Journal of Pure and Applied Algebra 219 (2015) 3030–3052

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Homotopy equivalences between p-subgroup categories ☆

Matthew Gelvin, Jesper M. Møller

Institut for Matematiske Fag, Universitetsparken 5, DK-2100 København, Denmark

ARTICLE INFO

Article history:
Received 1 January 2013
Received in revised form 26 August 2014
Available online 30 October 2014
Communicated by C.A. Weibel

MSC:
05E15; 20J15

ABSTRACT

Let $\mathcal{S}_G^*$ be the Brown poset of nonidentity p-subgroups of the finite group G ordered by inclusion. Results of Bouc and Quillen show that $\mathcal{S}_G^*$ is homotopy equivalent to its subposets $\mathcal{S}_G^{*+\mathrm{rad}}$ of nonidentity radical p-subgroups and $\mathcal{S}_G^{*+\mathrm{eab}}$ of nonidentity elementary abelian p-subgroups. In this note we extend these results for the Brown poset of G to other categories of p-subgroups of G, including the p-fusion system of G.

1. Introduction

Let p be a prime number and G a finite group of order divisible by p.

The *Brown poset* $\mathcal{S}_G^*$ consists of the nonidentity p-subgroups of G; this can be viewed topologically as the simplicial complex $|\mathcal{S}_G^*|$. Brown showed in Ref. [6] that consideration of the Euler characteristic $\chi(|\mathcal{S}_G^*|)$ of $\mathcal{S}_G^*$ leads to a sort of topological Sylow's Theorem: If $|G|_p$ is the maximal power of p dividing $|G|$, then $\chi(|\mathcal{S}_G^*|) \equiv 1$ modulo $|G|_p$. Thus, G gives rise to a combinatorial geometric object — the realization of a category — whose topology reflects some of the algebraic structure of G.

$\mathcal{S}_G^*$ is comparatively simple for a category, but there are other constructions we might consider to gain further understanding of G. Dwyer's theory of homology decompositions [8] shows that even recalling the natural G-action on $\mathcal{S}_G^*$ is enough to determine the p-homology of G, though not conversely. On the other hand, the *centric linking system* $\mathcal{L}_G^{\mathrm{sfc}}$ of Ref. [4], whose objects are the *p-self-centralzing* subgroups of G, is significantly more complicated than $\mathcal{S}_G^*$ but both determines and is determined by the p-completed classifying space of G.

In each of these examples, the topological data is overdetermined by the category: We could have obtained the same result with a smaller collections of p-subgroups of G. Quillen showed [18] that the realization of $\mathcal{S}_G^*$ has the same homotopy type (and hence the same Euler characteristic) as the full subcategory $\mathcal{S}_G^{*+\mathrm{eab}}$

☆ Supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92).
E-mail addresses: mgelvin@gmail.com (M. Gelvin), moller@math.ku.dk (J.M. Møller).
URL: http://www.math.ku.dk/~moller/ (J.M. Møller).

http://dx.doi.org/10.1016/j.jpaa.2014.10.002
0022-4049/

of nontrivial elementary abelian subgroups of G, and Bouc later proved [1] the dual result that the full subcategory $\mathcal{S}_G^{*+\mathrm{rad}}$ of G-radical p-subgroups is also homotopy equivalent. Dwyer's notion of an *ample* collection of subgroups is precisely the requirement that the p-homology of G can be recovered from the resulting sub-G-poset; both elementary abelian and G-radical subgroups form such a collection. And finally, the inclusion of the full subcategory of G-radical subgroups $\mathcal{L}_G^{\mathrm{sfc}+\mathrm{rad}} \subseteq \mathcal{L}_G^{\mathrm{sfc}}$ was shown to be a homotopy equivalence in Ref. [3].

In this paper we are interested in exploring the homotopy type of several such *p-subgroup categories*: Brown posets $\mathcal{S}_G^*$, transporter systems $\mathcal{T}_G^*$, linking systems $\mathcal{L}_G^*$, orbit systems $\mathcal{O}_G$, and the ambient-group free abstractions of these: Frobenius categories (or fusion systems) $\mathcal{F}$, exterior quotients of Frobenius categories $\widetilde{\mathcal{F}}$ (the fusion-theoretic analogue of an orbit category), and abstract linking systems $\mathcal{L}$. See Section 2 for definitions. More precisely, we are interested in identifying certain classes of subgroups that control the homotopy type of each of these p-subgroup categories, in the sense of the main results of Sections 6–10:

Theorem A. *The following inclusion functors are homotopy equivalences.*

(a) $\mathcal{S}_G^{*+\mathrm{eab}} \hookrightarrow \mathcal{S}_G^*$, $\mathcal{S}_G^{*+\mathrm{rad}} \hookrightarrow \mathcal{S}_G^*$, $\mathcal{S}_G^{\mathrm{sfc}+\mathrm{rad}} \hookrightarrow \mathcal{S}_G^{\mathrm{sfc}}$
(b) $\mathcal{T}_G^{*+\mathrm{eab}} \hookrightarrow \mathcal{T}_G^*$, $\mathcal{T}_G^{*+\mathrm{rad}} \hookrightarrow \mathcal{T}_G^*$, $\mathcal{T}_G^{\mathrm{sfc}+\mathrm{rad}} \hookrightarrow \mathcal{T}_G^{\mathrm{sfc}}$
(c) $\mathcal{F}^{*+\mathrm{eab}} \hookrightarrow \mathcal{F}^*$
(d) $\mathcal{O}_G^{\mathrm{rad}} \hookrightarrow \mathcal{O}_G$, $\mathcal{O}_G^{*+\mathrm{rad}} \hookrightarrow \mathcal{O}_G^*$, $\mathcal{O}_G^{\mathrm{sfc}+\mathrm{rad}} \hookrightarrow \mathcal{O}_G^{\mathrm{sfc}}$
(e) $\widetilde{\mathcal{F}}^{*+\mathrm{eab}} \hookrightarrow \widetilde{\mathcal{F}}^*$, $\widetilde{\mathcal{F}}^{\mathrm{sfc}+\mathrm{rad}} \hookrightarrow \widetilde{\mathcal{F}}^{\mathrm{sfc}}$
(f) $\mathcal{L}_G^{*+\mathrm{eab}} \hookrightarrow \mathcal{L}_G^*$, $\mathcal{L}^{\mathrm{sfc}+\mathrm{rad}} \hookrightarrow \mathcal{L}^{\mathrm{sfc}}$

Here and for the rest of the paper, a functor of categories a *homotopy equivalence* if the induced map of geometric realizations is a homotopy equivalence.

As indicated above, some of these results are well known in the literature, while others are new and provide new insight into the topological relationships between these categories. For instance, Theorem A, (c) and (e), together with the equality of categories $\mathcal{F}^{*+\mathrm{eab}} = \widetilde{\mathcal{F}}^{*+\mathrm{eab}}$ implies the unexpected

Corollary. *The quotient functor $\mathcal{F}^* \to \widetilde{\mathcal{F}}^*$ is a homotopy equivalence.*

More generally, we find it curious that the combinatorics of the Frobenius category $\mathcal{F}_G^*$, which is generally thought of as simply an organizing framework for the p-local data of G, can identify the elementary abelian p-subgroups of G. Similarly, the orbit category $\mathcal{O}_G$ is able to identify the G-radical and the cyclic subgroups, and the exterior quotient $\widetilde{\mathcal{F}}^{\mathrm{sfc}}$ of an abstract Frobenius category $\mathcal{F}$ is able to identify the $\mathcal{F}$-radical subgroups. We take this as evidence of a general theme that p-subgroup categories encode group structure in unexpected ways.

To understand the manner in which the shapes of our p-subgroup categories determine certain group-theoretic data, we must discuss our method of proof for Theorem A. Indeed, for us the method is at least as interesting as the final result. There are two interwoven threads to this story: In one, we make use of Leinster's theory of Euler characteristics for a general category [14] to identify a class of subgroups that is likely to control homotopy of the p-subgroup category; in the other, we make use of a special case of Quillen's celebrated Theorem A on homotopy equivalences of categories [17] to prove that our proposed class of subgroup actually does control the homotopy type. We now summarize these points.

Consider first the special case of an inclusion of posets $\iota : \mathcal{P} \subseteq \mathcal{Q}$. Here, Quillen's Theorem A says that $\mathcal{P}$ is homotopy equivalent to $\mathcal{Q}$ if every slice or coslice category of ι is contractible. In other words, it suffices to show that for every $q \in \mathcal{Q}$, we have $\mathcal{P}/q := \{p \in \mathcal{P} \mid p \leq q\}$ is contractible; or dually that for every $q \in \mathcal{Q}$, $q/\mathcal{P} := \{p \in \mathcal{P} \mid q \leq p\}$ is contractible. Rephrased slightly: In order for ι to be a homotopy equivalence, it is necessary that every object $q \in \mathcal{Q}$ whose *proper* slice category $\mathcal{P}//q := \{p \in \mathcal{P} \mid p \lneq q\}$ is *noncontractible* be an object of $\mathcal{P}$, or that the dual criterion hold. The first case led to Quillen's result $\mathcal{S}_G^{*+\mathrm{eab}} \simeq \mathcal{S}_G^*$, and the second to Bouc's homotopy equivalence $\mathcal{S}_G^{*+\mathrm{rad}} \simeq \mathcal{S}_G^*$.

When we generalize to our p-subgroup categories, it's important to note that we are not generalizing very far: All of the categories $\mathcal{C}$ we consider in this paper are finite *EI-categories*, so that every endomorphism of every object of $\mathcal{C}$ is an isomorphism. For us then, the role of proper slice category $\mathcal{C}//x$ of $x \in \mathcal{C}$ is filled by the category of *nonisomorphisms* with target x; the slice category is dually the category of nonisomorphisms with source x. See Section 3 for precise definitions. We then have our main technical tool for showing that a class of subgroups controls homotopy, appearing as Theorem 4.3:

Theorem B. *The homotopy type of a finite EI-category $\mathcal{C}$ is controlled by the set of objects whose proper slice categories are noncontractible, or dually those objects whose proper coslice categories are noncontractible.*

In other words, if all you care about is the homotopy type of $\mathcal{C}$, you might as well throw away all objects x such that $\mathcal{C}//x \simeq *$. This material is covered in Section 4, and the key technical results needed for implementation of Theorem B is collected in Section 5.

Of course, this is only helpful if we have a proposed class of subgroups that might control homotopy of the p-subgroup category. In order to direct our search we turn to Leinster's Euler characteristics for categories. This is a generalization of the Euler characteristic of a poset or a space, which relies on the notions of *weightings* and *coweightings* for a category $\mathcal{C}$. Roughly speaking, these are functions $k_{\mathcal{C}}^{\bullet}, k_{\bullet}^{\mathcal{C}} : \mathrm{Ob}(\mathcal{C}) \to \mathbf{Q}$ that serve as right and left "inverses" to the generalized incidence matrix, which records the number of morphisms between any two object of $\mathcal{C}$. If both a weighting and a coweighting for $\mathcal{C}$ exist, as is always the case for finite EI-categories, then the *Euler characteristic* $\chi(\mathcal{C})$ of $\mathcal{C}$ is the common sum of the values of either function and the *reduced Euler characteristic* of $\mathcal{C}$ is $\widetilde{\chi}(\mathcal{C}) := \chi(\mathcal{C}) - 1$. The connection with Theorem B comes from Theorem 3.7, whose key point is the following

Theorem C. *Let $\mathcal{C}$ be a finite EI-category. There is a weighting for $\mathcal{C}$ that on an object x takes a value proportional to the reduced Euler characteristic of the proper coslice category of x. Dually, there is a coweighting whose value on x is proportional to the reduced Euler characteristic of the proper slice category of x. Thus there are constants κ^x and κ_x such that*

$$k_{\mathcal{C}}^x = \kappa^x \cdot \widetilde{\chi}(x//\mathcal{C}) \quad \textit{and} \quad k_x^{\mathcal{C}} = \kappa_x \cdot \widetilde{\chi}(\mathcal{C}//x).$$

Thus the weighting is concentrated on the objects whose proper coslice categories have nonzero reduced Euler characteristic, which our intuition suggests must be noncontractible. While it appears to be an open question whether $\mathcal{C}$ contractible implies $\widetilde{\chi}(\mathcal{C}) = 0$ in general, we have from [14, Proposition 2.4(a)] that an adjunction between two categories forces equality of their Euler characteristics. In particular, every category with an initial or terminal object has trivial reduced Euler characteristic, which turns out to be the relevant consideration in this paper. This all suggests (but does not prove!) that the classes of subgroups we consider should be those with nonzero (co)weightings in our p-subgroup categories, which were computed in Ref. [13]. With our Euler characteristic calculations in hand, we conclude by applying Theorem B.

2. p-Subgroup categories

This section contains precise definitions of the p-subgroup categories occurring in this paper. By convention, maps act on elements from the right, and composition of morphisms is written in diagrammatic order. Likewise, functors act on categories from the right.

If a and b are objects in a category $\mathcal{C}$, we write $\mathcal{C}(a, b)$ for the set of $\mathcal{C}$-morphisms with domain a and codomain b, and $\mathcal{C}(a)$ is the monoid of $\mathcal{C}$-endomorphisms of a. All categories considered in this paper are EI-categories, so for us $\mathcal{C}(a)$ is in fact a group.

