}_q))|/|\mathrm{GL}_1(\mathbf{F}_q)| = -(q-1)^{r-1}$$

for all $r \geq 1$. $\square$

According to Proposition 2.5.(1), the first generating function, $F_1(x,q) = 1 + \sum_{n\geq 1} \widetilde{\chi}_1(n,q)x^n = 1-x$, is independent of q. We aim now for a recursion leading to the other generating functions $F_r(x,q)$ for $r>1$. The next lemma reduces the problem significantly.

Lemma 2.6. *Let A be an abelian subgroup of $\mathrm{GL}_n(\mathbf{F}_q)$ where $n>1$. If $\gcd(|A|,q)\neq 1$, then the $C_{\mathrm{GL}_n(\mathbf{F}_q)}(A)$-poset $C_{\mathrm{L}_n^*(\mathbf{F}_q)}(A)$ is $C_{\mathrm{GL}_n(\mathbf{F}_q)}(A)$-contractible.*

Proof. The assumption is that the abelian group A contains an element of order s, the characteristic of $\mathbf{F}_q$. Let $F = V_n(\mathbf{F}_q)^{O_s(A)}$ be the subspace of vectors in $V_n(\mathbf{F}_q)$ fixed by the nontrivial Sylow s-subgroup $O_s(A)$ of A. F is a nontrivial subspace since s-groups actions on $\mathbf{F}_s$-vector spaces fix a nonzero vector [6, Proposition VI.8.1]. F is a proper subspace since the nontrivial group $O_s(A)$ acts faithfully on $V_n(\mathbf{F}_q)$. F is invariant under A since $(vg)h=(vh)g=vg$ for all $v\in F$, $g\in A$, $h\in O_s(A)$. Thus F belongs to $C_{\mathrm{L}_n^*(\mathbf{F}_q)}(A)$. For any $U\in C_{\mathrm{L}_n^*(\mathbf{F}_q)}(A)$, $U\cap F = U^{O_s(A)}$ is of course proper and also nontrivial by [6, Proposition VI.8.1] again. Since $U \geq U\cap F \leq F$ for all $U\in C_{\mathrm{L}_n^*(\mathbf{F}_q)}(A)$, the poset $C_{\mathrm{L}_n^*(\mathbf{F}_q)}(A)$ is poset contractible [15, §1.5]. It is even $C_{\mathrm{GL}_n(\mathbf{F}_q)}(A)$-poset-contractible since $F^g=F$ and $U^g\cap F = U^g\cap F^g = (U\cap F)^g$ for all $g\in C_{\mathrm{GL}_n(\mathbf{F}_q)}(A)$, $U\in C_{\mathrm{L}_n^*(\mathbf{F}_q)}(A)$. $\square$

Corollary 2.7. *When $r,n\geq 1$, the $(r+1)$th equivariant Euler characteristic of the $\mathrm{GL}_n(\mathbf{F}_q)$-poset $\mathrm{L}_n^*(\mathbf{F}_q)$ is*

$$\widetilde{\chi}_{r+1}(n,q) = \sum_{\substack{[g]\in[\mathrm{GL}_n(\mathbf{F}_q)]\\ \gcd(q,|g|)=1}} \widetilde{\chi}_r(C_{\mathrm{L}_n^*(\mathbf{F}_q)}(g), C_{\mathrm{GL}_n(\mathbf{F}_q)}(g))$$

Proof. If $G=\mathrm{GL}_n(\mathbf{F}_q)$, $n\geq 2$, and $\Pi = \mathrm{L}_n^*(\mathbf{F}_q)$, only the conjugacy classes of order prime to q contribute to the sum of Lemma 2.3 according to (2.2) and Lemma 2.6. The corollary remains true for $n=1$ where $\mathrm{L}_1^*(\mathbf{F}_q)=\emptyset$. $\square$

3. Proofs of Theorem 1.4 and Corollary 1.5

Let $F(x,q) = 1 + \sum_{n\geq 1} a(n,q)x^n \in 1 + (x) \subseteq \mathbf{Z}[q][[x]]$ be a polynomial power series with constant term 1 and $A(q) = (A_1(q), A_2(q), \ldots, A_d(q), \ldots)$ a sequence of integers defined for every prime power q.

Definition 3.1. The $A(q)$-transform of the power series $F(x,q)$ is the power series

$$T_{A(q)}(F(x,q)) = \prod_{d\geq 1} F(x^d, q^d)^{A_d(q)}$$

Note that $T_{A(q)} \colon 1 + (x) \to 1 + (x)$ is multiplicative in the sense that

$$T_{A(q)}(1) = 1, \qquad T_{A(q)}(F_1(x,q)F_2(x,q)) = T_{A(q)}(F_1(x,q))T_{A(q)}(F_2(x,q)) \tag{3.2}$$

for any two polynomial power series $F_1(x,q), F_2(x,q) \in 1 + (x)$.

The $\mathrm{IM}(q)$-transform will be especially important. (See Section 1 for the definition of $\mathrm{IM}_d(q)$.) Finite field theory tells us that [13, Corollary 3.21, Theorem 3.25]

$$q^n - 1 = \sum_{d|n} d\,\mathrm{IM}_d(q), \qquad n\,\mathrm{IM}_n(q) = \sum_{d|n} \mu(n/d)(q^d - 1) \tag{3.3}$$

It is a little easier to calculate the transform with respect to the sequence $\overline{\mathrm{IM}}(q)$ where $\overline{\mathrm{IM}}_n(q)$ is the number of *all* Irreducible Monic polynomials $f \in \mathbf{F}_q[t]$ of degree $n \geq 1$. As the two sequences agree except in degree 1 where $\mathrm{IM}_1(q) = q - 1$ and $\overline{\mathrm{IM}}_1(q) = q$, the two transforms,

$$F(x,q)T_{\mathrm{IM}(q)}F(x,q) = T_{\overline{\mathrm{IM}}(q)}F(x,q) \tag{3.4}$$

are closely related.

Lemma 3.5. *$T_{\overline{\mathrm{IM}}(q)}((1-q^i x)^j) = (1-q^{i+1}x)^j$ and $T_{\mathrm{IM}(q)}((1-q^i x)^j) = \left(\frac{1-q^{i+1}x}{1-q^i x}\right)^j$ for any two integers i and j.*

Proof. It is immediate that

$$T_{\overline{\mathrm{IM}}(q)}\left(\frac{1}{1-q^i x}\right) = \prod_{d\geq 1} \frac{1}{(1-(q^i x)^d)^{\overline{\mathrm{IM}}_d(q)}} = \frac{1}{1-q^{i+1}x} \tag{3.6}$$

because the logarithm of the middle term is $\sum_{n\geq 1} q^n \frac{(q^i x)^n}{n} = \sum_{n\geq 1} \frac{(q^{i+1}x)^n}{n} = -\log(1 - q^{i+1}x)$ by the well-known Lemma 3.7 below and (3.3). The multiplicative property (3.2) and (3.4) now imply the result (cf. [16, Chp. 2]). □

Lemma 3.7. *Let $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$, and $(c_n)_{n\geq 1}$ be integer sequences such that*

$$\prod_{n\geq 1}(1-x^n)^{-b_n} = \exp\big(\sum_{n\geq 1} a_n \frac{x^n}{n}\big) = 1 + \sum_{n\geq 1} c_n x^n$$

Then

$$a_n = \sum_{d|n} d b_d, \qquad n b_n = \sum_{d|n} \mu(n/d) a_d, \qquad n c_n = \sum_{1\leq j\leq n} a_j c_{n-j}$$

where μ is the number theoretic Möbius function [11, Chp. 2, §2] and it is understood that $c_0 = 1$.

Proof. The first identity follows from

$$\sum_{n\geq 1} a_n x^n = \sum_{n\geq 1}\sum_{k\geq 1} n b_n x^{nk}$$

obtained by applying the operator $x\frac{d}{dx}\log$ to the given identity $\exp(\sum_{n\geq 1} a_n \frac{x^n}{n}) = \prod_{n\geq 1}(1-x^n)^{-b_n}$. Möbius inversion leads to the second identity. The third identity follows from

$$\big(1 + \sum_{n\geq 1} c_n x^n\big)\big(\sum_{n\geq 1} a_n x^n\big) = \sum_{n\geq 1} n c_n x^n$$

obtained by applying the operator $x\frac{d}{dx}$ to the given identity $\exp\big(\sum_{n\geq 1} a_n \frac{x^n}{n}\big) = 1 + \sum_{n\geq 1} c_n x^n$. $\square$

Since an element of $\mathrm{GL}_n(\mathbf{F}_q)$ is semi-simple if and only its order is prime to q [22, §2], it is precisely the semi-simple elements that contribute terms to the right side in Corollary 2.7.

An element $g \in \mathrm{GL}_n(\mathbf{F}_q)$ is semi-simple if and only the $\mathbf{F}_q[t]$-module $V_n(\mathbf{F}_q)$ with $tv = gv$ has the form

$$V_n(\mathbf{F}_q) \cong \bigoplus_f \underbrace{\mathbf{F}_q[t]/(f(t)) \oplus \cdots \oplus \mathbf{F}_q[t]/(f(t))}_{m_g(f)}$$

where the direct sum is over irreducible monic polynomials $f \in \mathbf{F}_q[t]$, $f(t) \neq t$, and the $m_g(f) \geq 0$ are natural numbers. (The irreducible polynomial $f(t) = t$ of degree 1 is excluded since we need t to act as an automorphism on $\mathbf{F}_q[t]/(f(t))$.) The Galois field $\mathbf{F}_q[t]/(f(t))$ has $q^{d(f)}$ elements where $d(f)$ is the degree of f. Thus there is a bijective correspondence

$$\prod_f f(t)^{m(f)} \longleftrightarrow \bigoplus_f (\mathbf{F}_q[t]/(f(t)))^{m(f)}$$

between the $q^{n-1}(q-1)$ monic polynomials in $\mathbf{F}_q[t]$ of degree n with nonzero constant term and the semi-simple classes in $\mathrm{GL}_n(\mathbf{F}_q)$. Here, the fs are monic irreducible polynomials with $f(0) \neq 0$, $m(f) \geq 0$ and $\sum_f d(f)m(f) = n$.

In the notation of [8, §1, §2], g is semi-simple if and only if all parts of the associated partitions $\nu_g(f)$ are 1 or 0. If $\nu(f) = (1, \dots, 1, 0, \dots)$ partitions $m(f)$, the matrix $U_{\nu(f)}(t)$ is the zero $(m(f) \times m(f))$-matrix and its module $V_{U_{\nu(f)}(t)}(q^{d(f)})$ is $V_{m(f)}(\mathbf{F}_{q^{d(f)}})$ as a $\mathbf{F}_{q^{d(f)}}$-vector space. The lattice of subspaces invariant under g in $V_n(\mathbf{F}_q)$ and the centralizer of g in $\mathrm{GL}_n(\mathbf{F}_q)$ are

$$C_{\mathrm{L}_n(\mathbf{F}_q)}(g) = \prod_f \mathrm{L}_{m_g(f)}(\mathbf{F}_{q^{d(f)}}), \qquad C_{\mathrm{GL}_n(\mathbf{F}_q)}(g) = \prod_f \mathrm{GL}_{m_g(f)}(\mathbf{F}_{q^{d(f)}})$$

according to [8, Lemma 2.1]. The semi-simple conjugacy class $[g]$ contributes

$$\widetilde{\chi}_{r-1}(C_{\mathrm{L}_n^*(\mathbf{F}_q)}(g), C_{\mathrm{GL}_n(\mathbf{F}_q)}(g)) = \prod_{f|c_g} \widetilde{\chi}_{r-1}(m_g(f), q^{d(f)}). \qquad c_g(t) = \prod_f f(t)^{m_g(f)}$$

to the sum of Corollary 2.7. The product runs over the irreducible monic factors f of c_g, the characteristic polynomial of g.

It is now immediate from an extended version of the Product formula [2, Theorem 8.5] for generating functions that Corollary 2.7 translates to the recurrence relation

$$F_{r+1}(x,q) = T_{\mathrm{IM}(q)} F_r(x,q), \qquad r \geq 1 \tag{3.8}$$

for the generating functions (1.3). The base function is $F_1(x,q) = 1 - x$ (Proposition 2.5.(1)).

Proof of Theorem 1.4. The sequence $F_{r+1}(x,q)$, $r \geq 0$, of Theorem 1.4 solves recurrence (3.8) since

$$\begin{aligned} T_{\mathrm{IM}(q)} \prod_{0\leq j\leq r} (1-q^j x)^{(-1)^{r-j}\binom{r}{j}} &= \frac{\prod_{0\leq j\leq r}(1-q^{j+1}x)^{(-1)^{r-j}\binom{r}{j}}}{\prod_{0\leq j\leq r}(1-q^j x)^{(-1)^{r-j}\binom{r}{j}}} \\ &= \prod_{0\leq j\leq r+1} (1-q^j x)^{(-1)^{r+1-j}\binom{r+1}{j}} \end{aligned}$$

for all $r \geq 0$ by Lemma 3.5. □

See [22, Proposition 4.1] for the case $r = 2$ where $F_2(x,q) = \frac{1-qx}{1-x}$. As $F_1(x,q) = 1-x$ and $F_2(x,q) = \frac{F_1(x,qx)}{F_1(x,q)}$,

$$F_{r+1}(x,q) = T_{\mathrm{IM}(q)} F_r(x,q) = \frac{T_{\mathrm{IM}(q)} F_{r-1}(x,qx)}{T_{\mathrm{IM}(q)} F_{r-1}(x,q)} = \frac{F_r(x,qx)}{F_r(x,q)}, \qquad r \geq 2 \tag{3.9}$$

by induction. This observation can be used to give another proof of Theorem 1.4.

Proof of Corollary 1.5. The logarithm of the $(r+1)$th generating function $F_{r+1}(x,q)$ is

$$\log F_{r+1}(x,q) = \sum_{0\le j\le r} (-1)^{r-j}\binom{r}{j}\log(1-q^jx) = \sum_{0\le j\le r}(-1)^{r-j}\binom{r}{j}\sum_{n\ge 1}-\frac{q^{nj}}{n}x^n$$
$$= -\sum_{n\ge 1}\frac{x^n}{n}\sum_{0\le j\le r}\binom{r}{j}(-1)^{r-j}q^{nj} = -\sum_{n\ge 1}(q^n-1)^r\frac{x^n}{n}$$

The corollary follows. □

We now write down explicitly the coefficient of x^n in the power series $F_{r+1}(x,q)$ of Theorem 1.4 (Corollary 3.10) and apply Lemma 3.7 to the power series of Corollary 1.5 (Corollary 3.11).

Corollary 3.10. *The $(r+1)$th, $r \ge 0$, equivariant reduced Euler characteristic of the $\mathrm{GL}_n(\mathbf{F}_q)$-poset $\mathrm{L}_n^*(\mathbf{F}_q)$ is*

$$\widetilde{\chi}_{r+1}(n,q) = (-1)^n \sum_{n_0+\cdots+n_r=n}\prod_{0\le j\le r}\binom{(-1)^{r-j}\binom{r}{j}}{n_j}q^{jn_j}$$

where the sum ranges over all $\binom{n+r}{n}$ weak compositions of n into $r+1$ parts [19, p. 15].

Corollary 3.11. *The $(r+1)$th, $r \ge 0$, equivariant reduced Euler characteristics satisfy the recursion*

$$\widetilde{\chi}_{r+1}(n,q) = \begin{cases} 1 & n=0 \\ -\frac{1}{n}\sum_{1\le j\le n}(q^j-1)^r\widetilde{\chi}_{r+1}(n-j,q) & n>0 \end{cases}$$

For example, $\widetilde{\chi}_1(n,q) = -\delta_{1,n}$, $\widetilde{\chi}_2(n,q) = -(q-1)$, $\widetilde{\chi}_3(n,q) = -n(q-1)^2q^{n-1}$ and

$$\widetilde{\chi}_4(n,q) = -(q-1)^3\big(1+\sum_{2\le j\le 2n-2}(-1)^jd(j)q^j\big) \qquad d(j) = \begin{cases}\binom{(j+1)/2}{2} & 2\nmid j \\ \binom{j/2+2}{2} & 2\mid j\end{cases}$$

for all $n \ge 1$ (with the understanding that $\widetilde{\chi}_4(1,q) = -(q-1)^3$).

3.1. Alternative presentations of the equivariant reduced Euler characteristics

One may equally well represent the equivariant reduced Euler characteristics $\widetilde{\chi}_{r+1}(n,q)$ by the generating functions

$$G_n(x,q) = \sum_{r\ge 0}\widetilde{\chi}_{r+1}(n,q)x^r = -\delta_{1,n} + \widetilde{\chi}_2(n,q)x + \widetilde{\chi}_3(n,q)x^2 + \cdots, \qquad n \ge 0 \quad (3.12)$$

where the parameter n is fixed rather than r as in $F_r(x,q)$ (1.3). Declaring $\widetilde{\chi}_{r+1}(0,q)$ to be 1 for all $r \geq 0$, we have $G_0(x,q) = x + x^2 + \cdots = \frac{x}{1-x}$.

The solution to the recursion of Corollary 3.11 involves integer partitions and the following terminology. A multiset λ is a base set $B(\lambda)$ with a multiplicity function $E(\lambda, b)$ defined for all $b \in B(\lambda)$. Representing the multiset as $\lambda = \{b^{E(\lambda,b)} \mid b \in B(\lambda)\}$, we let

$$|\lambda| = \sum_{b \in B(\lambda)} E(\lambda, b) \qquad n(\lambda) = \sum_{b \in B(\lambda)} bE(\lambda, b)$$

$$T(\lambda) = \frac{n(\lambda)!}{\prod_{b \in B(\lambda)} E(\lambda,b)! b^{E(\lambda,b)}} \qquad U(\lambda, q) = \prod_{b \in B(\lambda)} (q^b - 1)^{E(\lambda,b)} = \prod_{b \in \lambda} (q^b - 1)$$

so that $|\lambda|$ is the cardinality (number of parts) of λ, λ partitions n, $\lambda \vdash n$, if $n(\lambda) = n$, and $T(\lambda)$ is the number of elements in the symmetric group $\Sigma_{n(\lambda)}$ having cycle type λ [17, Proposition 1.1.1].