Fix a finite group G. The most fundamental p-subgroup category we consider is the poset $\mathcal{S}_G$ of all p-subgroups of G, ordered by inclusion. In other words, $\mathcal{S}_G$ is the category whose objects are all p-subgroups of G with one morphism $H \to K$ whenever $H \le K$ and no morphisms otherwise.

$\mathcal{S}_G$ forms the backbone for all of the p-subgroup categories we consider. With our finite group G still in mind, we consider the following categories, all of which have as objects the p-subgroups of G:

- $\mathcal{T}_G$: The transporter category of G; morphisms are elements of G conjugating one subgroup into another.
- $\mathcal{F}_G$: The Frobenius, or p-fusion, category of G [16,5]; morphisms are group homomorphisms between subgroups which are restrictions of G-conjugations.
- $\mathcal{L}_G$: The linking category of all p-subgroups G; an intermediary between $\mathcal{T}_G$ and $\mathcal{F}_G$, thought of as killing the p'-part of the kernel of the natural functor $\mathcal{T}_G \to \mathcal{F}_G$ [4].[1]
- $\mathcal{O}_G$: The p-orbit category of G; morphisms are G-maps between transitive G-sets with p-group isotropy.
- $\widetilde{\mathcal{F}}_G$: The exterior quotient of the Frobenius category $\mathcal{F}_G$; a fusion-theoretic analogue of $\mathcal{O}_G$ [16, 1.3, 4.8].

More explicitly, for any p-subgroups H and K of G, the morphisms of the above categories are given by:

$$\begin{aligned} &\mathcal{T}_G(H,K) = N_G(H,K) && \mathcal{L}_G(H,K) = O^pC_G(H)\backslash N_G(H,K) \\ &\mathcal{F}_G(H,K) = C_G(H)\backslash N_G(H,K) && \mathcal{O}_G(H,K) = N_G(H,K)/K \\ &\widetilde{\mathcal{F}}_G(H,K) = C_G(H)\backslash N_G(H,K)/K \end{aligned}$$

where $N_G(H,K)$ is the *transporter set* $\{g \in G \mid H^g \le K\}$ and O^pL denotes the minimal normal p-power index subgroup of L. Composition in these categories is induced by the group multiplication of G.

If $H \le G$ is a p-subgroup, the automorphism groups in these categories of G are given by

$$\begin{gathered} \mathcal{S}_G(H) = 1, \qquad \mathcal{T}_G(H) = N_G(H), \qquad \mathcal{L}_G(H) = O^pC_G(H)\backslash N_G(H), \\ \mathcal{F}_G(H) = C_G(H)\backslash N_G(H), \qquad \mathcal{O}_G(H) = N_G(H)/H, \qquad \widetilde{\mathcal{F}}_G(H) = C_G(H)\backslash N_G(H)/H. \end{gathered}$$

The six categories are related by the commutative diagram

$$\begin{tikzcd} \mathcal{S}_G \arrow[r, hook] & \mathcal{T}_G \arrow[r, two heads] \arrow[d, two heads] & \mathcal{L}_G \arrow[r, two heads] & \mathcal{F}_G \arrow[d, two heads] \\ & \mathcal{O}_G \arrow[rr, two heads] & & \widetilde{\mathcal{F}}_G \end{tikzcd}$$

of one faithful and five full functors.

$\mathcal{S}_G$ contains the identity subgroup as an initial object, so it is contractible. More topologically interesting is the *Brown poset* $\mathcal{S}_G^*$ of all nonidentity subgroups of G, which has long been an object of interest as a sort of geometry for the finite group G (cf. [6,18]). More generally, decorating one of our p-subgroup categories with an asterisk will denote the full subcategory of nonidentity subgroups: $\mathcal{T}_G^*, \mathcal{F}_G^*$, etc.

Sylow's Theorem implies that each of our p-subgroup categories (other than $\mathcal{S}_G$) is equivalent to its full subcategory with objects the subgroups of a fixed Sylow p-subgroup $P \in \mathrm{Syl}_p(G)$. We will prefer to work with these "pointed" versions, especially as this convention allows us to work with *abstract* p-subgroup categories that make no reference to an ambient group G:

Fix a finite nonidentity p-group P. A *Frobenius P-category*, or (*abstract*) *saturated fusion system over P*, is a category whose objects are the subgroups of P and whose morphisms satisfy a set of axioms [16,5] that

[1] Note that in much of the literature, only a full subcategory of $\mathcal{L}_G$ is considered, the *centric* linking system $\mathcal{L}_G^{\mathrm{sfc}}$. This full subcategory has much better homotopical properties than the full linking category, and we will concentrate on it once the notion of $\mathcal{F}$-self-centralizing, or $\mathcal{F}$-centric, subgroup is recalled below.

distill the properties of the Frobenius categories $\mathcal{F}_G$ coming from a group G. There are examples of abstract Frobenius P-categories $\mathcal{F}$ that are exotic in the sense that there is no finite group G with $\mathcal{F}_G$ equal to $\mathcal{F}$.

The *exterior quotient*, or *orbit category*, of $\mathcal{F}$ is the category $\widetilde{\mathcal{F}}$ whose objects are the subgroups of P and with morphism sets

$$\widetilde{\mathcal{F}}(H,K) = \mathcal{F}(H,K)/\mathcal{F}_K(K)$$

are $\mathcal{F}$-morphisms modulo inner automorphisms of the codomain. Composition in $\mathcal{F}$ induces composition in its quotient category $\widetilde{\mathcal{F}}$.

While the term “orbit category” reflects certain similarities between the exterior quotient $\widetilde{\mathcal{F}}$ and the category of p-orbits $\mathcal{O}_G$, these are not the same construction; even in the presence of an ambient group G, the categories $\mathcal{O}_G$ and $\widetilde{\mathcal{F}}_G$ are distinct.

Finally, we recall the fundamental notions of G- and $\mathcal{F}$-self-centralizing and G- and $\mathcal{F}$-radical subgroups. As usual, O_pK is the largest normal p-subgroup of the finite group K [11, Chp 6.3].

Definition 2.1. (See [16, 4.8.1], [5, Definition A.3], [1, Proposition 4].) The p-subgroup H of G is

- *p-self-centralizing in G* if the center $Z(H)$ of H is a Sylow p-subgroup of the centralizer $C_G(H)$ of H;
- *G-radical* if $O_p\mathcal{O}_G(H) = 1$, or, equivalently, $H = O_pN_G(H)$.

Definition 2.2. (See [16, 4.8], [5, Definition A.9].) An object H of $\mathcal{F}$ is

- *$\mathcal{F}$-self-centralizing* if $C_P(\varphi H) \leq \varphi H$ for every $\mathcal{F}$-morphism $\varphi \in \mathcal{F}(H,P)$ with domain H;
- *$\mathcal{F}$-radical* if $O_p\widetilde{\mathcal{F}}(H) = 1$.

If $p \mid |G|$, every p-self-centralizing subgroup of G is nontrivial, as is every $\mathcal{F}$-self-centralizing subgroup of P.

Let P be a Sylow p-subgroup of G and $\mathcal{F} = \mathcal{F}_G$ the induced Frobenius P-category. For every $H \leq P$,

$$H \text{ is } \mathcal{F}\text{-self-centralizing} \iff H \text{ is } p\text{-self-centralizing in } G$$
$$H \text{ is } \mathcal{F}\text{-self-centralizing and } \mathcal{F}\text{-radical} \implies H \text{ is } G\text{-radical}$$

and the second implication cannot be reversed.

According to Quillen [18, Proposition 2.4], we have

$$\mathcal{S}^*_K \text{ is noncontractible} \implies O_pK = 1 \tag{2.3}$$

for any finite group K. In the present context, this means that

$$\mathcal{S}^*_{\mathcal{O}_G(H)} \text{ is noncontractible} \implies H \text{ is } G\text{-radical} \tag{2.4}$$
$$\mathcal{S}^*_{\widetilde{\mathcal{F}}(H)} \text{ is noncontractible} \implies H \text{ is } \mathcal{F}\text{-radical} \tag{2.5}$$

Properties (2.4) and (2.5) will be very important in the proof of our homotopy equivalences. (Quillen conjectures in [18, Conjecture 2.9] that the reverse implication of (2.3) is true. If Quillen’s conjecture holds, then the reverse implications of (2.4) and (2.5) are true as well.)

3. Weightings and coweightings for EI-categories

Let $\mathcal{C}$ be a finite category and $[\mathcal{C}]$ the set of isomorphism classes of its objects. In this section we show that we can use coslice categories to define weightings in the sense of Leinster [14]. In particular, we will relate the Euler characteristic of $\mathcal{C}$ to the Euler characteristic of its coslice categories.

Definition 3.1. (See [14, Definitions 1.10, 2.2].) A *weighting* for $\mathcal{C}$ is a function $k^\bullet_\mathcal{C} \colon \mathrm{Ob}(\mathcal{C}) \to \mathbf{Q}$ so that

$$\forall a \in \mathrm{Ob}(\mathcal{C})\colon \qquad \sum_{b \in \mathrm{Ob}(\mathcal{C})} \big|\mathcal{C}(a,b)\big| k^b_\mathcal{C} = 1,$$

and a *coweighting* for $\mathcal{C}$ is a function $k_\bullet^\mathcal{C} \colon \mathrm{Ob}(\mathcal{C}) \to \mathbf{Q}$ so that

$$\forall b \in \mathrm{Ob}(\mathcal{C})\colon \qquad \sum_{a \in \mathrm{Ob}(\mathcal{C})} k_a^\mathcal{C} \big|\mathcal{C}(a,b)\big| = 1.$$

If $\mathcal{C}$ has both a weighting and a coweighting, then

$$\sum_{a \in \mathrm{Ob}(\mathcal{C})} k^a_\mathcal{C} = \sum_{b \in \mathrm{Ob}(\mathcal{C})} k_b^\mathcal{C} =: \chi(\mathcal{C})$$

is the *Euler characteristic* of $\mathcal{C}$. The *reduced* Euler characteristic of $\mathcal{C}$ is $\widetilde{\chi}(\mathcal{C}) = \chi(\mathcal{C}) - 1$.

A general finite category may not admit a weighting or coweighting, or it may have several. If at least one of each exists, the Euler characteristic is independent of the choice of weighting or coweighting. Moreover, if there are several (co)weightings, we will normalize by singling out the unique (co)weighting that is constant on each isomorphism class of our category.

If $\mathcal{C}$ is an EI-category, its objects can be arranged in such an order so that the matrix $[\mathcal{C}]$ is upper triangular. It follows that any finite EI-category has a *unique* weighting and a *unique* coweighting that are constant on isomorphism classes of objects [14, Lemma 1.3, Theorem 1.4, Lemma 1.12].

We will be interested in piecing together global (co)weightings from local data, to be described in terms of the (co)slice construction.

Definition 3.2 *(Coslice and slice categories).* Let $\mathcal{C}$ be a category with objects x and y, and $\mathcal{A}$ a full subcategory of $\mathcal{C}$.

- $x/\mathcal{A}$ is the category of $\mathcal{C}$-morphisms from x to an object of $\mathcal{A}$ (the *coslice* of $\mathcal{A}$ under x)
- $\mathcal{A}/y$ is the category of $\mathcal{C}$-morphisms from an object of $\mathcal{A}$ to y (the *slice* of $\mathcal{A}$ over y)
- $x//\mathcal{A}$ is the full subcategory of $x/\mathcal{A}$ with objects all nonisomorphisms from x to an object of $\mathcal{A}$ (the *proper coslice* of $\mathcal{A}$ under x)
- $\mathcal{A}//y$ is the full subcategory of $\mathcal{A}/y$ with objects all nonisomorphisms from an object of $\mathcal{A}$ to y (the *proper slice* of $\mathcal{A}$ over y)

Thus an object of $x/\mathcal{A}$ is a $\mathcal{C}$-morphism $\varphi_1 \in \mathcal{C}(x, a_1)$, and an $(x/\mathcal{C})$-morphism from φ_1 to $\varphi_2 \in \mathcal{C}(x, a_2)$ is any $\eta \in \mathcal{C}(a_1, a_2)$ that makes the following triangle commute in $\mathcal{C}$:

$$\begin{tikzcd}
& x \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] \arrow[d, phantom, "\circlearrowleft"] & \\
a_1 \arrow[rr, dotted, "\eta"'] & {} & a_2
\end{tikzcd}$$

The other (co)slice categories are defined similarly.

The (co)slice constructions define functors $\bullet/\mathcal{A} : \mathcal{C}^{\mathrm{op}} \to \mathrm{CAT}$ and $\mathcal{A}/\bullet : \mathcal{C} \to \mathrm{CAT}$ via (pre)composition of morphisms. The proper (co)slice constructions do not in general piece together to form a functor, for the simple reason that the composite of two nonisomorphisms can be an isomorphism. However, if $\mathcal{C}$ is an EI-category, this obstruction vanishes and we also have functors $\bullet//\mathcal{A}: \mathcal{C}^{\mathrm{op}} \to \mathrm{CAT}$ and $\mathcal{A}//\bullet: \mathcal{C} \to \mathrm{CAT}$. Again, all p-subgroup categories considered in this paper are EI-categories.

Definition 3.3. If $\mathcal{C}$ is an EI-category, we write

$$\begin{aligned}
\mathrm{supp}(\bullet/\mathcal{A}) &= \{x \in \mathrm{Ob}(\mathcal{C}) \mid x/\mathcal{A} \text{ is noncontractible}\} \\
\mathrm{supp}(\mathcal{A}/\bullet) &= \{y \in \mathrm{Ob}(\mathcal{C}) \mid \mathcal{A}/y \text{ is noncontractible}\} \\
\mathrm{supp}(\bullet//\mathcal{A}) &= \{x \in \mathrm{Ob}(\mathcal{C}) \mid x//\mathcal{A} \text{ is noncontractible}\} \\
\mathrm{supp}(\mathcal{A}//\bullet) &= \{y \in \mathrm{Ob}(\mathcal{C}) \mid \mathcal{A}//y \text{ is noncontractible}\}
\end{aligned}$$

for the *supports* of the coslice functors $\bullet/\mathcal{A}, \bullet//\mathcal{A}: \mathcal{C}^{\mathrm{op}} \to \mathrm{CAT}$ and slice functors $\mathcal{A}/\bullet, \mathcal{A}//\bullet: \mathcal{C} \to \mathrm{CAT}$.

The notation $\mathcal{A}//\bullet$ and $\bullet//\mathcal{A}$ is taken from [12, p. 269].

Lemma 3.4. *Let $\mathcal{C}$ be any finite category admitting a weighting $k_{\mathcal{C}}^{\bullet}: \mathrm{Ob}(\mathcal{C}) \to \mathbf{Q}$. Let a be any object of $\mathcal{C}$. The function*

$$k_{a/\mathcal{C}}^{\bullet} = k_{\mathcal{C}}^{\mathrm{cod}(\bullet)}: \mathrm{Ob}(a/\mathcal{C}) \to \mathbf{Q}, \qquad k_{a/\mathcal{C}}^{a \xrightarrow{\varphi} b} = k_{\mathcal{C}}^{b},$$

is a weighting for the coslice $a/\mathcal{C}$ of $\mathcal{C}$ under a.

Proof. The set of objects of $a/\mathcal{C}$, which is the set of $\mathcal{C}$-morphisms with domain a, is partitioned

$$\mathrm{Ob}(a/\mathcal{C}) = \coprod_{b \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(a, b) \tag{3.5}$$

according to codomains. Also, for any $\mathcal{C}$-morphism $\varphi \in \mathcal{C}(a, b)$ with codomain b and any $\mathcal{C}$-object c, the set of $\mathcal{C}$-morphisms from b to c is partitioned

$$\mathcal{C}(b, c) = \coprod_{\psi \in \mathcal{C}(a,c)} (a/\mathcal{C})(\varphi, \psi) \tag{3.6}$$

into $a/\mathcal{C}$-morphism sets with domain φ. The computation

$$\sum_{\psi \in \mathrm{Ob}(a/\mathcal{C})} |a/\mathcal{C}(\varphi, \psi)| k_{\mathcal{C}}^{\mathrm{cod}(\psi)} \overset{(3.5)}{=} \sum_{b \in \mathrm{Ob}(\mathcal{C})} \sum_{\psi \in \mathcal{C}(a,b)} |a/\mathcal{C}(\varphi, \psi)| k_{\mathcal{C}}^{\mathrm{cod}(\psi)} \overset{(3.6)}{=} \sum_{b \in \mathrm{Ob}(\mathcal{C})} |\mathcal{C}(\mathrm{cod}(\varphi), b)| k_{\mathcal{C}}^{b} = 1$$

shows that the function $k_{\mathcal{C}}^{\mathrm{cod}(\bullet)}$ is a weighting on $a/\mathcal{C}$. $\square$

We can simplify the computation of (co)weightings by concentrating on the case where each isomorphism class contains a single object. For $\mathcal{C}$ a finite EI-category with an object a, let $a^{\mathcal{C}}$ be the set of objects $\mathcal{C}$-isomorphic to a, $[\mathcal{C}]$ some fixed equivalent skeletal subcategory, and $[a]$ the unique object in $[\mathcal{C}]$ and $a^{\mathcal{C}}$. Then $\mathcal{C}$ has a weighting $k_{\mathcal{C}}^{\bullet}$ if and only if $[\mathcal{C}]$ has a weighting $k_{[\mathcal{C}]}^{\bullet}$, and we can construct one from the other:

$$k_{[\mathcal{C}]}^{[b]} = \sum_{y \in b^{\mathcal{C}}} k_{\mathcal{C}}^{y}, \qquad k_{\mathcal{C}}^{b} = \frac{1}{|b^{\mathcal{C}}|} \cdot k_{[\mathcal{C}]}^{[b]}, \quad b \in \mathrm{Ob}(\mathcal{C}),\ [b] \in \mathrm{Ob}([\mathcal{C}]).$$

Similarly, $\mathcal{C}$ has a coweighting if and only if $[\mathcal{C}]$ does. Note that the (co)weightings on $\mathcal{C}$ that arise in this manner are necessarily constant on $\mathcal{C}$-isomorphism classes of objects.

A full subcategory $\mathcal{I}$ of a category $\mathcal{C}$ is a *left ideal* if any $\mathcal{C}$-morphism whose domain is an object of $\mathcal{I}$ is an $\mathcal{I}$-morphism. For instance, if $\mathcal{C}$ is an EI-category and a an object of $\mathcal{C}$ then $a//\mathcal{C}$ is a left ideal in $a/\mathcal{C}$ by [14, Lemma 1.3].

Theorem 3.7. *Let $\mathcal{C}$ be a finite EI-category, and let $k^{\bullet}_{\mathcal{C}}$ and $k^{\mathcal{C}}_{\bullet}$ be the weighting and the coweighting on $\mathcal{C}$ that are constant on isomorphism classes of objects of $\mathcal{C}$. Then*

$$k^a_{\mathcal{C}} = \frac{-\widetilde{\chi}(a//\mathcal{C})}{|a^{\mathcal{C}}||\mathcal{C}(a)|}, \qquad k^{\mathcal{C}}_b = \frac{-\widetilde{\chi}(\mathcal{C}//b)}{|b^{\mathcal{C}}||\mathcal{C}(b)|}, \quad a, b \in \mathrm{Ob}(\mathcal{C}),$$

and the Euler characteristic of $\mathcal{C}$ is

$$\sum_{[a]\in[\mathcal{C}]} \frac{-\widetilde{\chi}(a//\mathcal{C})}{|\mathcal{C}(a)|} = \chi(\mathcal{C}) = \sum_{[b]\in[\mathcal{C}]} \frac{-\widetilde{\chi}(\mathcal{C}//b)}{|\mathcal{C}(b)|}$$

where the sums run over the set $[\mathcal{C}]$ of isomorphism classes of objects of $\mathcal{C}$.

Proof. We shall only prove the statement about the weighting since the statement about the coweighting is entirely dual. $\mathcal{C}$ is a finite EI-category, so it is easy to see that the coslice categories $a/\mathcal{C}$ and $a//\mathcal{C}$ are also finite EI-categories. Thus they admit weightings and coweightings, and have well-defined Euler characteristics. Since $a//\mathcal{C}$ is a left ideal in $a/\mathcal{C}$, the weighting for $a/\mathcal{C}$ from Lemma 3.4 restricts to a weighting for $a//\mathcal{C}$ [13, Remark 2.6]. The category $a/\mathcal{C}$ has an initial object, so it is contractible and has Euler characteristic 1. Therefore

$$1 = \sum_{\varphi\in\mathrm{Ob}(a/\mathcal{C})} k^{\mathrm{cod}(\varphi)}_{\mathcal{C}} = |a^{\mathcal{C}}||\mathcal{C}(a)|k^a_{\mathcal{C}} + \sum_{\varphi\in\mathrm{Ob}(a//\mathcal{C})} k^{\mathrm{cod}(\varphi)}_{\mathcal{C}} = |a^{\mathcal{C}}||\mathcal{C}(a)|k^a_{\mathcal{C}} + \chi(a//\mathcal{C})$$

because the weighting $k^{\bullet}_{\mathcal{C}}$ is assumed to be constant on the isomorphism class $[a]$ of a. $\square$

The rational functions

$$k^{[a]}_{[\mathcal{C}]} = \frac{-\widetilde{\chi}(a//\mathcal{C})}{|\mathcal{C}(a)|}, \qquad k^{[\mathcal{C}]}_{[b]} = \frac{-\widetilde{\chi}(\mathcal{C}//b)}{|\mathcal{C}(b)|}, \quad [a], [b] \in \big[\mathrm{Ob}(\mathcal{C})\big],$$

are the weighting and the coweighting for $[\mathcal{C}]$, respectively.

In the case $\mathcal{S}$ is a poset, we sometimes write ${}_{a\leq}\mathcal{S}$, ${}_{a<}\mathcal{S}$, $\mathcal{S}_{\leq b}$, $\mathcal{S}_{<b}$ for $a/\mathcal{S}$, $a//\mathcal{S}$, $\mathcal{S}/b$, $\mathcal{S}//b$, respectively. Using this notation, the last part of Theorem 3.7 takes the following form.

Corollary 3.8. *The Euler characteristic of a finite poset $\mathcal{S}$ is the sum*

$$\sum_{a\in\mathrm{Ob}(\mathcal{S})} -\widetilde{\chi}({}_{a<}\mathcal{S}) = \chi(\mathcal{S}) = \sum_{b\in\mathrm{Ob}(\mathcal{S})} -\widetilde{\chi}(\mathcal{S}_{<b})$$

of the negatives of the local reduced Euler characteristics.

This reproduces a well-known result from the combinatorial theory of posets.

4. Homotopy equivalences between categories

The famous Quillen's Theorem A provides a sufficiency criterion for a functor between two categories to be a homotopy equivalence. We quote this theorem here not in its full generality, but only for the special case that is of interest to us.

Theorem 4.1 *(Quillen's Theorem A for inclusions of categories). (See [17, Theorem A].) Let $\mathcal{C}$ be a category and $\mathcal{A}$ a full subcategory. The inclusion $\mathcal{A} \hookrightarrow \mathcal{C}$ is a homotopy equivalence if either* $\mathrm{supp}(\bullet/\mathcal{A})$ *or* $\mathrm{supp}(\mathcal{A}/\bullet)$ *is empty.*

We also quote a perhaps less well-known result of Bouc providing a sufficient condition for an inclusion of posets to be a homotopy equivalence.

Theorem 4.2 *(Bouc's theorem for posets). (See [1].) Let $\mathcal{S}$ be a finite poset and $\mathcal{A}$ a subposet. The inclusion $\mathcal{A} \hookrightarrow \mathcal{S}$ is a homotopy equivalence if either* $\mathrm{supp}(\bullet//\mathcal{S})$ *or* $\mathrm{supp}(\mathcal{S}//\bullet)$ *is contained in* $\mathrm{Ob}(\mathcal{A})$.

In this section we generalize Bouc's theorem for poset inclusions to finite EI-category inclusions. This boils down to a repurposing of Quillen's Theorem A in terms of our notion of support, and should be thought of as a statement about what sort of objects control the homotopy type of a finite EI-category.

Theorem 4.3 *(Bouc's theorem for finite EI-categories). Let $\mathcal{C}$ be a finite EI-category and $\mathcal{A}$ a full subcategory that is closed under isomorphisms. The inclusion of $\mathcal{A} \hookrightarrow \mathcal{C}$ is a homotopy equivalence if either* $\mathrm{supp}(\bullet//\mathcal{C})$ *or* $\mathrm{supp}(\mathcal{C}//\bullet)$ *is contained in* $\mathrm{Ob}(\mathcal{A})$.

Proof. Assume that $\mathrm{Ob}(\mathcal{A})$ contains the support $\mathrm{supp}(\bullet//\mathcal{C})$ of the functor $\bullet//\mathcal{C}$. The claim is that the inclusion functor $\iota: \mathcal{A} \to \mathcal{C}$ is a homotopy equivalence. It suffices to show that the coslice $x/\mathcal{A}$ of $\mathcal{A}$ is contractible for every object x of $\mathcal{C}$ (Theorem 4.1).

For any object x of $\mathcal{C}$ define the *height* of x, $\mathrm{ht}(x)$, to be the maximal length of any path

$$x_0 \to x_1 \to \cdots \to x_h = x$$

of nonisomorphisms in $\mathcal{C}$ terminating at x. The height of x is finite since there are no circuits in paths of nonisomorphisms [14, Lemma 1.3]. If there is a nonisomorphism from x_0 to x_1, then $\mathrm{ht}(x_0) < \mathrm{ht}(x_1)$. Define $\mathrm{ht}(\mathcal{C})$ to be the maximal height of any object of $\mathcal{C}$.

Suppose that x is an object of $\mathcal{C}$ of maximal height, $\mathrm{ht}(\mathcal{C})$. Then $x//\mathcal{C}$ is the empty category because there is no nonisomorphism from x to any object of $\mathcal{C}$. The empty category is not contractible, so $x \in \mathrm{supp}(\bullet//\mathcal{C}) \subset \mathrm{Ob}(\mathcal{A})$ is an object of $\mathcal{A}$. Then $x/\mathcal{A}$ is contractible with the identity of x as an initial object.

Let now x be any object of $\mathcal{C}$ such that the coslice $y/\mathcal{A}$ of $\mathcal{A}$ is contractible for all objects y of height greater than $\mathrm{ht}(x)$. Then the functor

$$x//\iota: x//\mathcal{A} \to x//\mathcal{C}$$

is a homotopy equivalence by Theorem 4.1 because the category

$$(x \to y)/(x//\iota) = y/\mathcal{A}$$

is contractible for every object $x \to y$ of $x//\mathcal{C}$. In the case x is an object of $\mathcal{A}$, $x/\mathcal{A}$ is contractible as before. In the case x is not an object of $\mathcal{A}$, $x/\mathcal{A} = x//\mathcal{A}$ because there can be no isomorphism from x to an object of $\mathcal{A}$ as $\mathcal{A}$ is closed under isomorphisms. We now have

$$x/\mathcal{A} = x//\mathcal{A} \simeq x//\mathcal{C}$$

and $x//\mathcal{C}$ is contractible since $x \notin \mathrm{supp}(\bullet//\mathcal{C})$. Thus $x/\mathcal{A}$ is also contractible.