Proposition 3.13. *For $r \geq 0$ and $n \geq 1$,*

$$\widetilde{\chi}_{r+1}(n,q) = \frac{1}{n!} \sum_{\lambda \vdash n} (-1)^{|\lambda|} T(\lambda) U(\lambda, q)^r \qquad G_n(x,q) = \frac{1}{n!} \sum_{\lambda \vdash n} (-1)^{|\lambda|} \frac{T(\lambda)}{1 - xU(\lambda, q)}$$

Proof. The sequence $(\widetilde{\chi}_{r+1}(n,q))_{n \geq 1}$ as defined in the proposition solves the recursion of Corollary 3.11. $\square$

Examples of Proposition 3.13 are $1!G_1(x,q) = -\frac{1}{1-x(q-1)}$, $2!G_2(x,q) = -\frac{1}{1-x(q^2-1)} + \frac{1}{1-x(q-1)^2}$ and

$$3!G_3(x,q) = -\frac{2}{1 - x(q^3 - 1)} + \frac{3}{1 - x(q^2 - 1)(q - 1)} - \frac{1}{1 - x(q - 1)^3}$$

Corollary 3.14. *The polynomial $\widetilde{\chi}_{r+1}(n,q) \in \mathbf{Z}[q]$, $r \geq 0$, $n \geq 1$, is divisible by $(q-1)^r$ and also by q^{n-1} when r is even. The sum of the coefficients in the quotient polynomial $-\widetilde{\chi}_{r+1}(n,q)/(q-1)^r$ is n^{r-1}.*

Proof. Since $U(\lambda, q)$ is divisible by $(q-1)^{|\lambda|}$ for all $\lambda \vdash n$, Proposition 3.13 implies $(q-1)^r \mid \widetilde{\chi}_{r+1}(n,q)$.

For even r, the weak partitions $(n_0, n_1, \ldots, n_r)$ of n with $n_0 > 1$ contribute 0 to the sum in Corollary 3.10 since

$$\binom{(-1)^r \binom{r}{0}}{n_0} = \binom{1}{n_0} = 0, \qquad n_0 > 1$$

The remaining weak partitions with $n_0 \leq 1$ contribute 0 or a polynomial of degree $\sum jn_j = n_1 + 2n_2 + \cdots + rn_r \geq n_1 + n_2 + \cdots + n_r \geq n - 1$. Thus $q^{n-1} \mid \widetilde{\chi}_{r+1}(n,q)$.

The quotient $U(\lambda,q)/(q-1)$ evaluates to 0 at $q=1$ unless $\lambda=\{n^1\}$ where the evaluation is n. Thus $\widetilde{\chi}_{r+1}(n,q)/(q-1)^r$ at $q=1$ is $-\frac{1}{n!}T(\{n^1\})n^r = -\frac{1}{n!}\frac{n!}{n}n^r = -n^{r-1}$. $\square$

Remark 3.15. Let $\overline{\chi}_{r+1}(n,q)$ denote the quotient polynomial $-\widetilde{\chi}_{r+1}(n,q)/D$ where $D=(q-1)^r$ for odd r and $D=q^{n-1}(q-1)^r$ for even r. Then $\overline{\chi}_2(n,q)=1$ and $\overline{\chi}_3(n,q)=n$. The polynomial $\overline{\chi}_7(2,q)=6q^4+20q^2+6=2(3q^2+1)(q^2+3)$ is reducible but I do not know of any reducible $\overline{\chi}_r(n,q)\in\mathbf{Q}[q]$ with $n>2$. For example,

$$\begin{aligned}\overline{\chi}_6(4,q) &= 35q^{12}-45q^{11}+150q^{10}-170q^9+290q^8-235q^7+270q^6-100q^5+60q^4\\ &\quad -10q^3+10q^2+1\end{aligned}$$

is irreducible in $\mathbf{Q}[q]$ by Eisenstein's irreducibility criterion [11, p. 78]. The coefficient sum is 4^4.

The reciprocal of $F_{r+1}(x,q)$,

$$F_{r+1}(x,q)^{-1} = \prod_{0\le j\le r}(1-q^jx)^{(-1)^{r+1-j}\binom{r}{j}} = \exp\Big(\sum_{n\ge 1}(q^n-1)^r\frac{x^n}{n}\Big) = \prod_{k\ge 1}(1-x^k)^{-a_{r+1}(k,q)}$$

satisfies by (3.2) the recursion $F_{r+1}(x,q)^{-1}=T_{\mathrm{IM}(q)}F_r(x,q)^{-1}=T_{-\mathrm{IM}(q)}F_r(x,q)$. Let $\widetilde{\chi}^{-1}_{r+1}(n,q)\in\mathbf{Z}[q]$ denote the coefficient of x^n in $F_{r+1}(x,q)^{-1}$. In particular, $\widetilde{\chi}_2^{-1}(n,q)$ is, by construction, the number of semi-simple classes in $\mathrm{GL}_n(\mathbf{F}_q)$. We have, as above,

$$1+\sum_{n\ge 1}\widetilde{\chi}^{-1}_{r+1}(n,q)x^n = F_{r+1}(x,q)^{-1} \qquad 1+\sum_{r\ge 1}\widetilde{\chi}^{-1}_{r+1}(n,q)x^r = \frac{1}{n!}\sum_{\lambda\vdash n}\frac{T(\lambda)}{1-xU(\lambda,q)} \tag{3.16}$$

$$\widetilde{\chi}^{-1}_{r+1}(n,q) = \frac{1}{n!}\sum_{\lambda\vdash n}T(\lambda)U(\lambda,q)^r = (-1)^n\sum_{n_0+\cdots+n_r=n}\prod_{0\le j\le r}\binom{(-1)^{r+1-j}\binom{r}{j}}{n_j}q^{jn_j} \tag{3.17}$$

Special cases are $\widetilde{\chi}_1^{-1}(n,q)=1$, $\widetilde{\chi}_2^{-1}(n,q)=q^n-q^{n-1}$, $\widetilde{\chi}_3^{-1}(n,q)=\frac{q-1}{q+1}(q^{2n}-1)$ and

$$\widetilde{\chi}_4^{-1}(n,q) = (q-1)^3q^{n-1}\big(q^{2n-2}+\sum_{2\le j\le 2n-2}(-1)^jd(j)q^{2n-2-j}\big)$$

$$d(j) = \begin{cases}\binom{(j+1)/2}{2} & 2\nmid j\\ \binom{j/2+2}{2} & 2\mid j\end{cases}$$

with the understanding that $\widetilde{\chi}_4^{-1}(1,q)=(q-1)^3$. For $n=1,2,3$, $r>0$, $1!\widetilde{\chi}^{\pm 1}_{r+1}(1,q)=\mp(q-1)^r$, $2!\widetilde{\chi}^{\pm 1}_{r+1}(2,q)=\mp(q^2-1)^r+(q-1)^{2r}$ and $3!\widetilde{\chi}^{\pm 1}_{r+1}(3,q)=\mp 2(q^3-1)^r+3(q^2-1)^r(q-1)^r\mp(q-1)^{3r}$ by Proposition 3.13 and (3.17).

3.2. Polynomial identities for partitions

The polynomial identities [22, Theorem A, B] are parts of a greater hierarchy of polynomial identities.

Let M_n, $n \geq 1$, be the set of all finite multisets $\lambda = \{(m_1,d_1)^{e(m_1,d_1)},\cdots,(m_s,d_s)^{e(m_s,d_s)}\}$ of pairs of natural numbers (m_i,d_i) with multiplicities $e(m_i,d_i)$ such that the multiset $\{(m_1d_1)^{e(m_1,d_1)},\cdots,(m_sd_s)^{e(m_s,d_s)}\}$ is a partition of n. The coefficient of x^n in the $A(q)$-transform (Definition 3.1) $T_{A(q)}F(x,q)$ of $F(x,q) = 1+\sum_{n\geq 1} a(n,q)x^n$ is

$$\sum_{\lambda\in M_n} \prod_{\{d \mid \exists m\colon\ (m,d)\in B(\lambda)\}} \binom{A_d(q)}{[E(\lambda,(m,d)) \mid (m,d)\in B(\lambda)]} \times \prod_{\{m \mid (m,d)\in B(\lambda)\}} a(m,q^d)^{E(\lambda,e(m,d))}$$

where

- the first product extends over the set of all second coordinates of the multiset λ
- $[E(\lambda,(m,d)) \mid (m,d)\in B(\lambda)]$ is the multiset of multiplicities of elements of λ with d as second coordinate
- the multinomial coefficient

$$\binom{n}{k_1,\ldots,k_s} = \frac{n(n-1)\cdots(n+1-\sum k_i)}{k_1!k_2!\cdots k_s!}$$

For instance, the multiset $\{(1,1)^2,(2,1)^2,(1,2)^2\}$ from M_{10} contributes the term

$$\binom{A_1(q)}{2,2} a(1,q)^2 a(2,q)^2 \binom{A_2(q)}{2} a(2,q^2)^2$$

to the sum over all the 244 multisets in M_{10}.

The ordinary generating function for the number of elements in M_n is

$$1+\sum_{n\geq 1}|M_n|x^n = \prod_{k\geq 1}(1-x^k)^{-\tau(k)}$$

where $\tau(k)$ is the number of divisors of k. The first terms are $|M_n| = 1,3,5,11,17,34,52,94,145,244,\ldots$.