By finite downward induction on $\mathrm{ht}(x)$ we see that $x/\mathcal{A}$ is contractible for all objects x of $\mathcal{C}$. □

Remark 4.4. Theorem 4.3 is the main technical tool of this paper, but it should also be thought of as a descriptive statement about control of homotopy type of finite EI-categories. We will use the following reinterpretation: If $\mathcal{C}$ is a finite EI-category, then either (a) those objects whose proper slice categories are contractible do not, as a whole, contribute to the overall homotopy type of $\mathcal{C}$, or (b) the same holds for those objects with contractible proper coslice categories. (Note that the union of these two classes cannot be discarded without affecting the homotopy type, as the example of the poset $x < y$ shows.) In our search for homotopy equivalences between p-subgroup categories, we will therefore concentrate on identifying those objects with contractible proper (co)slices. Morally speaking, the reduced Euler characteristic of a contractible category should vanish, although this remains an open question as of this writing. Nevertheless, our intuition and the combinatorics developed in the previous section will guide our search in what follows, ultimately leading to success.

5. Subgroup categories for p-groups

In this section we collect several technical examples that will allow us to apply Theorem 4.3 more generally.

For any small category $\mathcal{C}$ and any set $D \subset \mathrm{Ob}(\mathcal{C})$ of objects of $\mathcal{C}$, we let $\mathcal{C}^D$ denote the full subcategory of $\mathcal{C}$ generated by the objects in the set D. For instance, if $H \lneq K$ are p-subgroups of G, then $\mathcal{F}_G^{[H,K)}$ denotes the full subcategory of $\mathcal{F}_G$ with objects the set of all subgroups L of G for which $H \leq L \lneq K$.

In the following lemma we consider

$\mathcal{S}_P^{(1,P)}$: the poset of nonidentity proper subgroups of P
$\mathcal{O}_P^{[1,P)}$: the full subcategory of $\mathcal{O}_P$ with objects all proper subgroups of P
$\widetilde{\mathcal{F}}_P^{(1,P)}$: the full subcategory of $\widetilde{\mathcal{F}}_P$ with objects all nonidentity proper subgroups of P

for P a nonidentity p-group. We write μ for the Möbius function of the poset $\mathcal{S}_P$ [20, §3.7], and we abbreviate $\mu(1,K)$ to $\mu(K)$ for any subgroup K of P.

Lemma 5.1. *Let P be a nonidentity p-subgroup. Then*

(a) • $\widetilde{\chi}(\mathcal{S}_P^{(1,P)}) = \mu(P)$
• $\widetilde{\chi}(\widetilde{\mathcal{F}}_P^{(1,P)}) = \widetilde{\chi}(\mathcal{F}_P^{(1,P)}) = \frac{\mu(P)}{|P:Z(P)|}$
• $\chi(\mathcal{O}_P^{[1,P)}) = \begin{cases} p^{-1} & P \text{ is cyclic} \\ 1 & \text{else} \end{cases}$

(b) $\mathcal{S}_P^{(1,P)}$ *is noncontractible* $\Longleftrightarrow$ *P is elementary abelian.*
(c) $\mathcal{O}_P^{[1,P)}$ *is homotopy equivalent to* $\mathcal{O}_V^{[1,V)}$*, where $V = P/\Phi(P)$ is the Frattini quotient of P.*
(d) $\widetilde{\mathcal{F}}_P^{(1,P)}$ *is noncontractible* $\Longleftrightarrow$ *P is elementary abelian.*

Proof. (a) It is well known that $\widetilde{\chi}(\mathcal{S}_P^{(1,P)}) = \widetilde{\chi}(1,P) = \mu(P)$ ([20, 3.8.5, 3.8.6], [13, 2.3]). The formulas for $\widetilde{\chi}(\widetilde{\mathcal{F}}_P^{(1,P)})$ and $\chi(\mathcal{O}_P^{[1,P)})$ follow from [13, Remark 2.6, Example 3.7, Theorem 7.7, Theorem 4.1]: If $k_\bullet^{\widetilde{\mathcal{F}}}$ is a coweighting for $\widetilde{\mathcal{F}}_P^{(1,P]}$ then

$$1 = \chi(\widetilde{\mathcal{F}}_P^{(1,P]}) = \chi(\widetilde{\mathcal{F}}_P^{(1,P)}) + k_P^{\widetilde{\mathcal{F}}} = \chi(\widetilde{\mathcal{F}}_P^{(1,P)}) + \frac{-\mu(P)}{|P:Z(P)|}$$

because $\widetilde{\mathcal{F}}_P^{(1,P]}$ is contractible, with P as a terminal object, containing the right ideal $\widetilde{\mathcal{F}}_P^{(1,P)}$. Similarly, if $k_\bullet^{\mathcal{O}}$ is a coweighting for $\mathcal{O}_P$ then

$$1 = \chi(\mathcal{O}_P) = \chi\big(\mathcal{O}_P^{[1,P)}\big) + k_P^{\mathcal{O}} = \chi\big(\mathcal{O}_P^{[1,P)}\big) + \begin{cases} 1 - \frac{1}{p} & P \text{ is cyclic} \\ 0 & P \text{ is not cyclic} \end{cases}$$

and the expression for the Euler characteristic of $\mathcal{O}_P^{[1,P)}$ follows.

(b) If P is elementary abelian, $\mathcal{S}_P^{(1,P)}$ is well known to be a bouquet of spheres [18, §10], and is therefore not contractible. If P is not elementary abelian, the Frattini subgroup $\Phi(P)$ is nontrivial [11, Chp 5, Theorem 1.3]. There are adjoint functors

$$\begin{tikzcd} \mathcal{S}_P^{(1,P)} \arrow[r, shift left, "\ell"] & \mathcal{S}_P^{[\Phi(P),P)} \arrow[l, shift left, "r"] \arrow[r, shift left, "r"] & \mathcal{S}_P^{[1,P)} \arrow[l, shift left, "\ell"] \end{tikzcd}$$

where $Q\ell = Q\Phi(P)$ and $Qr = Q$ for $Q \le P$. Observe that $Q \lneqq P \implies Q\Phi(P) \lneqq P$ because the Frattini subgroup is the group of nongenerators of P. The poset on the right, $\mathcal{S}_P^{[1,P)}$, is contractible with the trivial group as an initial object. The poset on the left, $\mathcal{S}_P^{(1,P)}$, is therefore also contractible. Alternatively, the natural transformations $Q \le Q\Phi(P) \ge \Phi(P)$, $1 \le Q \lneqq P$, define a homotopy from the identity of $\mathcal{S}_P^{(1,P)}$ to a constant map.

(c) There are functors

$$\begin{tikzcd} \mathcal{O}_P^{[1,P)} \arrow[r, shift left, "\ell"] & \mathcal{O}_P^{[\Phi(P),P)} \arrow[l, shift left, "r"] & \mathcal{O}_{P/\Phi(P)}^{[1,P/\Phi(P))} \arrow[l, "\cong"', "u"] \end{tikzcd}$$

where r and ℓ are adjoint functors and u is an isomorphism. The functors r and ℓ are given by $Q\ell = Q\Phi(P)$ and $Qr = Q$ for $Q \le P$. The category in the middle, $\mathcal{O}_P^{[\Phi(P),P)}$, is isomorphic to the category $\mathcal{O}_{P/\Phi(P)}^{[1,P/\Phi(P))}$. To see this, observe that all supergroups of the Frattini subgroup $\Phi(P)$ are normal, so that $\mathcal{O}_P(Q_1, Q_2) = P/Q_2 = \frac{P/\Phi(P)}{Q_2/\Phi(P)} = \mathcal{O}_{P/\Phi(P)}(Q_1/\Phi(P), Q_2/\Phi(P))$ when Q_1 and Q_2 both contain $\Phi(P)$.

(d) If P is elementary abelian, then $\widetilde{\mathcal{F}}_P = \mathcal{S}_P$ and $\widetilde{\mathcal{F}}_P^{(1,P)} = \mathcal{S}_P^{(1,P)}$, is noncontractible by (b). If P is not elementary abelian, the Frattini subgroup $\Phi(P)$ is a nontrivial normal subgroup and so is its intersection with the center $Z(P)$ of P [19, 5.2.1]. There are adjoint equivalences of categories

$$\begin{tikzcd} \widetilde{\mathcal{F}}_P^{(1,P)} \arrow[r, shift left, "\ell"] & \widetilde{\mathcal{F}}_P^{[\Phi(P)\cap Z(P),P)} \arrow[l, shift left, "r"] \arrow[r, shift left, "r"] & \widetilde{\mathcal{F}}_P^{[1,P)} \arrow[l, shift left, "\ell"] \end{tikzcd}$$

where $Q\ell = Q\Phi(P)$ and $Qr = Q$ for $Q \lneqq P$. The category to the right, $\widetilde{\mathcal{F}}_P^{[1,P)}$, is contractible because it has the trivial group as an initial object. The category to the left, $\widetilde{\mathcal{F}}_P^{(1,P)}$, is therefore also contractible. $\square$

One might be led by Lemma 5.1(a) to suspect that, for any nonidentity p-group P,

$$\mathcal{O}_P^{[1,P)} \text{ is noncontractible} \quad \Longrightarrow \quad P \text{ is cyclic}$$

or, equivalently, for any nonidentity elementary abelian p-group V,

$$\mathcal{O}_V^{[1,V)} \text{ is noncontractible} \quad \Longrightarrow \quad \operatorname{rank}(V) = 1$$

To see that these two statements are equivalent, recall that the Frattini quotient of P is cyclic precisely when P itself is cyclic [11, Chp 5, Corollary 1.2] and use Lemma 5.1(c). However, Example 5.2 demonstrates that these statements are false.

Example 5.2. Let $V = C_p^r$ be the elementary abelian p-group of rank $r \geq 1$. The objects of the category $\mathcal{O}_V^{[1,V)}$ are the proper subgroups of V, and the set of morphisms from $H \lneqq V$ to $K \lneqq V$ is

$$\mathcal{O}_V^{[1,V)}(H,K) = \begin{cases} V/K & \text{if } H \leq K \\ \emptyset & \text{otherwise} \end{cases}$$

with composition in this category induced from composition in the abelian group V.

If the rank $r = 1$, then the category $\mathcal{O}_V^{[1,V)} = \mathcal{O}_V^{\{1\}}$ is the cyclic group V, which is not contractible.

Let us now explore the category $\mathcal{O}_V^{[1,V)}$ in the case where the rank $r > 1$. There is an obvious functor

$$\pi\colon \mathcal{O}_V^{[1,V)} \to \mathcal{S}_V^{[1,V)}$$

to the poset of proper subgroups of V. For any proper subgroup K of V, the π-slice over K is $\pi/K = \mathcal{O}_V^{[1,K]}$. There is an adjunction

$$\mathcal{O}_V^{[1,K]} \underset{r}{\overset{\ell}{\rightleftarrows}} \mathcal{O}_V^{\{K\}}, \qquad r\ell \overset{\varepsilon}{=} 1_{\mathcal{O}_V^{\{K\}}}, \qquad 1_{\mathcal{O}_V^{[1,K]}} \overset{\eta}{\Rightarrow} \ell r,$$

where $H\ell = K$ and $Kr = K$. The functor r includes the full subcategory of $\mathcal{O}_V$ with K as its only object into the full subcategory of all subgroups of K. The functor ℓ is the projection $\mathcal{O}_V(H_1, H_2) = V/H_2 \to V/K = \mathcal{O}_V(K,K)$, $H_1 \leq H_2 \leq K$. Thus the category $\mathcal{O}_V^{[1,K]}$ is homotopy equivalent to the category $\mathcal{O}_V^{\{K\}}$ which is the group V/K. The composite functor spectral sequence ([10, pp. 155–157], [9, Proof of Proposition 2.3])

$$E^2_{st} = H_s\big(\mathcal{S}_V^{[1,V)}; H_t(V/\bullet; \mathbf{F}_p)\big) \quad\Longrightarrow\quad H_{s+t}\big(\mathcal{O}_V^{[1,V)}; \mathbf{F}_p\big) \tag{5.3}$$

associated to the functor π provides information about the homology groups of the category $\mathcal{O}_V^{[1,V)}$. Here, we write $H_s(\mathcal{S}_V^{[1,V)}; H_t(V/\bullet))$ for the sth left derived of the functor $\operatorname{colim} H_t(V/\bullet)$. In concrete terms, these groups are the homology groups of the normalized chain complex [15, Theorem VIII.6.1] of the simplicial abelian group $\coprod_* H_t(V/\bullet)$ [2, XII.5.5],

$$0 \leftarrow \bigoplus_{0\leq L_0<V} H_t(V/L_0) \xleftarrow{\partial_1} \bigoplus_{0\leq L_0<L_1<V} H_t(V/L_0) \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_s} \bigoplus_{0\leq L_0<L_1\cdots<L_s<V} H_t(V/L_0) \xleftarrow{\partial_{s+1}} \cdots$$

with boundary homomorphism ∂_s is defined by deleting single entries of the s-flag $L_0 < L_1 < \cdots < L_s$ and applying $H_t(V/L_0) \to H_t(V/L_1)$ in the case of deletion of the first entry. This chain complex is trivial in degrees $> r-1$ so that the spectral sequence (5.3) is concentrated in the vertical band $0 \leq s \leq r-1$.