Proposition 3.13, (3.17) and the recursive relations $F_{r+1}(x,q)^{\pm 1} = T_{\mathrm{IM}(q)}F_r(x,q)^{\pm 1} = T_{-\,\mathrm{IM}(q)}F_r(x,q)^{\mp 1}$ give a sequence of polynomial identities

$$\widetilde{\chi}_{r+1}^{-\varepsilon_1}(n,q) = \sum_{\lambda\in M_n} \prod_{\{d \mid \exists m\colon\ (m,d)\in B(\lambda)\}} \binom{\varepsilon_1\varepsilon_2\,\mathrm{IM}_d(q)}{[E(\lambda,(m,d)) \mid (m,d)\in B(\lambda)]}$$

$$\times \prod_{\{m|(m,d)\in B(\lambda)\}} \widetilde{\chi}_r^{-\varepsilon_2}(m,q^d), \qquad r \geq 1$$

where $\widetilde{\chi}_{r+1}^{-\varepsilon_1}(n,q) = \frac{1}{n!}\sum_{\lambda\vdash n} \varepsilon_1^{|\lambda|}T(\lambda)U(\lambda,q)^r$, $\widetilde{\chi}_r^{-\varepsilon_2}(m,q^d) = \sum_{\mu\vdash m} \varepsilon_2^{|\mu|}T(\mu)U(\mu,q^d)^{r-1}$ for $\varepsilon_1,\varepsilon_2 = \pm 1$. Taking $r = 1$ and $r = 2$ we get the polynomial identities

$$\widetilde{\chi}_2^{-\varepsilon_1}(n,q) = \frac{1}{n!}\sum_{\lambda\vdash n} \varepsilon_1^{|\lambda|}T(\lambda)U(\lambda,q)$$
$$= \begin{cases} \sum_{\lambda\vdash n}(-1)^{|\lambda|}\prod_{d\in B(\lambda)}\binom{-\varepsilon_1 \operatorname{IM}_d(q)}{E(\lambda,d)} & \varepsilon_2 = -1 \\ \sum_{\lambda\in M_n}\prod_d \binom{\varepsilon_1 \operatorname{IM}_d(q)}{[E(\lambda,(m,d)) \mid (m,d)\in B(\lambda)]} & \varepsilon_2 = +1 \end{cases}$$

$$\widetilde{\chi}_3^{-\varepsilon_1}(n,q) = \frac{1}{n!}\sum_{\lambda\vdash n} \varepsilon_1^{|\lambda|}T(\lambda)U(\lambda,q)^2$$
$$= \begin{cases} \sum_{\lambda\in M_n}\prod_d \binom{-\varepsilon_1 \operatorname{IM}_d(q)}{[E(\lambda,(m,d)) \mid (m,d)\in B(\lambda)]}\prod_m (1-q^d)^{E(\lambda,(m,d))} \\ \quad \varepsilon_2 = -1 \\ \sum_{\lambda\in M_n}\prod_d \binom{\varepsilon_1 \operatorname{IM}_d(q)}{[E(\lambda,(m,d)) \mid (m,d)\in B(\lambda)]}\prod_m (q^{dm}-q^{d(m-1)})^{E(\lambda,(d,m))} \\ \quad \varepsilon_2 = +1 \end{cases}$$

where we used that $\widetilde{\chi}_1(m,q^d) = -\delta_{1,m}$ (Proposition 2.5.(1)) contributes only for $m = 1$, $\widetilde{\chi}_1^{-1}(m,q^d) = 1$, $\widetilde{\chi}_2(m,q^d) = 1-q^d$ and $\widetilde{\chi}_2^{-1}(m,q^d) = q^{dm}-q^{d(m-1)}$. The left sides above are

$$\widetilde{\chi}_2^{-\varepsilon_1}(n,q) = \begin{cases} q^n - q^{n-1} & \varepsilon_1 = +1 \\ 1-q & \varepsilon_1 = -1 \end{cases} \qquad \widetilde{\chi}_3^{-\varepsilon_1}(n,q) = \begin{cases} \frac{q-1}{q+1}(q^{2n}-1) & \varepsilon_1 = +1 \\ -nq^{n-1}(q-1)^2 & \varepsilon_1 = -1 \end{cases}$$

The polynomial identities [22, Theorem A, B] are the identities at $r = 1$ for $\widetilde{\chi}_2^{\pm}(n,q)$. The polynomial identities for $r > 1$ and the identities involving Corollary 3.13 seem to be new.

Specializing further to $n = 3$, the index set

$$M_3 = \{\{(3,1)\},\{(1,3)\},\{(1,1),(1,2)\},\{(1,1),(2,1)\},\{(1,1)^3\}\}$$

contains 5 multisets and the above identities for $\widetilde{\chi}_2^{\pm\varepsilon_1}(3,q)$ are

$$\widetilde{\chi}_2^{-1}(3,q) = q^3 - q^2 = \frac{1}{6}(2(q^3-1) + 3(q^2-1)(q-1) + (q-1)^3)$$
$$\overset{\varepsilon_2=-1}{\underset{\text{A}}{=}} -\binom{-\operatorname{IM}_3(q)}{1} + \binom{-\operatorname{IM}_2(q)}{1}\binom{-\operatorname{IM}_1(q)}{1} - \binom{-\operatorname{IM}_1(q)}{3}$$

$$\overset{\varepsilon_2=+1}{=} \binom{\mathrm{IM}_3(q)}{1} + \binom{\mathrm{IM}_1(q)}{1} + \binom{\mathrm{IM}_1(q)}{1}\binom{\mathrm{IM}_2(q)}{1} + \binom{\mathrm{IM}_1(q)}{1,1} + \binom{\mathrm{IM}_1(q)}{3}$$

$$\widetilde{\chi}_2(3,q) = 1-q = \frac{1}{6}(-2(q^3-1)+3(q^2-1)(q-1)-(q-1)^3)$$

$$\underset{\mathrm{B}}{\overset{\varepsilon_2=-1}{\equiv}} -\binom{\mathrm{IM}_3(q)}{1} + \binom{\mathrm{IM}_2(q)}{1}\binom{\mathrm{IM}_1(q)}{1} - \binom{\mathrm{IM}_1(q)}{3}$$

$$\overset{\varepsilon_2=+1}{=} \binom{-\mathrm{IM}_3(q)}{1} + \binom{-\mathrm{IM}_1(q)}{1} + \binom{-\mathrm{IM}_1(q)}{1}\binom{-\mathrm{IM}_2(q)}{1} + \binom{-\mathrm{IM}_1(q)}{1,1} + \binom{-\mathrm{IM}_1(q)}{3}$$

while for $\widetilde{\chi}_3^{\pm\varepsilon_1}(3,q)$ they are

$$\widetilde{\chi}_3^{-1}(3,q) = \frac{q-1}{q+1}(q^6-1) = \frac{1}{6}(2(q^3-1)^2+3(q^2-1)^2(q-1)^2+(q-1)^6)$$

$$\overset{\varepsilon_2=-1}{=} \binom{-\mathrm{IM}_3(q)}{1}(1-q^3) + \binom{-\mathrm{IM}_1(q)}{1}(1-q) + \binom{-\mathrm{IM}_1(q)}{1}\binom{-\mathrm{IM}_2(q)}{1}(1-q)(1-q^2) + \binom{-\mathrm{IM}_1(q)}{1,1}(1-q)^2 + \binom{-\mathrm{IM}_1(q)}{3}(1-q)^3$$

$$\overset{\varepsilon_2=+1}{=} \binom{\mathrm{IM}_3(q)}{1}(q^3-1) + \binom{\mathrm{IM}_1(q)}{1}(q^3-q^2) + \binom{\mathrm{IM}_1(q)}{1}\binom{\mathrm{IM}_2(q)}{1}(q-1)(q^2-1) + \binom{\mathrm{IM}_1(q)}{1,1}(q-1)(q^2-q) + \binom{\mathrm{IM}_1(q)}{3}(q-1)^3$$

$$\widetilde{\chi}_3(3,q) = -3(q-1)^2q^2 = \frac{1}{6}(-2(q^3-1)^2+3(q^2-1)^2(q-1)^2-(q-1)^6)$$

$$\overset{\varepsilon_2=-1}{=} \binom{\mathrm{IM}_1(q)}{1}(1-q) + \binom{\mathrm{IM}_3(q)}{1}(1-q^3) + \binom{\mathrm{IM}_1(q)}{1}\binom{\mathrm{IM}_2(q)}{1}(1-q)(1-q^2) + \binom{\mathrm{IM}_1(q)}{1,1}(1-q)^2 + \binom{\mathrm{IM}_1(q)}{3}(1-q)^3$$

$$\overset{\varepsilon_2=+1}{=} \binom{-\mathrm{IM}_3(q)}{1}(q^3-1) + \binom{-\mathrm{IM}_1(q)}{1}(q^3-q^2)$$

$$+\binom{-\operatorname{IM}_1(q)}{1}\binom{-\operatorname{IM}_2(q)}{1}(q-1)(q^2-1)$$
$$+\binom{-\operatorname{IM}_1(q)}{1,1}(q-1)(q^2-q)+\binom{-\operatorname{IM}_1(q)}{3}(q-1)^3$$

4. The p-primary equivariant reduced Euler characteristic

The *rth p-primary equivariant reduced Euler characteristic* of the G-poset Π is the normalized sum [21, (1-5)]

$$\widetilde{\chi}_r(\Pi, G, p) = \frac{1}{|G|} \sum_{X \in \operatorname{Hom}(\mathbf{Z}\times\mathbf{Z}_p^{r-1}, G)} \widetilde{\chi}(C_\Pi(X(\mathbf{Z}\times\mathbf{Z}_p^{r-1}))) \tag{4.1}$$

of the reduced Euler characteristics of the $X(\mathbf{Z}\times\mathbf{Z}_p^{r-1})$-fixed Π-subposets as X ranges over the set of all homomorphisms of $\mathbf{Z}\times\mathbf{Z}_p^{r-1}$ to G. When G acts trivially on Π, $\widetilde{\chi}_r(\Pi, G, p) = \widetilde{\chi}(\Pi)|\operatorname{Hom}(\mathbf{Z}_p^{r-1}, G)/G|$ is proportional to the number of conjugacy classes of commuting $(r-1)$-tuples of p-singular elements of G [9, Lemma 4.13]. (A group element is *p-singular* if its order is a power of p [7, Definition 40.2, §82.1].) When p does not divide the order of G, there are no nontrivial p-singular elements in G and $\widetilde{\chi}_r(\Pi, G, p) = \widetilde{\chi}_1(\Pi, G)$ does not depend on r. In particular, the primary equivariant Euler characteristics of the $\operatorname{GL}_n(\mathbf{F}_q)$-poset $\operatorname{L}_n^*(\mathbf{F}_q)$ are defined as follows.