Take $r = 2$ and $p = 2$ and consider the category $\mathcal{O}_V^{[1,V)}$ where V is the Klein 4-group. The objects of $\mathcal{O}_V^{[1,V)}$ are the identity subgroup, $\{0\}$, and three subgroups, L_1, L_2, and L_3, of order 2. The category $\mathcal{O}_V^{[1,V)}$ is

$$\begin{tikzcd} L_1 \arrow[loop above, "V/L_1"] & L_2 \arrow[loop above, "V/L_2"] & L_3 \arrow[loop above, "V/L_3"] \\ & \{0\} \arrow[ul, "V/L_1"] \arrow[u, "V/L_2"'] \arrow[ur, "V/L_3"'] \arrow[loop below, "V"] \end{tikzcd} \qquad \zeta = \begin{pmatrix} 4 & 2 & 2 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \qquad k_\bullet^{\mathcal{O}} = (1/4, 1/4, 1/4, 1/4) \qquad \chi\big(\mathcal{O}_V^{[1,V)}\big) = 1$$

with composition induced from addition in the abelian group V. Here ζ is the generalized incidence matrix that records the number of morphisms between objects. The first quadrant spectral sequence (5.3) is concentrated on the two vertical lines $s = 0$ and $s = 1$ so that all differentials are trivial. The groups $E^2_{0t} = E^\infty_{0t}$ and $E^2_{1t} = E^\infty_{1t}$ are the homology groups of the normalized simplicial replacement chain complex

$$\cdots \leftarrow 0 \leftarrow H_t(V/0) \oplus H_t(V/L_1) \oplus H_t(V/L_2) \oplus H_t(V/L_3) \leftarrow H_t(V/0) \oplus H_t(V/0) \oplus H_t(V/0) \leftarrow 0 \to\leftarrow$$

concentrated in degrees 0 and 1. Since $H_t(V/0)$ has dimension $t+1$, $H_t(V/L_i)$, $i = 1, 2, 3$, is 1-dimensional, the term $E^\infty_{1,t}$ has dimension at least $2t-1$, and consequently $\dim_{\mathbf{F}_2} H_{t+1}(\mathcal{O}_V^{[1,V)}; \mathbf{F}_2) \geq 2t-1$ for all degrees $t \geq 1$.

The above argument is easily seen to work for any prime p and we conclude that $\dim_{\mathbf{F}_p} H_{t+1}(\mathcal{O}_V^{[1,V)}; \mathbf{F}_p) \geq pt - 1$ for all degrees $t \geq 1$ when the rank $r = 2$. Thus $\mathcal{O}_V^{[1,V)}$ is noncontractible when V has rank $r = 2$.

Here are few remarks about the spectral sequence (5.3) for arbitrary prime p and rank $r \geq 2$. When $t = 0$, $E^2_{s0} = H_s(\mathcal{S}_V^{[1,V)}; \mathbf{F}_p)$, so that $E^2_{00} = \mathbf{F}_p$ and $E^2_{s0} = 0$ for $s > 0$, as $\mathcal{S}_V^{[1,V)}$ is contractible. When $t > 0$, we conjecture, based on computer calculations, that $E^2_{st} = 0$ except for $s = r - 1$. We have not been able to prove this conjecture.

6. Brown posets and transporter categories

We now begin the process of proving the results summarized in Theorem A, which will take up the remainder of the paper.

Let G be a finite group of order divisible by p and $\mathcal{S}_G$ the poset of p-subgroups of G. The Brown poset for G is the subposet $\mathcal{S}_G^* = \mathcal{S}_G^{(1,G]}$ of nonidentity p-subgroups of G. We show that the homotopy type of $\mathcal{S}_G^*$ is determined by either the elementary abelian p-subgroups of G, or the G-radical subgroups of G, as well as showing that the full subcategory of p-self-centralizing subgroups of G has its homotopy determined by the G-radical, p-self-centralizing subgroups. The results of this section are not new, but they provide the template of our argument, which we outline now:

For each claim of Theorem 6.1, we break the proof into two separate parts: The computation of the Euler characteristic of the general (co)slice of the larger category, and actual proof of homotopy equivalence through an application of Bouc's Theorem 4.3. These parts are labelled **[EC]** and **[HE]**, respectively. The truth of the result is shown in the second part, whereas the Euler characteristics calculation is not, strictly speaking, necessary for the proof of the theorem. Instead, it is offered as a moral argument as to why we should expect the result to be true, following Remark 4.4.

In fact, the work of the **[HE]** sections lies in establishing an adjunction between the (co)slice categories of interest and other categories with whose contractibility is well understood. As adjunctions preserve Euler characteristic [14, Proposition 2.4(a)], we could reinterpret **[HE]** as an alternate derivation of certain Euler characteristic computations from Ref. [13].

Let us consider as a toy example Part (a) of Theorem 6.1, which is Quillen's result [18] that the homotopy type of $\mathcal{S}_G^*$ is determined by the subposet $\mathcal{S}_G^{*+\mathrm{eab}}$ of elementary abelian p-subgroups. By Theorem 4.3, we must find those objects K of $\mathcal{S}_G^*$ whose proper slice categories $\mathcal{S}_G^*//K$ are not contractible. We will ultimately find that these are precisely the elementary abelian p-subgroups of G (the "second part" of the proof), but first we pretend not to know this and ask what sort of subgroups we should consider. A good first guess would be those objects whose proper slice categories have nonzero reduced Euler characteristic, as those proper slice categories should be (and, in fact, are) noncontractible. This does not guarantee that all other objects have contractible proper slice categories, but with a little more work, this turns out to be the case. This basic chain of reasoning will be repeated for all of the similar results that follow.

Theorem 6.1. *(See [1,18].) The following inclusions are homotopy equivalences:*

$$\textbf{(a)}\quad \mathcal{S}_G^{*+\mathrm{eab}} \hookrightarrow \mathcal{S}_G^* \qquad \textbf{(b)}\quad \mathcal{S}_G^{*+\mathrm{rad}} \hookrightarrow \mathcal{S}_G^* \qquad \textbf{(c)}\quad \mathcal{S}_G^{\mathrm{sfc}+\mathrm{rad}} \hookrightarrow \mathcal{S}_G^{\mathrm{sfc}}$$

Proof. We now flesh out the details of the above paragraph:

(a) **[EC]** The coweighting for $\mathcal{S}_G^*$ can be expressed in two different ways by [13, Theorem 1.1.(1)] and Theorem 3.7,

$$-\widetilde{\chi}\big(\mathcal{S}_K^{(1,K)}\big) = k^K = -\widetilde{\chi}\big(\mathcal{S}_G^*//K\big),$$

so that the categories $\mathcal{S}_G^*//K$ and $\mathcal{S}_K^{(1,K)}$ have identical Euler characteristics. By Lemma 5.1(b), $\widetilde{\chi}(\mathcal{S}_K^{(1,K)}) = 0$ unless K is elementary abelian. This suggests that the class of subgroups with non-contractible proper slice categories is precisely the elementary abelian p-subgroups of G.
[HE] Indeed, the categories $\mathcal{S}_G^*//K$ and $\mathcal{S}_K^{(1,K)}$ not only have the same Euler characteristics, they are themselves identical! Lemma 5.1(b) then gives us more information:

$$\mathrm{supp}\big(\mathcal{S}_G^*//\bullet\big) = \big\{k \in \mathrm{Ob}\big(\mathcal{S}_G^*\big) \bigm| K \text{ is elementary abelian}\big\} = \mathrm{Ob}\big(\mathcal{S}_G^{*+\mathrm{eab}}\big)$$

and Bouc's Theorem 4.3 shows that the inclusion of $\mathcal{S}_G^{*+\mathrm{eab}}$ into $\mathcal{S}_G^*$ is a homotopy equivalence.

(b) **[EC]** The weighting for $\mathcal{S}_G^*$ can be expressed in two different ways by [13, Theorem 1.3.(1)] and Theorem 3.7:

$$-\widetilde{\chi}\big(\mathcal{S}_{\mathcal{O}_G(H)}^*\big) = k_{\mathcal{S}}^H = -\widetilde{\chi}\big(H//\mathcal{S}_G^*\big).$$

In particular, the categories $H//\mathcal{S}_G^*$ and $\mathcal{S}_{\mathcal{O}_G(H)}^*$ have identical Euler characteristics. By property (2.4), if $\mathcal{S}_{\mathcal{O}_G(H)}^*$ is not contractible, then H must be G-radical. Therefore the class of subgroups whose proper coslice category has nonzero Euler characteristic is contained in the class of G-radical subgroups.
[HE] We show that this equality of reduced Euler characteristics reflects a homotopy equivalence $H//\mathcal{S}_G^* \simeq \mathcal{S}_{\mathcal{O}_G(H)}^*$. For any nonidentity p-subgroup H of G, there are functors

$$H/\mathcal{S}_G^* \underset{i_H}{\overset{r_H}{\rightleftarrows}} \mathcal{S}_{\mathcal{O}_G(H)}, \qquad H//\mathcal{S}_G^* \underset{i_H}{\overset{r_H}{\rightleftarrows}} \mathcal{S}_{\mathcal{O}_G(H)}^*$$

given by $Kr_H = N_K(H)/H$ for all p-supergroups K of H and $\overline{K}i_H = K$ when $\overline{K} = K/H$ and $H \le K \le N_G(H)$ [18, Lemma 6.1]. The composite functor $i_H r_H$ is the identity of $\mathcal{S}_{\mathcal{O}_G(H)}$ and there is a natural transformation from $r_H i_H : K \mapsto N_K(H)$ to the identity functor of $H/\mathcal{S}_G^*$. This shows that these functors are homotopy equivalences of categories. By property (2.4),

$$\begin{aligned}\mathrm{supp}\big(\bullet//\mathcal{S}_G^*\big) &= \big\{H \in \mathrm{Ob}\big(\mathcal{S}_G^*\big) \bigm| \mathcal{S}_{\mathcal{O}_G(H)}^* \text{ is noncontractible}\big\}\\ &\subseteq \big\{H \in \mathrm{Ob}\big(\mathcal{S}_G^*\big) \bigm| H \text{ is } G\text{-radical}\big\} = \mathrm{Ob}\big(\mathcal{S}_G^{*+\mathrm{rad}}\big)\end{aligned}$$

and Bouc's Theorem 4.3 shows that the inclusion of $\mathcal{S}_G^{*+\mathrm{rad}}$ into $\mathcal{S}_G^*$ is a homotopy equivalence.

(c) Since any p-supergroup of a p-self-centralizing G-subgroup is itself p-self-centralizing, $H//\mathcal{S}_G^{\mathrm{sfc}} = H//\mathcal{S}_G^*$ for any p-self-centralizing subgroup H of G. The result then follows from Part (b). □

Example 6.2. If $G = C_2 \times \Sigma_3$ and $p = 2$, then $\mathcal{S}_G^{\mathrm{sfc}}$ is a discrete poset consisting of the 3 Sylow 2-subgroups, while $\mathcal{S}_G^*$ is contractible since $O_2G = C_2$ is nontrivial. Thus the inclusion $\mathcal{S}_G^{\mathrm{sfc}} \to \mathcal{S}_G^*$ is *not* a homotopy equivalence.

The following proposition points out that the largest normal p-subgroup is the smallest G-radical p-subgroup. It implies that the poset $\mathcal{S}_G^{*+\mathrm{rad}}$ has a smallest element in the case O_pG is nontrivial. In light of Theorem 6.1(b), this could be thought of as the essential ingredient that goes into Quillen's property (2.3). (We thank Andy Chermak for the proof.)

Proposition 6.3. *Any G-radical p-subgroup of G contains the G-radical p-subgroup O_pG.*

Proof. It is clear that O_pG is a normal G-radical p-subgroup. Let H be a p-subgroup of G not containing O_pG. The normalizer of H in the p-subgroup $(O_pG)H$ is normal in $N_G(H)$ for any element of G normalizing H normalizes $(O_pG)H$. Since $N_{(O_pG)H}(H)$ is a normal p-subgroup of $N_G(H)$ strictly larger than H, the p-subgroup H is not G-radical. □

We close this section by moving from p-subgroup posets to more general EI-categories. Let $\mathcal{T}_G$ be the transporter category of p-subgroups of G.

Theorem 6.4. *The following inclusions are homotopy equivalences:*

$$(\mathbf{a})\quad \mathcal{T}_G^{*+\mathrm{eab}} \hookrightarrow \mathcal{T}_G^{*} \qquad (\mathbf{b})\quad \mathcal{T}_G^{*+\mathrm{rad}} \hookrightarrow \mathcal{T}_G^{*} \qquad (\mathbf{c})\quad \mathcal{T}_G^{\mathrm{sfc}+\mathrm{rad}} \hookrightarrow \mathcal{T}_G^{\mathrm{sfc}}$$

Proof. Every morphism of $\mathcal{T}_G^*$ is both epi and mono, so it follows that the (co)slice categories of objects should be identifiable with the Brown posets of certain groups relating those objects to G. With this in mind, the argument follows that of Theorem 6.1 closely.

(a) **[EC]** The coweighting on $[\mathcal{T}_G^*]$ is computed in [13, Theorem 1.1.(2)]. Theorem 3.7 gives an alternate calculation of the coweighting in terms of Euler characteristics of proper slice categories:

$$\frac{-\widetilde{\chi}(\mathcal{S}_K^{(1,K)})}{|\mathcal{T}_G^*(K)|} = k_{[K]}^{[\mathcal{T}]} = \frac{-\widetilde{\chi}(\mathcal{T}_G^*//K)}{|\mathcal{T}_G^*(K)|}.$$

Lemma 5.1(a) then implies that $\widetilde{\chi}(\mathcal{T}_G^*//K)$ is nonzero iff K is elementary abelian.
[HE] In fact, Lemma 5.1 says more: $\mathcal{S}_K^{(1,K)}$ is *noncontractible* iff K is elementary abelian. Our goal is then to show that the equality of reduced Euler characteristics $\widetilde{\chi}(\mathcal{S}_K^{(1,K)}) = \widetilde{\chi}(\mathcal{T}_G^*//K)$ reflects a homotopy equivalence $\mathcal{S}_K^{(1,K)} = \mathcal{T}_G^*//K$; once this has been accomplished, Theorem 4.3 will complete the result. There are functors

$$\mathcal{S}_K^{(1,K]} \underset{i_K}{\overset{r_K}{\rightleftarrows}} \mathcal{T}_G^*/K, \qquad \mathcal{S}_K^{(1,K)} \underset{i_K}{\overset{r_K}{\rightleftarrows}} \mathcal{T}_G^*//K,$$

given by $Hr_K = (H \xrightarrow{1} K)$ and $(H \xrightarrow{g} K)i_K = H^g$. Clearly these are equivalences of categories, so we have our desired homotopy equivalence $\mathcal{S}_K^{(1,K)} = \mathcal{T}_G^*//K$ and the result is proved.

(b) **[EC]** The weighting of $[\mathcal{T}_G^*]$ was computed in [13, Theorem 1.3.(2)]. Comparing this to the alternate calculation of the weighting Theorem 3.7, we have

$$\frac{-\widetilde{\chi}(\mathcal{S}_{\mathcal{O}_G(H)}^*)}{|\mathcal{T}_G^*(H)|} = k_{[\mathcal{T}]}^{[H]} = \frac{-\widetilde{\chi}(H//\mathcal{T}_G^*)}{|\mathcal{T}_G^*(H)|}.$$

Property (2.4) implies that $\widetilde{\chi}(H//\mathcal{T}_G^*) \neq 0$ implies that H is G-radical.

[HE] If we can show that there is a homotopy equivalence $\mathcal{S}^*_{\mathcal{O}_G(H)} \simeq H//\mathcal{T}^*_G$, the full strength of property (2.4) will yield $\mathrm{supp}(\bullet//\mathcal{T}^*_G)$ is contained in the class of G-radical subgroup, so Theorem 4.3 will give the result. There are functors

$$H/\mathcal{T}^*_G \underset{i_H}{\overset{r_H}{\rightleftarrows}} \mathcal{S}_{\mathcal{O}_G(H)}, \qquad H//\mathcal{T}^*_G \underset{i_H}{\overset{r_H}{\rightleftarrows}} \mathcal{S}^*_{\mathcal{O}_G(H)}$$

given by $(H \xrightarrow{g} K)r_H = N_K(H^g)^{g^{-1}}/H$ and $\overline{K}i_H = (H \xrightarrow{1} K)$ where $\overline{K} = K/H$ and we have $H \leq K \leq N_G(H)$. Clearly $i_H r_h = \mathrm{id}_{\mathcal{S}^*_{\mathcal{O}_G(H)}}$, and we have a natural transformation $\eta : r_H i_H \Rightarrow \mathrm{id}_{H//\mathcal{T}^*_G}$ induced by the inclusion $N_K(H^g)^{g^{-1}} \leq N_G(H)$. Thus the two categories are homotopy equivalent, and the result is proved.

(c) Follows from Part (b) and the observation that supergroups of p-self-centralizing subgroups of G are themselves p-self-centralizing. □

Suppose that $\mathcal{C}$ is a small category and $X, Y : \mathcal{C} \to \mathrm{CAT}$ are functors with values in the category CAT of small categories. If there is a natural transformation from X to Y with components $X(c) \to Y(c)$, $c \in \mathrm{Ob}(\mathcal{C})$, that are all homotopy equivalences, then the induced functor $\int_{\mathcal{C}} X \to \int_{\mathcal{C}} Y$ of Grothendieck constructions is a homotopy equivalence. This follows from Thomason's homotopy colimit theorem [21] and homotopy invariance of the homotopy colimit [2, Ch. XII, §4, Homotopy Lemma 4.2]. As the inclusions of Theorem 6.1 are G-equivariant inclusions of G-categories and $\mathcal{T}^*_G$ is the Grothendieck construction of the G-action on $\mathcal{S}^*_G$, etc., we obtain an alternative proof of Proposition 6.4. Similarly, if O_pG is nontrivial, there is a homotopy equivalence $G \hookrightarrow \mathcal{T}^{*+\mathrm{rad}}_G$ induced by the G-equivariant homotopy equivalence $* \hookrightarrow \mathcal{S}^{*+\mathrm{rad}}_G$ of Proposition 6.3.

7. Frobenius categories

Let P be a finite p-group and $\mathcal{F}$ a Frobenius P-category. In this section we show that the homotopy type of $\mathcal{F}^*$ is determined by the elementary abelian subgroups of P.

We will need the following facts:

- All morphisms in $\mathcal{F}$ are monomorphisms, which implies
- For any $K \leq P$, the categories $\mathcal{F}^*/K$ and $\mathcal{F}^*//K$ are *thin*, i.e., there is at most one morphism between any two objects
- The coweighting for $\mathcal{F}^*$ vanishes off the elementary abelian subgroups [13, Theorem 7.5]

Theorem 7.1. *The inclusion $\mathcal{F}^{*+\mathrm{eab}} \to \mathcal{F}^*$ is a homotopy equivalence.*

Proof.

[EC] We compute the coweighting on $[\mathcal{F}^*]$ using both [13, Theorem 7.5] and Theorem 3.7:

$$\frac{-\widetilde{\chi}(\mathcal{S}_K^{(1,K)})}{|\mathcal{F}^*(K)|} = k^{[\mathcal{F}^*]}_{[K]} = \frac{-\widetilde{\chi}(\mathcal{F}^*//K)}{|\mathcal{F}^*(K)|}$$

Therefore $\mathcal{S}_K^{(1,K)}$ and $\mathcal{F}^*//K$ have identical Euler characteristics. By Lemma 5.1(b), $\widetilde{\chi}(\mathcal{F}^*//K) \neq 0$ implies K is elementary abelian.

[HE] Indeed, there are functors

$$\mathcal{F}^*/K \underset{i_K}{\overset{r_K}{\rightleftarrows}} \mathcal{S}_K^{(1,K]}, \qquad \mathcal{F}^*//K \underset{i_K}{\overset{r_K}{\rightleftarrows}} \mathcal{S}_K^{(1,K)}$$

The functor r_K takes $\varphi \in \mathcal{F}^*(H,K)$ to its image H^φ in K. The functor i_K takes $H \leq K$ to the inclusion $H \hookrightarrow K$ of H into K. Obviously, $i_K r_K$ is the identity functor of $\mathcal{S}_K^{(1,K]}$, and there is a natural transformation from the identity functor to the endofunctor $r_K i_K : (H \xrightarrow{\varphi} K) \mapsto (H^\varphi \hookrightarrow K)$ of $\mathcal{F}^*/K$. This shows that r_K and i_K are homotopy equivalences between $\mathcal{F}^*/K$ and $\mathcal{S}_K^{(1,K]}$. Their restrictions are homotopy equivalences between $\mathcal{F}^*//K$ and $\mathcal{S}_K^{(1,K)}$. By the full strength of Lemma 5.1(b),

$$\operatorname{supp}\bigl(\mathcal{F}^*//\bullet\bigr) = \operatorname{Ob}\bigl(\mathcal{F}^{*+\mathrm{eab}}\bigr)$$

and Bouc's Theorem 4.3 shows that the inclusion of $\mathcal{F}^{*+\mathrm{eab}}$ into $\mathcal{F}^*$ is a homotopy equivalence. □

In the course of the proof of Theorem 7.1 we saw that the homotopy type of the category $\mathcal{F}^*//K$ of $\mathcal{F}^*$-nonisomorphisms to K depends only on K, not on $\mathcal{F}$. This reflects the curious fact that the shape of the Frobenius P-category is able to detect some algebraic information of the underlying p-group.

We know of no formula for the weighting of a *general* Frobenius category $\mathcal{F}$. There is an explicit formula in [13, Theorem 1.3.(3)] for the weighting of the Frobenius category $\mathcal{F}_G$ associated to a finite group G, but we have not been able to determine the support of this weighting or describe the categories $H//\mathcal{F}_G^*$.

8. Orbit categories

Let G be a finite group of order divisible by p and $\mathcal{O}_G$ the orbit category of p-subgroups of G.

We will need the following facts:

- The trivial subgroup is not initial in $\mathcal{O}_G$
- All morphisms in $\mathcal{O}_G$ are epimorphisms, therefore
- The categories $H/\mathcal{O}_G^*$ and $H//\mathcal{O}_G^*$ are thin
- The weighting for $\mathcal{O}_G$ vanishes off the G-radical p-subgroups of G [13, Proposition 3.14]
- The coweighting for $\mathcal{O}_G$ vanishes off the cyclic p-subgroups [13, Theorem 4.1]

Theorem 8.1. *The following inclusions are homotopy equivalences:*

$$(\mathbf{a}) \quad \mathcal{O}_G^{\mathrm{rad}} \hookrightarrow \mathcal{O}_G \qquad (\mathbf{b}) \quad \mathcal{O}_G^{*+\mathrm{rad}} \hookrightarrow \mathcal{O}_G^* \qquad (\mathbf{c}) \quad \mathcal{O}_G^{\mathrm{sfc}+\mathrm{rad}} \hookrightarrow \mathcal{O}_G^{\mathrm{sfc}}$$

Proof. The setup for each claim is identical.

[EC] The two expressions for the weighting for $[\mathcal{O}_G]$ from [13, Eq. (3.15)] and Theorem 3.7 yield

$$\frac{-\widetilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G(H)})}{|\mathcal{O}_G(H)|} = k^{[H]}_{[\mathcal{O}_G]} = \frac{-\widetilde{\chi}(H//\mathcal{O}_G)}{|\mathcal{O}_G(H)|},$$

so that $\mathcal{S}^*_{\mathcal{O}_G(H)}$ and $H//\mathcal{O}_G$ have identical Euler characteristics. Since $\widetilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G(H)}) \neq 0$ implies H is G-radical (property (2.4)), each claim is at least plausible.

[HE] We show that the equality of reduced Euler characteristics reflects a homotopy equivalence $\mathcal{S}^*_{\mathcal{O}_G(H)} \simeq H//\mathcal{O}_G$; the result will then follow from the contractibility of $\mathcal{S}^*_{\mathcal{O}_G(H)}$ by property (2.4) and Theorem 4.3. For any nonidentity p-subgroup H of G, there are functors

$$r_H\colon H/\mathcal{O}_G \to \mathcal{S}_{\mathcal{O}_G(H)}, \qquad r_H\colon H//\mathcal{O}_G \to \mathcal{S}^*_{\mathcal{O}_G(H)}.$$

The functor r_H takes $gK \in \mathcal{O}_G(H,K) = N_G(H,K)/K$ to the subgroup $N_{{}^gK}(H)/H$ of $\mathcal{O}_G(H) = N_G(H)/H$. Let L be a p-subgroup such that $H \le L \le N_G(H)$ and let $\overline{L} = L/H$ be the image of L in $N_G(H)/H = \mathcal{O}_G(H)$. The category $\overline{L}/r_H$ is the full subcategory of $\mathcal{O}_G/H$ generated by all morphisms $gK \in \mathcal{O}_G(H,K)$ such that $L \le N_{{}^gK}(H)$. The inclusion of H into L is an object of $\overline{L}/r_H$ as $L = N_L(H)$. Note that the morphism $gK\colon H \to K$ extends to a morphism $gK\colon L \to K$ because $L^g \le N_{{}^gK}(H)^g = N_K(H^g) \le K$. There is thus a morphism

$$\begin{tikzcd} & H \arrow[dl, hook'] \arrow[dr, "gK"] & \\ L \arrow[rr, "gK"'] & & K \end{tikzcd}$$

in $\overline{L}/r_H$. This shows that the inclusion $H \hookrightarrow L$ is an initial object of $\overline{L}/r_H$. By Quillen's Theorem A (Theorem 4.1), the functor r_H is a homotopy equivalence from $H/\mathcal{O}_G$ to $\mathcal{S}_{\mathcal{O}_G(H)}$. The same argument shows that r_H restricts to a homotopy equivalence from $H//\mathcal{O}_G$ to $\mathcal{S}^*_{\mathcal{O}_G(H)}$. Thus $\mathrm{supp}(\bullet//\mathcal{O}_G) \subset \mathrm{Ob}(\mathcal{O}_G^{\mathrm{rad}})$ and Part (a) is proved.