Definition 4.2. The rth, $r \geq 1$, p-primary equivariant reduced Euler characteristic of the $\operatorname{GL}_n(\mathbf{F}_q)$-poset $\operatorname{L}_n^*(\mathbf{F}_q)$ is the normalized sum

$$\widetilde{\chi}_r(\operatorname{L}_n^*(\mathbf{F}_q), \operatorname{GL}_n(\mathbf{F}_q), p) = \frac{1}{|\operatorname{GL}_n(\mathbf{F}_q)|} \sum_{X \in \operatorname{Hom}(\mathbf{Z}\times\mathbf{Z}_p^{r-1}, \operatorname{GL}_n(\mathbf{F}_q))} \widetilde{\chi}(C_{\operatorname{L}_n^*(q)}(X(\mathbf{Z}\times\mathbf{Z}_p^{r-1})))$$

of reduced Euler characteristics.

In this section we calculate the p-primary generating functions $F_r(x, q, p) = 1 + \sum_{n\geq 1} \widetilde{\chi}_r(n, q, p)x^n$ (1.6) for the p-primary equivariant reduced Euler characteristics, $\widetilde{\chi}_r(n, q, p) = \widetilde{\chi}_r(\operatorname{L}_n^*(\mathbf{F}_q), \operatorname{GL}_n(\mathbf{F}_q), p)$.

Proposition 4.3. *Suppose that $r = 1$ or $n = 1$.*

(1) When $r = 1$, $\widetilde{\chi}_1(n, q, p) = \widetilde{\chi}_1(n, q) = -\delta_{1,n}$ is -1 for $n = 1$ and 0 for $n > 1$.
(2) When $n = 1$, $\widetilde{\chi}_r(1, q, p) = -(q-1)_p^{r-1}$ for all p, q, and $r \geq 1$.

Proof. When $r = 1$, the p-primary equivariant reduced Euler characteristic and the equivariant reduced Euler characteristic agree by Definition 4.2 and we refer to Proposition 2.5.(1). When $n = 1$,

$$\begin{aligned}\widetilde{\chi}_r(1,q,p) &= -|\operatorname{Hom}(\mathbf{Z}\times\mathbf{Z}_p^{r-1},\operatorname{GL}_1(\mathbf{F}_q))|/|\operatorname{GL}_1(\mathbf{F}_q)| = -(q-1)(q-1)_p^{r-1}/(q-1)\\ &= -(q-1)_p^{r-1}\end{aligned}$$

since $\mathrm{L}_1^*(\mathbf{F}_q)=\emptyset$ and $\widetilde{\chi}(\emptyset)=-1$. □

According to Proposition 4.3.(1), the first p-primary generating function $F_1(x,q,p)=1-x$ is independent of p and q. In fact, $F_r(x,q,p)=1-x$ for all $r\geq 1$ if q is a power of p by Lemma 2.6. The interesting case is thus when $p\nmid q$ where the first terms in the rth generating function are $F_r(x,q,p)=1-(q-1)_p^{r-1}x+\cdots$.

The analogue of Corollary 2.7, proved exactly as before, asserts that

$$\widetilde{\chi}_r(n,q,p)=\sum_{[g]\in[\operatorname{GL}_n(\mathbf{F}_q)_p]}\widetilde{\chi}_{r-1}(C_{\mathrm{L}_n^*(\mathbf{F}_q)}(g),C_{\operatorname{GL}_n(\mathbf{F}_q)}(g),p)$$

where the sum is extended over the set $[\operatorname{GL}_n(\mathbf{F}_q)_p]$ of p-singular conjugacy classes in $\operatorname{GL}_n(\mathbf{F}_q)$.

The *order* of a polynomial $f\in\mathbf{F}_q[t]$ with $f(0)\neq 0$ is the least positive integer e for which $f(t)$ divides t^e-1 [13, Definition 3.2].

Lemma 4.4. *A semi-simple element of $\operatorname{GL}_n(\mathbf{F}_q)$ is p-singular if and only if all irreducible factors of its characteristic polynomial have p-power order.*

Proof. Is enough to show that multiplication by t on $\mathbf{F}_q[t]/(f(t))$, where $f(t)$ is an irreducible monic polynomial with $f(0)\neq 0$, has p-power order if and only if f has p-power order. But multiplication by t has p-power order if and only if $f(t)$ divides t^c-1 for some p-power c if and only if the order of f divides c by [13, Lemma 3.6]. □

As in Section 3 we conclude from Lemma 4.4 that the p-primary generating functions obey the recurrence relation

$$F_{r+1}(x,q,p)=T_{\mathrm{IM}(q,p)}F_r(x,q,p),\qquad r\geq 1 \tag{4.5}$$

with base function $F_1(x,q,p)=1-x$.

We need a little preparation before we can solve (4.5). The following observation is the p-primary analogue of a fundamental classical result.

Theorem 4.6. *The product of all monic irreducible polynomials in $\mathbf{F}_q[t]$ with nonzero constant term, p-power order, and degree dividing $n\geq 1$ is $t^{(q^n-1)_p}-1$.*

Proof. We already know from the classical theorem [13, Theorem 3.20] that each irreducible factor in $t^{q^n-1}-1$ occurs exactly once in the factorization. If f is an irreducible factor of $t^{(q^n-1)_p}-1$, then the order of f divides $(q^n-1)_p$ by [13, Corollary 3.7]. Conversely, let f, $f(0)\neq 0$, be a monic irreducible polynomial of degree dividing n and of order p^e for some $e \geq 0$. Then f divides $t^c - 1$ where $c = \gcd(p^e, q^n - 1)$ and hence f also divides $t^{(q^n-1)_p} - 1$ [13, Lemma 3.6, Corollary 3.7]. $\square$

By comparing the degree of $t^{(q^n-1)_p} - 1$ with the total degree of its canonical factorization [13, Theorem 1.59] we obtain p-primary versions

$$(q^n - 1)_p = \sum_{d|n} d\,\mathrm{IM}_d(q,p), \qquad n\,\mathrm{IM}_n(q,p) = \sum_{d|n} \mu(n/d)(q^d - 1)_p \tag{4.7}$$

of the classical relations (3.3). See Section 1 for the definition of $\mathrm{IM}_d(p,q)$.

We are now ready to prove Theorem 1.7. The present proof, a tremendous improvement of the original lengthy case-by-case checking, is due to an anonymous referee. A similar argument can be used to prove Theorem 1.4.

Proof of Theorem 1.7. We must show that the power series

$$F_r(x,q,p) = \prod_{n\geq 1}(1-x^n)^{a_r(n,q,p)}, \qquad a_r(n,q,p) = \frac{1}{n}\sum_{d|n}\mu(n/d)(q^d-1)_p^{r-1}$$

solve recurrence (4.5). Indeed, the $\mathrm{IM}(q,p)$-transform of $F_r(x,q,p)$ equals $F_{r+1}(x,q,p)$ because in the product

$$\begin{aligned} T_{\mathrm{IM}(q,p)}F_r(x,q,p) &= \prod_{d\geq 1} F_r(x^d, q^d, p)^{\mathrm{IM}_d(q,p)} = \prod_{d,n\geq 1}(1-x^{nd})^{\mathrm{IM}_d(q,p)a_r(n,q^d,p)} \\ &= \prod_{N\geq 1}(1-x^N)^{\sum_{d|N} a_r(N/d,q^d,p)\,\mathrm{IM}_d(q,p)} \end{aligned}$$

the exponent of the $(1-x^N)$-factor is

$$\begin{aligned} \sum_{d|N} a_r(N/d,q^d,p)\,\mathrm{IM}_d(q,p) &= \sum_{d|N}\frac{d}{N}\sum_{e|(N/d)}\mu(N/de)(q^{de}-1)_p^{r-1}\,\mathrm{IM}_d(q,p) \\ &\overset{(4.7)}{=} \frac{1}{N}\sum_{d|N}\sum_{e|(N/d)}\mu(N/de)(q^{de}-1)_p^{r-1}\sum_{f|d}\mu(d/f)(q^f-1)_p \\ &= \frac{1}{N}\sum_{f|d_1|d_2|N}\mu(N/d_2)(q^{d_2}-1)_p^{r-1}\mu(d_1/f)(q^f-1)_p \\ &= \frac{1}{N}\sum_{d|N}\mu(N/d)(q^d-1)_p^{r-1}(q^d-1)_p = \frac{1}{N}\sum_{d|N}\mu(N/d)(q^d-1)_p^r = a_{r+1}(N,q,p) \end{aligned}$$