Since $\mathcal{O}_G^*$ and $\mathcal{O}_G^{\mathrm{sfc}}$ are left ideals in $\mathcal{O}_G$, $H//\mathcal{O}_G^* = H//\mathcal{O}_G$ and $H//\mathcal{O}_G^{\mathrm{sfc}} = H//\mathcal{O}_G$ for any nonidentity, respectively, p-self-centralizing subgroup H of G. By property (2.4),

$$\mathrm{supp}\big(\bullet//\mathcal{O}_G^*\big) \subset \mathrm{Ob}\big(\mathcal{O}_G^{*+\mathrm{rad}}\big), \qquad \mathrm{supp}\big(\bullet//\mathcal{O}_G^{\mathrm{sfc}}\big) \subset \mathrm{Ob}\big(\mathcal{O}_G^{\mathrm{sfc}+\mathrm{rad}}\big)$$

proving (b) and (c). $\square$

It seems that there should be a dual result to Theorem 8.1 involving certain "small" subgroups in place of the "large" G-radical class. More precisely, there should be a theorem whose proof uses slices in place of the coslices of the previous argument. The relevant class of subgroups to consider would then be those contained in $\mathrm{supp}(\mathcal{O}_G//\bullet)$. However, we cannot identify this class of subgroups at this point, and indeed experimental evidence leads us to conjecture that *all* p-subgroups will necessarily be contained in the support. If this conjecture holds, the dual theorem would reduce to the tautology $\mathcal{O}_G \simeq \mathcal{O}_G$, which would not be particularly enlightening.

9. Exterior quotients of Frobenius categories

Let P be a nonidentity finite p-group, $\mathcal{F}$ a Frobenius P-category, and $\widetilde{\mathcal{F}}$ the exterior quotient of $\mathcal{F}$ [16, 1.3, 2.6, 4.8]. In this section we examine the homotopy types of $\widetilde{\mathcal{F}}^*$ and $\widetilde{\mathcal{F}}^{\mathrm{sfc}}$.

We begin with $\widetilde{\mathcal{F}}^*$, searching for a class of "small" subgroups that control the homotopy type. For our Euler characteristic intuition-building, the essential fact here is that the coweighting for $[\widetilde{\mathcal{F}}^*]$ vanishes off of the elementary abelian subgroups by [13, Theorem 7.7].

Theorem 9.1. *The inclusion $\widetilde{\mathcal{F}}^{*+\mathrm{eab}} \hookrightarrow \widetilde{\mathcal{F}}^*$ is a homotopy equivalence.*

Proof.

[EC] Comparing the reduced Euler characteristic expression for the coweighting of $[\widetilde{\mathcal{F}}^*]$ from Theorem 3.7 to [13, Theorem 7.7] yields

$$\frac{-\widetilde{\chi}(\widetilde{\mathcal{F}}_K^{(1,K)})}{|\widetilde{\mathcal{F}}^*(K)|} = k_{[K]}^{[\widetilde{\mathcal{F}}^*]} = \frac{-\widetilde{\chi}(\widetilde{\mathcal{F}}^*//K)}{|\widetilde{\mathcal{F}}^*(K)|}.$$

Therefore $\widetilde{\mathcal{F}}_K^{(1,K)}$ and $\widetilde{\mathcal{F}}^*//K$ have identical Euler characteristics for any object K of $\widetilde{\mathcal{F}}^*$. By Lemma 5.1(a) and (d), $\widetilde{\chi}(\widetilde{\mathcal{F}}^*//K)$ can only be nonzero if K is elementary abelian.

[HE] In fact, there are equivalences of categories

$$i_K\colon \widetilde{\mathcal{F}}_K^{(1,K]} \to \widetilde{\mathcal{F}}^*/K, \qquad i_K\colon \widetilde{\mathcal{F}}_K^{(1,K)} \to \widetilde{\mathcal{F}}^*//K$$

On an object $H \leq K$, we have $Hi_K = [\iota_K^H] \in \widetilde{\mathcal{F}}^*(H,K)$ is the class of the inclusion $\iota_K^H \in \mathcal{F}^*(H,K)$ of H into K. Observe that there is an obvious identification of morphism sets

$$\widetilde{\mathcal{F}}_K^*(H_1,H_2) = \big(\widetilde{\mathcal{F}}^*/K\big)(H_1 i_K, H_2 i_K),$$

which defines the effect of i_K on morphism sets. Thus i_K is full and faithful. It is also easily seen to be essentially surjective on objects, hence an equivalence of categories.

Combining the homotopy equivalence $\widetilde{\mathcal{F}}_K^{(1,K)} \simeq \mathcal{F}^*//K$ with Lemma 5.1(d), we have

$$\mathrm{supp}\big(\widetilde{\mathcal{F}}^*//\bullet\big) = \mathrm{Ob}\big(\widetilde{\mathcal{F}}^{*+\mathrm{eab}}\big)$$

and Bouc's Theorem 4.3 shows that the inclusion of $\widetilde{\mathcal{F}}^{*+\mathrm{eab}}$ into $\widetilde{\mathcal{F}}^*$ is a homotopy equivalence. □

We now turn to the question of finding a "large" collection of subgroups that controls the homotopy type of the exterior quotient, in some sense dual to the elementary abelian subgroups of Theorem 9.1. There is a new technical difficult we must take into consideration here: We lack a good understanding of the full exterior quotient of a Frobenius P-category. Much more is known about the $\mathcal{F}$-self-centralizing subcategory $\widetilde{\mathcal{F}}^{\mathrm{sfc}}$, where we can make use of the following facts:

- All morphisms in $\widetilde{\mathcal{F}}^{\mathrm{sfc}}$ are epimorphisms [16, Corollary 4.9], therefore
- The categories $H/\widetilde{\mathcal{F}}^{\mathrm{sfc}}$ are thin
- The weighting for $\widetilde{\mathcal{F}}_G^{\mathrm{sfc}}$ vanishes off the $\mathcal{F}_G$-radical subgroups [13, Corollary 8.6]

We will also need the following technical result, which is a reformulation of [7, Proposition 2.4]:

Lemma 9.2. *Let H, N, and K be objects of $\mathcal{F}$ such that H is $\mathcal{F}$-self-centralizing and $H \leq N \leq N_P(H)$. An $\mathcal{F}$-morphism $\varphi\colon H \to K$ extends to an $\mathcal{F}$-morphism $\psi\colon N \to K$ if and only if $\mathcal{F}_N(H)^\varphi \leq \mathcal{F}_K(H^\varphi)$.*

Proof. We prove the "if" implication, as the converse is clear. Since H is $\mathcal{F}$-self-centralizing, the same is true of H^φ and thus H^φ is fully centralized in $\mathcal{F}$ [16, 4.8]. By the Extension Axiom for Frobenius P-categories and our assumption, $\varphi\colon H \to K$ extends to a morphism $\rho\colon N \to P$ [16, 2.10.1]. We claim that $(x)\rho \in K$ for all $x \in N$. By assumption, there is some $y \in K$ such that conjugation with $(x)\rho$ and with y has the same effect on H^φ. This means that $(x)\rho y^{-1} \in C_P(H^\varphi) \leq Z(H^\varphi) \leq H^\varphi \leq K$, and thus $(x)\rho \in K$. The corestriction $\psi = K|\rho\colon N \to K$ of $\psi\colon N \to P$ extends $\varphi\colon H \to K$. □

Consequently,

$$\widetilde{\mathcal{F}}(N,K) = \widetilde{\mathcal{F}}(H,K)^{\mathcal{F}_N(H)}$$

under the assumptions of Lemma 9.2.

We are now ready to prove:

Theorem 9.3. *The inclusion $\widetilde{\mathcal{F}}^{\mathrm{sfc+rad}} \hookrightarrow \widetilde{\mathcal{F}}^{\mathrm{sfc}}$ is a homotopy equivalence.*

Proof. Fix an $\mathcal{F}$-self-centralizing subgroup $H \le P$, and let $G := \widetilde{\mathcal{F}}(H)$ be the automorphism group of H in the exterior quotient category.

[EC] Consider the special case that $\widetilde{\mathcal{F}}^{\mathrm{sfc}} = \widetilde{\mathcal{F}}^{\mathrm{sfc}}_G$ for some finite group G inducing the exterior quotient category $\widetilde{\mathcal{F}}$. The weighting for $\widetilde{\mathcal{F}}^{\mathrm{sfc}}_G$ was computed in [13, Proposition 8.5]; comparison with Theorem 3.7 yields

$$\frac{-\widetilde{\chi}(\mathcal{S}^*_{\widetilde{\mathcal{F}}^{\mathrm{sfc}}_G(H)})}{|\widetilde{\mathcal{F}}^{\mathrm{sfc}}_G(H)|} = k^{[H]}_{[\widetilde{\mathcal{F}}^{\mathrm{sfc}}_G]} = \frac{-\widetilde{\chi}(H//\widetilde{\mathcal{F}}^{\mathrm{sfc}}_G)}{|\widetilde{\mathcal{F}}^{\mathrm{sfc}}_G(H)|}.$$

Thus the proper coslice category $H//\widetilde{\mathcal{F}}^{\mathrm{sfc}}_G$ and the poset $\mathcal{S}^*_{\widetilde{\mathcal{F}}^{\mathrm{sfc}}_G(H)}$ have identical Euler characteristics for any object H of $\widetilde{\mathcal{F}}^{\mathrm{sfc}}$.

We believe that an abstract version of [13, Proposition 8.5] that does not reference an ambient finite group is true as well. Such a result would imply that $H//\widetilde{\mathcal{F}}^{\mathrm{sfc}}$ and $\mathcal{S}^*_{\widetilde{\mathcal{F}}^{\mathrm{sfc}}(H)}$ should in general have identical reduced Euler characteristics. (In fact, this result will follow from the homotopy equivalence of the next paragraph.) As the Euler characteristic computation serves primarily to direct our attention toward the class of subgroups that control the homotopy type, this special case is already enough to suggest that we should consider the $\mathcal{F}$-radical subgroups. That is where we will focus our attention.

[HE] We claim there is a homotopy equivalence $H//\widetilde{\mathcal{F}} \simeq \mathcal{S}^*_{\widetilde{\mathcal{F}}(H)}$. There are functors

$$r_H \colon H/\widetilde{\mathcal{F}} \to \mathcal{S}_{\widetilde{\mathcal{F}}(H)}, \qquad r_H \colon H//\widetilde{\mathcal{F}} \to \mathcal{S}^*_{\widetilde{\mathcal{F}}(H)} \tag{9.4}$$

There is no loss of generality in assuming that H is fully normalized in $\widetilde{\mathcal{F}}$, so that the order of the P-normalizer of H is maximal in its $\widetilde{\mathcal{F}}$-isomorphism class. The functor r_H is defined by $[\varphi]r_H = {}^{\varphi}\widetilde{\mathcal{F}}_K(H^{\varphi})$, where $[\varphi] = \varphi\mathcal{F}_K(K) \in \widetilde{\mathcal{F}}(H,K) = \mathcal{F}(H,K)/\mathcal{F}_K(K)$ is an object of $H/\widetilde{\mathcal{F}}$. Note that this is well-defined even though φ is only defined up to conjugacy in K. The group

$$\widetilde{\mathcal{F}}_K(H^{\varphi}) = C_K(H^{\varphi})\backslash N_K(H^{\varphi})/H^{\varphi} = Z(H^{\varphi})\backslash N_K(H^{\varphi})/H^{\varphi} = N_K(H^{\varphi})/H^{\varphi}$$

and the isomorphic group $r_H(\varphi\mathcal{F}_K(K)) = {}^{\varphi}\widetilde{\mathcal{F}}_K(H^{\varphi})$ are related by the commutative diagram

$$\begin{tikzcd} H \arrow[r, "\varphi", "\cong"'] \arrow[d, dotted, "{}^{\varphi}\widetilde{\mathcal{F}}_K(H^{\varphi})"'] & H^{\varphi} \arrow[d, "\widetilde{\mathcal{F}}_K(H^{\varphi})=N_K(H^{\varphi})/H^{\varphi}"] \\ H \arrow[r, "\varphi", "\cong"'] & H^{\varphi} \end{tikzcd}$$

It is clear that ${}^{\varphi}\widetilde{\mathcal{F}}_K(H^{\varphi})$ is a p-subgroup of $\widetilde{\mathcal{F}}(H)$ and that $r_H(\varphi_1) \le r_H(\varphi_2)$ whenever there is an $\widetilde{\mathcal{F}}$-morphism

$$\begin{tikzcd} & H \arrow[dl, "\varphi_1\mathcal{F}_{K_1}(K_1)"'] \arrow[dr, "\varphi_2\mathcal{F}_{K_2}(K_2)"] & \\ K_1 \arrow[rr] & & K_2 \end{tikzcd}$$

under H. Thus r_H is a functor. We now want to use Quillen's Theorem A to show that r_H is an equivalence of categories.