In this calculation we used that, for fixed f and d_2, the sum

$$\sum_{d_1:\ f|d_1|d_2}\mu(d_1/f) = \begin{cases} 1 & f = d_2 \\ 0 & f < d_2 \end{cases}$$

contributes only when $f = d_1 = d_2$. □

Corollary 4.8. *The $(r+1)$th, $r \geq 0$, p-primary equivariant reduced Euler characteristics satisfy the recursion*

$$\widetilde{\chi}_{r+1}(n,q,p) = \begin{cases} 1 & n = 0 \\ -\dfrac{1}{n} \displaystyle\sum_{1 \leq j \leq n} (q^j - 1)_p^r \widetilde{\chi}_{r+1}(n-j,q,p) & n > 0 \end{cases}$$

Proof. Apply Lemma 3.7 to the formula of Theorem 1.7. □

We now look more closely at the sequence $\mathrm{IM}(q,p) = (\mathrm{IM}_d(q,p))_{d \geq 1}$ recording the number of irreducible monic polynomials of p-power order, nonzero constant term, and degree d in $\mathbf{F}_q[t]$. If q is a power of p, $\mathrm{IM}_1(q,p) = 1$ and $\mathrm{IM}_n(q,p) = 0$ for all $n > 1$, as the only polynomial that fulfills the requirements is $f(t) = t - 1$ [13, Corollary 3.2]. In the more interesting case where p and q are prime, consider the subgroup $\langle q \rangle$ of $\mathbf{Z}_p^\times$ generated by q in the unit topological group $\mathbf{Z}_p^\times$ of the ring $\mathbf{Z}_p$ of p-adic integers.

Lemma 4.9. *When $p \nmid q$, the sequence $\mathrm{IM}(q,p)$ and the function $F_r(x,q,p)$, $r \geq 1$, depend only on the closure $\overline{\langle q \rangle}$ in $\mathbf{Z}_p^\times$ of $\langle q \rangle$.*

Proof. The integer $\mathrm{IM}_d(q,p)$ depends only on the images of $\langle q \rangle$ under the continuous [18, Chp. 1, §3] homomorphisms $\mathbf{Z}_p^\times \to (\mathbf{Z}/p^n\mathbf{Z})^\times$, $n \geq 1$. But $\langle q \rangle$ and $\overline{\langle q \rangle}$ have the same image in the discrete topological space $(\mathbf{Z}/p^n\mathbf{Z})^\times$. □

We say that q_1 and q_2, prime powers prime to p, are p-equivalent if $\overline{\langle q_1 \rangle} = \overline{\langle q_2 \rangle}$ in $\mathbf{Z}_p^\times$. More explicitly, q_1 and q_2 are p-equivalent if and only if $O(q_1,p) = O(q_2,p)$ [5, §3] where, for a prime power q prime to p, $O(q,p)$ denotes the integer pair

$$O(q,p) = \begin{cases} (q \bmod 8, \nu_2(q^2-1)) & p = 2 \\ (\mathrm{ord}_p(q), \nu_p(q^{\mathrm{ord}_p(q)} - 1)) & p > 2 \end{cases} \tag{4.10}$$

The multiplicative order, $\mathrm{ord}_p(q)$, was defined in Section 1. The following well-known lemma can be used to calculate p-adic valuations.

Lemma 4.11 *(Lifting the Exponent). Let p be any prime and $n \geq 1$ any natural number.*

(1) If $a \equiv b \not\equiv 0 \bmod p$ and $\gcd(p,n) = 1$ then $\nu_p(a^n - b^n) = \nu_p(a-b)$.
(2) If p is odd and $a \equiv b \not\equiv 0 \bmod p$ then $\nu_p(a^n - b^n) = \nu_p(a-b) + \nu_p(n)$.
(3) If a and b are odd and n even then $\nu_2(a^n - b^n) = \nu_2(a-b) + \nu_2(a+b) + \nu_2(n) - 1$.
(4) If a and b are odd and $a \equiv b \bmod 4$ then $\nu_2(a^n - b^n) = \nu_2(a-b) + \nu_2(n)$.

We first consider the situation when p is an *odd* prime. Let g be a prime primitive root mod p^2 [11, Definition p 41]. Such a prime g always exists by the Dirichlet Density Theorem [11, Chp. 16, §1, Theorem 1] and the congruence class of g generates $(\mathbf{Z}/p^n\mathbf{Z})^\times$ for all $n \geq 1$ [11, Chp. 4, §1, Theorem 2]. By [4, Lemma 1.11.(a)] it suffices to consider p-primary generating functions $F_r(x, (g^s)^{p^e}, p)$ at the prime powers $(g^s)^{p^e}$ where s divides $p-1$ and $e \geq 0$.

Lemma 4.12. *Let p be an odd prime and $q = g^{p-1}$. For all $n \geq 1$ and $r \geq 0$,*

$$F_{r+1}(x,q,p) = \exp(-\sum_{n\geq 1}(pn)_p^r\frac{x^n}{n}), \qquad \mathrm{IM}_n(q,p) = \begin{cases} p & n=1 \\ p-1 & n = n_p > 1 \\ 0 & \textit{otherwise} \end{cases}$$

and, for any $e \geq 0$, $\mathrm{IM}_n(q^{p^e},p) = p^e\,\mathrm{IM}_n(q,p)$ and $F_{r+1}(x,q^{p^e},p) = F_{r+1}(x,q,p)^{p^{re}}$.

Proof. Since $(q^n-1)_p = pn_p$, Theorem 1.7 immediately gives the formula for $F_{r+1}(x,q,p)$. An elementary calculation verifies $\sum_{d|n} d\,\mathrm{IM}_d(q,p) = (q^n-1)_p$ when the integers $\mathrm{IM}_d(q,p)$ are defined as in the lemma. Since $(q^{dp^e}-1)_p = p^e(q^d-1)_p$, we get $\mathrm{IM}_n(q,p^{p^e}) = p^e\,\mathrm{IM}_n(q,p)$ by (4.7) and $F_{r+1}(x,q^{p^e},p) = F_{r+1}(x,p,q)^{p^e}$ by Theorem 1.7. □

Lemma 4.13. $\exp(-\sum_{n\geq 1}(pn)_p^r\frac{x^n}{n}) = \prod_{n\geq 0}\Big(\frac{(1-x^{p^n})^p}{1-(x^{p^n})^p}\Big)^{p^{(r-1)(n+1)}}$ *for any prime p.*

Proof. Let $F(x) = \exp(-\sum_{n\geq 1}(pn)_p^r\frac{x^n}{n})$. The rewriting

$$-\sum_{n\geq 1}(pn)_p^r\frac{x^n}{n} = -\sum_{p\nmid n}p^r\frac{x^n}{n} - \sum_{p|n}(pn)_p^r\frac{x^n}{n} = -p^{r-1}\sum_{p\nmid n}p\frac{x^n}{n} - p^{r-1}\sum_{n\geq 1}(pn)_p^r\frac{(x^p)^n}{n}$$

translates to the functional equation

$$F(x) = \Big(\frac{(1-x)^p}{1-x^p}\Big)^{p^{r-1}} F(x^p)^{p^{r-1}}$$

Repeated use of this relation leads to the product expansion of the lemma. □

Corollary 4.14. *Let p be an odd prime. When $(q-1)_p = p$, i.e. $O(q,p) = (1,1)$, and $r \geq 0$*

$$F_{r+1}(x,q,p) = \prod_{n\geq 0}\Big(\frac{(1-x^{p^n})^p}{1-(x^{p^n})^p}\Big)^{p^{(r-1)(n+1)}}$$

Proof. Combine Lemma 4.12 and Lemma 4.13. □

Lemma 4.15. *Let p be an odd prime and $q = g^s$ where $s \geq 1$ and $st = p - 1$ for some $t > 1$. Then*

$$F_{r+1}(x, q^{p^e}, p)^t = \frac{(1-x)^t}{1-x^t} F_{r+1}(x^t, (g^{p-1})^{p^e}, p),$$

$$\mathrm{IM}_n(q^{p^e}, p) = \begin{cases} 1 & n = 1 \\ (p^{1+e} - 1)/t & n = t \\ sp^e & t \mid n, n/t = (n/t)_p > 1 \\ 0 & \textit{otherwise} \end{cases}$$

for all $e \geq 0$ and $r \geq 0$.

Proof. Note that $(q^{np^e} - 1)_p$ equals $p^e(pn/t)_p$ if n is divisible by t and 1 if not. The tth power of the generating function $F_{r+1}(x, q^{p^e}, p)$ from Theorem 1.7 is

$$\begin{aligned} F_{r+1}(x, q^{p^e}, p)^t &= \exp(-t \sum_{n \geq 1} (q^{np^e} - 1)_p^r \frac{x^n}{n}) = \exp(-t \sum_{t \nmid n} \frac{x^n}{n} - t \sum_{t \mid n} p^{re} (pn)_p^r \frac{x^n}{n}) \\ &= \exp(-t \sum_{t \nmid n} \frac{x^n}{n} - \sum_{n \geq 1} p^{re} (pn)_p^r \frac{(x^t)^n}{n}) = \frac{(1-x)^t}{1-x^t} F_{r+1}(x^t, (g^{p-1})^{p^e}, p) \end{aligned}$$

An elementary calculation confirms that $\sum_{d|n} d\, \mathrm{IM}_d(q, p) = (q^n - 1)_p$ when the integers $\mathrm{IM}_d(q, p)$ are defined as in the lemma. □

Example 4.16. The 3-equivalence classes of prime powers prime to 3 are represented by 2^{3^e} and 4^{3^e} with $O(2^{3^e}, 3) = (2, 1+e)$ and $O(4^{3^e}, 3) = (1, 1+e)$, $e \geq 0$ [4, Lemma 1.11.(a)] as 2 is a primitive root modulo 9. The 3-equivalence classes of 2, 2^3, 4, and 4^3 contain the prime powers

$$\begin{array}{ll} 2, 5, 11, 23, 29, 32, 41, 47, 59, \dots & 8, 17, 71, 89, 125, 179, 197, 233, \dots \\ 4, 7, 13, 16, 25, 31, 43, 49, 61, \dots & 19, 37, 64, 73, 127, 181, 199, 289, \dots \end{array}$$

The 3-primary generating functions satisfy

$$\begin{aligned} F_{r+1}(x, 4^{3^e}, 3) &= \exp(-\sum_{n \geq 1} (3n)_3^r \frac{x^n}{n})^{3^{re}} = \prod_{n \geq 0} \left(\frac{(1 - x^{3^n})^3}{1 - (x^{3^n})^3} \right)^{3^{(r-1)(n+1)+re}} \\ F_{r+1}(x, 2^{3^e}, 3)^2 &= \frac{1-x}{1+x} F_{r+1}(x^2, 4^{3^e}, 3) \end{aligned}$$

according to Lemma 4.12, 4.13, 4.15.

When p is odd, $p \nmid q$ and $\mathrm{ord}_p(q)$ is big, many p-primary equivariant reduced Euler characteristics vanish.

Table 3
2-primary equivariant generating functions $F_{r+1}(x,q,2)$ for $r \geq 0$.

	$(q^n-1)_2$	$\mathrm{IM}_n(q,2)$	$F_{r+1}(x,q,2)$
$q=-3$	$(4n)_2$	$\begin{cases} 4 & n=1 \\ 2 & n=n_2>1 \\ 0 & \text{otherwise} \end{cases}$	$\exp(-\sum_{n\geq 1}(4n)_2^r \frac{x^n}{n})$
$q=+3^{2^e}, e>0$	$2^e(4n)_2$	$2^e\,\mathrm{IM}_n(2,-3)$	$F_{r+1}(x,-3,2)^{2^{re}}$
$q=3$	$\begin{cases} 2 & 2\nmid n \\ (4n)_2 & 2\mid n \end{cases}$	$\begin{cases} 2 & n=1 \\ 3 & n=2 \\ 2 & n=n_2>2 \\ 0 & \text{otherwise} \end{cases}$	$\left(\frac{1-x}{1+x}\right)^{2^{r-1}} F_{r+1}(x^2,-3,2)^{2^{r-1}}$
$q=-3^{2^e}, e>0$	$\begin{cases} 2 & 2\nmid n \\ 2^e(4n)_2 & 2\mid n \end{cases}$	$\begin{cases} 2 & n=1 \\ 2^{2+e}-1 & n=2 \\ 2^{1+e} & n=n_2>2 \\ 0 & \text{otherwise} \end{cases}$	$\left(\frac{1-x}{1+x}\right)^{2^{r-1}} F_{r+1}(x^2,3^{2^e},2)^{2^{r-1}}$

Proposition 4.17. *Assume p is odd, prime to q and $\mathrm{ord}_p(q)>2$. Then $\widetilde{\chi}_{r+1}(n,q,p)=0$ unless $n\equiv 0,1 \bmod d$, $\widetilde{\chi}_{r+1}(n,q,p)+\widetilde{\chi}_{r+1}(n+1,q,p)=0$ when $n\equiv 0 \bmod d$, and $\widetilde{\chi}_{r+1}(d,q,p)=-\frac{1}{d}((q^d-1)_p^r-1)$ where $d=\mathrm{ord}_p(q)$ and $r\geq 0$.*

Proof. For $n=0$, $\widetilde{\chi}_{r+1}(0,q,p)=1$ by convention. For $n=1$, $\widetilde{\chi}_{r+1}(1,q,p)=-(q-1)_p^r=-1$ by Proposition 4.3.(2). Lemma 4.11 shows that $(q^{md}-1)_p=(q^d-1)_p m_p$ for any $m\geq 1$ while $(q^k-1)_p=1$ for any $k\geq 1$ not a multiplum of d. By Corollar 4.8, $-2\widetilde{\chi}_{r+1}(2,q,p)=(q-1)_p^r\widetilde{\chi}_{r+1}(1,q,p)+(q^2-1)_p^r\widetilde{\chi}_{r+1}(0,q,p)=-1+1=0$. It is now clear that we can proceed by induction using Corollary 4.8. □

Next, we consider the case $p=2$. The 2-equivalence classes of odd prime powers are represented by the 2-adic numbers $\pm 3^{2^e}$ [4, Lemma 1.11.(b)] with

$$O(\pm 3^{2^e},2)=\begin{cases}(\pm 3,3) & e=0\\ (\pm 1,3+e) & e>0\end{cases}$$

The 2-classes of $-3, 3^2, 3, -3^2$ contain the prime powers

$$\begin{array}{ll} 5,13,29,37,53,61,101,109,125,\dots & 9,25,41,73,89,121,137,169,233,\dots \\ 3,11,19,27,43,59,67,83,107,131,\dots & 7,23,71,103,151,167,199,263,\dots \end{array}$$

The results for $p=2$ are summarized in Table 3. When $q=-3$, $((-3)^n-1)_2=(4n)_2$ and $F_{r+1}(x,-3,2)$ given by Theorem 1.7. An elementary calculation shows that $\sum_{d\mid n} d\,\mathrm{IM}_d(-3,2)=((-3)^n-1)_2$ when $\mathrm{IM}_d(-3,2)$ is as in Table 3. When $q=3$, the rewriting

$$-\sum_{n\geq 1}(3^n-1)_2^r\frac{x^n}{n}=-\sum_{2\nmid n}2^r\frac{x^n}{n}-\sum_{2\mid n}(4n)_2^r\frac{x^n}{n}=-2^{r-1}\sum_{2\nmid n}2\frac{x^n}{n}-2^{r-1}\sum_{n\geq 1}(4n)_2^r\frac{(x^2)^n}{n}$$

translates to

$$F_{r+1}(x,3,2)=\left(\frac{1-x}{1+x}\right)^{2^{r-1}}F_{r+1}(x^2,-3,2)^{2^{r-1}}$$

An elementary calculation shows that $\sum_{d|n} d\,\mathrm{IM}_d(3,2)=(3^n-1)_2$ when $\mathrm{IM}_d(3,2)$ is as in Table 3. The other cases are similar. Lemma 4.13 gives product expansions of the 2-primary generating functions.

4.1. Alternative presentations of the p-primary equivariant reduced Euler characteristics

Consider the p-primary version of the generating function (3.12),

$$G_n(x,q,p)=\sum_{r\geq 0}\widetilde{\chi}_{r+1}(n,q,p)x^r=-\delta_{1,n}+\widetilde{\chi}_2(n,q,p)x+\widetilde{\chi}_3(n,q,p)x^2+\cdots,\quad n\geq 0 \tag{4.18}$$

where the coefficient of x^r is the $(r+1)$th p-primary reduced Euler characteristics $\widetilde{\chi}_{r+1}(n,q,p)$. (Declare $\widetilde{\chi}_{r+1}(0,q,p)$ to be 1 for all $r\geq 0$.) We have $G_0(x,p,q)=\frac{x}{1-x}$ and $G_1(x,q,p)=-\frac{1}{1-x(q-1)_p}$ by Proposition 4.3.(2). If p is odd, $p\nmid q$ and $d=\mathrm{ord}_p(q)>2$, $G_n(x,q,p)=0$ unless $n\equiv 0,1 \bmod d$ and $G_d(x,q,p)=-\frac{1}{d}\left(\frac{1}{1-x(q^d-1)_p}-\frac{1}{1-x}\right)=-G_{d+1}(x,q,p)$ by Proposition 4.17. The following description of the power series $G_n(x,q,p)$ is obtained exactly as in Proposition 3.13 (and $|\lambda|$, $T(\lambda)$, and $U(\lambda,q)$ are as there).