Let $\overline{L}$ be a p-subgroup of $\widetilde{\mathcal{F}}(H) = \mathcal{F}(H)/\mathcal{F}_H(H)$. We may assume that $\overline{L}$ is contained in the Sylow p-subgroup $\widetilde{\mathcal{F}}_P(H) = N_P(H)/H$ of $\widetilde{\mathcal{F}}(H)$ (which is known to be Sylow by the assumption that H is fully normalized in $\widetilde{\mathcal{F}}$). There is a unique p-subgroup $L \in [P, N_P(H)]$ such that $\overline{L} = L/H$. The category $\overline{L}/r_H$ is the full subcategory of $H/\widetilde{\mathcal{F}}$ generated by all objects $\varphi\mathcal{F}_K(K) \in \widetilde{\mathcal{F}}(H,K)$ such that $\overline{L}^\varphi \leq \widetilde{\mathcal{F}}_K(H^\varphi) = N_K(H^\varphi)/H^\varphi$, or, equivalently, $L^\varphi \leq N_K(H^\varphi)$. Here is an attempt to visualize this relationship:

$$\begin{tikzcd} H \arrow[r, "\varphi", "\cong"'] \arrow[d, "L/H"'] & H^\varphi \arrow[d, "N_K(H^\varphi)/H^\varphi"] \\ H \arrow[r, "\varphi", "\cong"'] & H^\varphi \end{tikzcd}$$

The inclusion $\iota_L^H : H \hookrightarrow L$ of H into L represents both a morphism in $\widetilde{\mathcal{F}}(H,L)$ and an object of $\overline{L}/r_H$ because $\overline{L}$ is contained in $(\iota_L^H \mathcal{F}_L(L))r_H = N_L(H)/H = L/H = \overline{L}$. By Lemma 9.2 there is an extension in $\widetilde{\mathcal{F}}$ of $\varphi : H \to K$

$$\begin{tikzcd} & H \arrow[dl, hook, "\iota_L^H"'] \arrow[dr, "\varphi"] & \\ L \arrow[rr, dotted] & & K \end{tikzcd}$$

to a morphism $L \to K$. We have now shown that $\iota_L^H \mathcal{F}_L(L)$ is an initial object of $\overline{L}/r_H$ for any object $\overline{L}$ of $\mathcal{S}_{\widetilde{\mathcal{F}}(H)}$. According to Quillen's Theorem A (Theorem 4.1), r_H is a homotopy equivalence of categories.

Since the functor r_H takes nonisomorphisms $\varphi\mathcal{F}_K(K) \in \widetilde{\mathcal{F}}(H,K) \subset \mathrm{Ob}(H/\widetilde{\mathcal{F}})$ to nonidentity p-subgroups of $\widetilde{\mathcal{F}}(H)$, it restricts to a functor $r_H : H//\widetilde{\mathcal{F}} \to \mathcal{S}^*_{\widetilde{\mathcal{F}}(H)}$ of $H//\widetilde{\mathcal{F}}$ into the Brown poset of the automorphism group of H. But since $\overline{L}/r_H$ is contractible for any nonidentity p-subgroup $\overline{L}$ of $\widetilde{\mathcal{F}}(H)$, we already know that the restricted functor r_H is a homotopy equivalence of categories. By property (2.5),

$$\mathrm{supp}\big(\bullet//\widetilde{\mathcal{F}}^{\mathrm{sfc}}\big) \subset \mathrm{Ob}\big(\widetilde{\mathcal{F}}^{\mathrm{sfc+rad}}\big)$$

and Bouc's Theorem 4.3 shows that the inclusion of $\widetilde{\mathcal{F}}^{\mathrm{sfc+rad}}$ into $\widetilde{\mathcal{F}}^{\mathrm{sfc}}$ is a homotopy equivalence. □

10. Linking categories

Let $\mathcal{L}^{\mathrm{sfc}}$ be the centric linking system associated to a Frobenius P-category $\mathcal{F}$ [5, Definition 1.7]. We will prove that the homotopy type of $\mathcal{L}^{\mathrm{sfc}}$ is controlled by the $\mathcal{F}$-radical subgroups. This result is part of [3, Theorem B], the full strength of which would be accessible by our methods if we were to consider the more general notion of a *quasi*centric linking system.

We will need the following facts:

- All morphisms in $\mathcal{L}^{\mathrm{sfc}}$ are monomorphisms and epimorphisms [16, Proposition 24.2]
- The weighting for $\mathcal{L}^{\mathrm{sfc}}$ vanishes off the $\mathcal{F}$-radical subgroups [13, Proposition 8.5]

Theorem 10.1. *The inclusion functor $\mathcal{L}^{\mathrm{sfc+rad}} \to \mathcal{L}^{\mathrm{sfc}}$ is a homotopy equivalence.*

Proof. Let H be an $\mathcal{F}$-self-centralizing object of $\mathcal{F}$. The functor $\widetilde{\pi}: \mathcal{L}^{\mathrm{sfc}} \to \widetilde{\mathcal{F}}^{\mathrm{sfc}}$ is bijective on objects and $|K|$-to-1 for on morphism sets $\mathcal{L}^{\mathrm{sfc}}(H,K) \to \widetilde{\mathcal{F}}^{\mathrm{sfc}}(H,K)$ with codomain $K \in \mathrm{Ob}(\mathcal{F}^{\mathrm{sfc}})$. $K = \mathcal{L}_K(K) \leq \mathcal{L}(K)$ acts freely from the right on $\mathcal{L}(H,K)$ with quotient $\mathcal{L}^{\mathrm{sfc}}(H,K)/K = \widetilde{\mathcal{F}}^{\mathrm{sfc}}(H,K)$ [5, Lemma 1.10]. This implies that if $\varphi_1 \in \mathcal{L}(H,K_1)$, $\varphi_2 \in \mathcal{L}(H,K_2)$, and the commutative $\widetilde{\mathcal{F}}$-diagram to the right has a solution

$$\begin{tikzcd} & H \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] \arrow[d, phantom, "\mathcal{L}"] & \\ K_1 \arrow[rr, dotted] & & K_2 \end{tikzcd} \qquad \begin{tikzcd} & H \arrow[dl, "(\varphi_1)\pi"'] \arrow[dr, "(\varphi_2)\pi"] \arrow[d, phantom, "\widetilde{\mathcal{F}}"] & \\ K_1 \arrow[rr, dotted] & & K_2 \end{tikzcd}$$

then the commutative $\mathcal{L}$-diagram to the left has a unique solution [5, Lemma 1.10]. Consider the functor

$$H/\widetilde{\pi}: H/\mathcal{L}^{\mathrm{sfc}} \to H/\widetilde{\mathcal{F}}^{\mathrm{sfc}}$$

induced by the functor $\widetilde{\pi}: \mathcal{L}^{\mathrm{sfc}} \to \widetilde{\mathcal{F}}^{\mathrm{sfc}}$. The above considerations mean that any $\varphi \in \mathcal{L}(H,K) \subset \mathrm{Ob}(H/\mathcal{L})$ is initial in the category $(\varphi)\widetilde{\pi}/H/\widetilde{\pi}$. By Quillen's Theorem A (Theorem 4.1), $H/\widetilde{\pi}$ is a homotopy equivalence.

Restricting to the nonisomorphisms we get a homotopy equivalence $H//\widetilde{\pi}: H//\mathcal{L} \to H//\widetilde{\mathcal{F}}$. Compose these homotopy equivalences with the homotopy equivalences of (9.4) to get homotopy equivalences

$$H/\mathcal{L}^{\mathrm{sfc}} \to \mathcal{S}_{\widetilde{\mathcal{F}}(H)}, \qquad H//\mathcal{L}^{\mathrm{sfc}} \to \mathcal{S}^*_{\widetilde{\mathcal{F}}(H)} \tag{10.2}$$

By property (2.5),

$$\mathrm{supp}\big(\bullet//\mathcal{L}^{\mathrm{sfc}}\big) \subset \mathrm{Ob}\big(\mathcal{L}^{\mathrm{sfc+rad}}\big)$$

and Bouc's Theorem 4.3 shows that the inclusion of $\mathcal{L}^{\mathrm{sfc+rad}}$ into $\mathcal{L}^{\mathrm{sfc}}$ is a homotopy equivalence. □

It is worth noting that the main connection between the theory of Frobenius P-categories and topology comes from the classifying space of $\mathcal{L}^{\mathrm{sfc}}$, which should be thought of as a generalization of the p-completion of the classifying space of a finite group. What is interesting in the preceding proof is that we are able to show that control of homotopy for the linking system actually comes from the seemingly less natural question about control of homotopy in the exterior quotient category $\widetilde{\mathcal{F}}^{\mathrm{sfc}}$.

Finally, we close with the dual statement, where the homotopy type is controlled by the "small" nonidentity elementary abelian groups subgroups of P. As there is currently no clear definition for an abstract linking system which has all nonidentity subgroups of P as objects, we will restrict our attention to the case where an actual finite group induces $\mathcal{L}^*_G$.

Proposition 10.3. *The inclusion $\mathcal{L}^{*+\mathrm{eab}}_G \to \mathcal{L}^*_G$ is a homotopy equivalence.*

Proof. The two expressions, from [13, Theorem 1.1.(2)], and Lemma 5.1(a) and Theorem 3.7, for the coweighting for $[\mathcal{L}^*_G]$

$$\frac{-\widetilde{\chi}(\mathcal{S}_K^{(1,K)})}{|\mathcal{L}^*_G(K)|} = k_{[K]}^{[\mathcal{L}^*_G]} = \frac{-\widetilde{\chi}(\mathcal{L}^*_G//K)}{|\mathcal{L}^*_G(K)|}$$

show that $\mathcal{S}_K^{(1,K)}$ and $\mathcal{L}_G^*//K$ have identical Euler characteristics for any object K of $\mathcal{L}_G^*$. In fact they are homotopy equivalent as we see in much the same way as in the proof of Theorem 7.1. The proof now follows from Bouc's Theorem 4.3 because $\mathrm{supp}(\mathcal{L}_G^*//K) \subset \mathrm{Ob}(\mathcal{L}_G^{*+\mathrm{eab}})$ by Lemma 5.1(b). $\square$

References

[1] Serge Bouc, Homologie de certains ensembles ordonnés, C. R. Acad. Sci. Paris Sér. I Math. 299 (2) (1984) 49–52, MR 756517 (85k:20150).
[2] A.K. Bousfield, D.M. Kan, Homotopy Limits, Completions and Localizations, 2nd ed., Lect. Notes Math., vol. 304, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1987.
[3] Carles Broto, Natàlia Castellana, Jesper Grodal, Ran Levi, Bob Oliver, Subgroup families controlling p-local finite groups, Proc. Lond. Math. Soc. (3) 91 (2) (2005) 325–354, MR MR2167090 (2007e:20111).
[4] Carles Broto, Ran Levi, Bob Oliver, Homotopy equivalences of p-completed classifying spaces of finite groups, Invent. Math. 151 (3) (2003) 611–664, MR MR1961340 (2004c:55031).
[5] Carles Broto, Ran Levi, Bob Oliver, The homotopy theory of fusion systems, J. Am. Math. Soc. 16 (4) (2003) 779–856 (electronic), MR 1 992 826.
[6] Kenneth S. Brown, Euler characteristics of groups: the p-fractional part, Invent. Math. 29 (1) (1975) 1–5, MR 0385008 (52 #5878).
[7] Antonio Díaz, Assaf Libman, The Burnside ring of fusion systems, Adv. Math. 222 (6) (2009) 1943–1963, MR 2562769 (2011a:20039).
[8] W.G. Dwyer, Homology decompositions for classifying spaces of finite groups, Topology 36 (4) (1997) 783–804, MR 1432421 (97m:55016).
[9] W.G. Dwyer, C.W. Wilkerson, A cohomology decomposition theorem, Topology 31 (2) (1992) 433–443, MR MR1167181 (93h:55008).
[10] P. Gabriel, M. Zisman, Calculus of Fractions and Homotopy Theory, Ergeb. Math. Ihrer Grenzgeb., vol. 35, Springer-Verlag New York, Inc., New York, 1967, MR 0210125 (35 #1019).
[11] Daniel Gorenstein, Finite Groups, Harper & Row Publishers, New York, 1968, MR MR0231903 (38 #229).
[12] Stefan Jackowski, Jolanta Słomińska, G-functors, G-posets and homotopy decompositions of G-spaces, Fundam. Math. 169 (3) (2001) 249–287, MR 1852128 (2002h:55017).
[13] Martin Wedel Jacobsen, Jesper M. Møller, Euler characteristics and Möbius algebras of p-subgroup categories, J. Pure Appl. Algebra 216 (12) (2012) 2665–2696, MR 2943749.
[14] Tom Leinster, The Euler characteristic of a category, Doc. Math. 13 (2008) 21–49, MR MR2393085.
[15] Saunders MacLane, Homology, first ed., Grundlehren Math. Wiss., Band 114, Springer-Verlag, Berlin, 1967, MR MR0349792 (50 #2285).
[16] Lluís Puig, Frobenius Categories versus Brauer Blocks: The Grothendieck Group of the Frobenius Category of a Brauer Block, Prog. Math., vol. 274, Birkhäuser Verlag, Basel, 2009, MR MR2502803.
[17] Daniel Quillen, Higher algebraic K-theory. I, in: Algebraic K-theory, I: Higher K-theories, Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972, in: Lect. Notes Math., vol. 341, Springer, Berlin, 1973, pp. 85–147, MR 0338129 (49 #2895).
[18] Daniel Quillen, Homotopy properties of the poset of nontrivial p-subgroups of a group, Adv. Math. 28 (2) (1978) 101–128, MR MR493916 (80k:20049).
[19] Derek J.S. Robinson, A Course in the Theory of Groups, second ed., Springer-Verlag, New York, 1996, MR 96f:20001.
[20] Richard P. Stanley, Enumerative Combinatorics, vol. 1, Camb. Stud. Adv. Math., vol. 49, Cambridge University Press, Cambridge, 1997, With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original, MR MR1442260 (98a:05001).
[21] R.W. Thomason, Homotopy colimits in the category of small categories, Math. Proc. Camb. Philos. Soc. 85 (1) (1979) 91–109, MR 510404 (80b:18015).