Proposition 4.19. *For $r\geq 0$ and $n\geq 1$,*

$$\widetilde{\chi}_{r+1}(n,q,p)=\frac{1}{n!}\sum_{\lambda\vdash n}(-1)^{|\lambda|}T(\lambda)U(\lambda,q)_p^r,\qquad G_n(x,q,p)=\frac{1}{n!}\sum_{\lambda\vdash n}(-1)^{|\lambda|}\frac{T(\lambda)}{1-xU(\lambda,q)_p}$$

Examples of Proposition 4.19 with $p=2$ and $q=\pm 3^{2^e}$ are

$$1!G_1(x,3^{2^e},2)=\begin{cases}\frac{-1}{1-2x} & e=0\\ \frac{-1}{1-2^{2+e}x} & e>0\end{cases}$$

$$1!G_1(x,-3^{2^e},2)=\begin{cases}\frac{-1}{1-2^2x} & e=0\\ \frac{-1}{1-2x} & e>0\end{cases}$$

$$2!G_2(x,3^{2^e},2)=\begin{cases}\frac{-1}{1-2^3x}+\frac{1}{1-2^2x} & e=0\\ \frac{-1}{1-2^{3+e}x}+\frac{1}{1-2^{4+2e}x} & e>0\end{cases}$$

$$2!G_2(x,-3^{2^e},2)=\begin{cases}\frac{-1}{1-2^3x}+\frac{1}{1-2^4x} & e=0\\ \frac{-1}{1-2^{3+e}x}+\frac{1}{1-2^2x} & e>0\end{cases}$$

$$3!G_3(x, 3^{2^e}, 2) = \begin{cases} \frac{-2}{1-2x} + \frac{3}{1-2^4x} + \frac{-1}{1-2^3x} & e = 0 \\ \frac{-2}{1-2^{2+e}x} + \frac{3}{1-2^{5+2e}x} + \frac{-1}{1-2^{6+3e}x} & e > 0 \end{cases}$$

$$3!G_3(x, -3^{2^e}, 2) = \begin{cases} \frac{-2}{1-2^2x} + \frac{3}{1-2^5x} + \frac{-1}{1-2^6x} & e = 0 \\ \frac{-2}{1-2x} + \frac{3}{1-2^{4+e}x} + \frac{-1}{1-2^3x} & e > 0 \end{cases}$$

Define the reciprocal p-primary equivariant reduced Euler characteristic, $\widetilde{\chi}_{r+1}^{-1}(n, q, p)$, to be the coefficient of x^n in the reciprocal of $F_{r+1}(x, q, p)$. Then

$$1 + \sum_{n \geq 1} \widetilde{\chi}_{r+1}^{-1}(n, q, p)x^n = F_{r+1}(x, q, p)^{-1} \tag{4.20}$$

$$1 + \sum_{r \geq 1} \widetilde{\chi}_{r+1}^{-1}(n, q, p)x^r = \frac{1}{n!} \sum_{\lambda \vdash n} \frac{T(\lambda)}{1 - xU(\lambda, q)_p} \qquad \widetilde{\chi}_{r+1}^{-1}(n, q, p) = \frac{1}{n!} \sum_{\lambda \vdash n} T(\lambda)U(\lambda, q)_p^r \tag{4.21}$$

For instance, the multiplicative order $\mathrm{ord}_5(2) = 4$ and the sequences of (reciprocal) third 5-primary reduced equivariant Euler characteristics

$$(\widetilde{\chi}_3(n, 2, 5))_{n \geq 0} = (1, -1, 0, 0, -6, 6, 0, 0, 15, -15, 0, 0, -20, 20, 0, 0, 15, -15, 0, 0, -36 \cdots)$$
$$(\widetilde{\chi}_3^{-1}(n, 2, 5))_{n \geq 0} = (1, 1, 1, 1, 7, 7, 7, 7, 28, 28, 28, 28, 84, 84, 84, 84, 210, 210, 210, 210, 492, \ldots)$$

illustrate Proposition 4.17 and Definition (4.20).

We noted in Section 3.1 that $\widetilde{\chi}_2^{-1}(n, q) = q^{n-1}(q-1)$ counts semi-simple classes in $\mathrm{GL}_n(\mathbf{F}_q)$. The following corollary is the p-primary analogue.

Corollary 4.22. *The coefficient of x^n in the power series*

$$F_2(x, q, p)^{-1} = T_{\mathrm{IM}(q,p)}(1 - x)^{-1} = \exp\Big(\sum_{n \geq 1} (q^n - 1)_p \frac{x^n}{n}\Big) = \prod_{d \geq 1} (1 - x^d)^{-\mathrm{IM}_d(q,p)}$$

is the number of p-singular semi-simple classes in $\mathrm{GL}_n(\mathbf{F}_q)$.

The recursive relations $F_{r+1}(x, q, p)^{\pm 1} = T_{\mathrm{IM}(q,p)} F_r(x, q, p)^{\pm 1} = T_{-\mathrm{IM}(q,p)} \times F_r(x, q, p)^{\mp 1}$ give a sequence of p-primary polynomial identities

$$\widetilde{\chi}_{r+1}^{-\varepsilon_1}(n, q, p) = \sum_{\lambda \in M_n} \prod_{\{d \mid \exists m:\ (m,d) \in B(\lambda)\}} \binom{\varepsilon_1\varepsilon_2 \,\mathrm{IM}_d(q, p)}{[E(\lambda, (m, d)) \mid (m, d) \in B(\lambda)]} \times \prod_{\{m \mid (m,d) \in B(\lambda)\}} \widetilde{\chi}_r^{-\varepsilon_2}(m, q^d, p), \qquad r \geq 1$$

where $\widetilde{\chi}_{r+1}^{-\varepsilon_1}(n,q,p) = \frac{1}{n!}\sum_{\lambda\vdash n}\varepsilon_1^{|\lambda|}T(\lambda)U(\lambda,q)_p^r$, $\widetilde{\chi}_r^{-\varepsilon_2}(m,q^d,p) = \sum_{\mu\vdash m}\varepsilon_2^{|\mu|}T(\mu)\times U(\mu,q^d)_p^{r-1}$ for $\varepsilon_1,\varepsilon_2 = \pm 1$. The $r = 1$ term in this sequence, a p-primary version of [22, Theorem A, B], is

$$\begin{aligned}\widetilde{\chi}_2^{-\varepsilon_1}(n,q,p) &= \frac{1}{n!}\sum_{\lambda\vdash n}\varepsilon_1^{|\lambda|}T(\lambda)U(\lambda,q)_p \\ &= \begin{cases}\displaystyle\sum_{\lambda\vdash n}(-1)^{|\lambda|}\prod_{d\in B(\lambda)}\binom{-\varepsilon_1\,\mathrm{IM}_d(q,p)}{E(\lambda,d)} & \varepsilon_2=-1\\ \displaystyle\sum_{\lambda\in M_n}\prod_{d}\binom{\varepsilon_1\,\mathrm{IM}_d(q,p)}{[E(\lambda,(m,d))\mid (m,d)\in B(\lambda)]} & \varepsilon_2=+1\end{cases}\end{aligned}$$

where we used that $\widetilde{\chi}_1(m,q^d,p) = -\delta_{1,m}$ (Proposition 4.3.(1)) contributes only for $m = 1$ and $\widetilde{\chi}_1^{-1}(m,q^d,p) = 1$ for all $m \geq 1$.

When q is a power of p, the identity is the only p-singular semi-simple element in $\mathrm{GL}_n(\mathbf{F}_q)$ and the generating function of Corollary 4.22 is $F_2(x,q,p)^{-1} = (1-x)^{-1} = 1+\sum_{n\geq 1}x^n$. When $p \nmid q$, all p-singular classes are semi-simple so

$$1+\sum_{n\geq 1}|\,\mathrm{GL}_n(\mathbf{F}_q)_p/\,\mathrm{GL}_n(\mathbf{F}_q)|x^n = F_2(x,q,p)^{-1}$$

where $|\,\mathrm{GL}_n(\mathbf{F}_q)_p/\,\mathrm{GL}_n(\mathbf{F}_q)|$ is the number of p-singular classes. For instance, the groups $\mathrm{GL}_2(\mathbf{F}_q)$ and $\mathrm{GL}_3(\mathbf{F}_q)$ contain

$$\begin{aligned}\widetilde{\chi}_2^{-1}(2,q,p) &= \frac{1}{2!}((q^2-1)_p+(q-1)_p^2) \overset{\varepsilon_2=-1}{=} \binom{\mathrm{IM}_1(q,p)}{2} - \binom{-\,\mathrm{IM}_2(q,p)}{1} \\ &\overset{\varepsilon_2=+1}{=} \binom{\mathrm{IM}_2(q,p)}{1} + \binom{\mathrm{IM}_1(q,p)}{1} + \binom{\mathrm{IM}_1(q,p)}{2} \\ \widetilde{\chi}_2^{-1}(3,q,p) &= \frac{1}{3!}(2(q^3-1)_p+3(q-1)_p(q^2-1)_p+(q-1)_p^3) \\ &\overset{\varepsilon_2=-1}{=} -\binom{-\,\mathrm{IM}_3(q,p)}{1} + \binom{-\,\mathrm{IM}_2(q,p)}{1}\binom{-\,\mathrm{IM}_1(q,p)}{1} - \binom{-\,\mathrm{IM}_1(q,p)}{3} \\ &\overset{\varepsilon_2=+1}{=} \binom{\mathrm{IM}_3(q,p)}{1} + \binom{\mathrm{IM}_1(q,p)}{1} + \binom{\mathrm{IM}_1(q,p)}{1}\binom{\mathrm{IM}_2(q,p)}{1} \\ &+ \binom{\mathrm{IM}_1(q,p)}{1,1} + \binom{\mathrm{IM}_1(q,p)}{3}\end{aligned}$$

p-singular conjugacy classes when $p \nmid q$.

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