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How can you tell two spaces apart when they have the same n-type for all n ?

By C.A.McGibbon and J.M.Møller*

In memory of J.F.Adams

In 1957, Adams gave the first example of two different homotopy types, say X and Y, whose Postnikov approximations, $X^{(n)}$ and $Y^{(n)}$, are homotopy equivalent for each n. He did this in response to a question posed by J.H.C.Whitehead. Adams gave an explicit description of both spaces and showed they are different, up to homotopy, by noting that one contains a sphere as a retract whereas the other does not, [1]. Recently, in our study of infinite dimensional spaces, we have had to confront the same problem. Often we can prove that for a given space, e.g., $X = S^3 \times K(\mathbf{Z},3)$, there are many other spaces, up to homotopy, with the same n-type as X for all n. But when asked to describe one of them, we had to plead ignorance. To correct this situation we began to look for explicit descriptions and for homotopy invariants that are not determined by finite approximations. In doing so, we found some new examples and a new answer, involving automorphism groups, to the question posed in the title. This paper deals with those examples. We find them intriguing, in part, because they indicate an unexpected lack of homogeneity among spaces of the same n -type for all n. Nevertheless, as examples go, Adams's original one remains one of our favorites because of its simplicity and explicit nature. We commend it to the reader.

To describe our results, we first recall Wilkerson's classification theorem. In [13] , he classified, in the following sense, all spaces having the same n-type,

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for all n, as a given space X.

Theorem 1. Given a connected CW space X, let $SNT(X)$ denote the set of all homotopy types $[Y]$ such that $Y^{(n)} \simeq X^{(n)}$ for all n. Then there is a bijection of pointed sets,

$$SNT(X) \approx \varprojlim{}^1 AutX^{(n)}$$

where $AutX^{(n)}$ is the group of homotopy classes of homotopy self-equivalences of $X^{(n)}$. □

Recently we used this theorem to show that if X is a nilpotent space with finite type over some subring of the rationals, then the set $SNT(X)$ is either uncountably large or it has just one member; namely $[X]$. For spaces that are rationally equivalent to H-spaces, i.e., H_0-spaces, one can determine which alternative holds by using the following result from [5].

Theorem 2. Let X be a 1-connected, H_0-space with finite type over $\mathbf{Z}_P$ for some set of primes P. Then the following statements are equivalent:
(i) $SNT(X) = *$.
(ii) the canonical map $AutX \longrightarrow AutX^{(n)}$, has a finite cokernel for all n.
(iii)the map $AutX \xrightarrow{f \mapsto f^*} AutH^{\leq n}(X; \mathbf{Z}_P)$ has a finite cokernel for all integers n. □

Using this result we were able to determine, for example, the cardinality of $SNT(X)$ when X is the classifying space of a connected compact Lie group.

Theorem 3. Let G be a connected compact Lie group. Then $SNT(BG) = *$ if and only if $G = T^k$, $SU(n)$ or $PSU(n)$ when $k \geq 0$ and $n = 2$ or 3. □

In other words, except for a few special cases, $SNT(BG)$ is almost always uncountably large. In contrast, we have yet to find a finite complex K for which $SNT(\Omega K)$ is nontrivial! Some partial results on this loop space problem are given in [5]. For example, it is shown there if K is a compact Lie group or a complex Stiefel manifold, then $SNT(\Omega K)$ is the one element set. The Eckmann-Hilton dual problem is considered in [6].

The results just quoted deal only with the cardinality of $SNT(X)$ and do not actually address the question posed in the title. As mentioned earlier, we now seek ways to distinguish different members of $SNT(X)$. What tools should be used in this task ? It might be worth pointing out that mod p cohomology is of little use here. Members of $SNT(X)$ cannot be distinguished, a priori, by their cohomology - even as algebras over the Steenrod algebra - when the spaces involved are nilpotent and of finite type. The reason involves the profinite completion functor, $X \mapsto \widehat{X}$. On the one hand, when X has finite type this map induces an equivalence in mod p cohomology while on the other, for such X, $SNT(\widehat{X})$ has just one element, as Wilkerson showed in [13], Corollary IIc.

The situation for K-theory appears to be different. Indeed, Notbohm claimed at the 1990 Barcelona conference that if X and Y are in $SNT(BG)$ where G is simply connected compact Lie, then X and Y are homotopy equivalent if and only if $K(X)$ and $K(Y)$ are isomorphic as λ-rings. We have not seen the details yet, but the results in [10] make this sound plausible. If this is true, then there may be a computable K-theory invariant (perhaps a power series of some sort) that classifies $SNT(BG)$. On the other hand, obtaining such an invariant may be difficult. As a λ-ring, $K(X)$, for X in $SNT(BG)$, currently seems complicated and intractible. We are certain that Adams and Wilkerson would have loved to have had a K-theory version of their embedding theorem for spaces of this kind. One can safely assume that their failure to achieve such a result was not for lack of trying. For an indication of how complicated this question gets, even in the rank 1 case (how *not* to do it, if you will), see [4].

Having dispensed with something that doesn't work (cohomology), and mentioned something that might work in special cases (K-theory), let us turn to another invariant; the discrete group, $Aut(X)$. Of course, one would not expect $Aut(\ )$ to separate points of $SNT(X)$, in general. Indeed, we find it curious that $Aut(Y)$ is not constant, as Y varies over $SNT(X)$. This is what our examples will illustrate. First recall from [2] that the Postnikov decomposition of X determines the following short exact sequence of groups,

$$0 \longrightarrow \varprojlim{}^1 \pi_1 aut X^{(n)} \longrightarrow Aut(X) \longrightarrow \varprojlim Aut X^{(n)} \longrightarrow 1$$

In our first example, the $\varprojlim^1$ term on the left happens to be zero. In this case, $Aut(Y)$ and $Aut(X)$, where Y is in $SNT(X)$, are isomorphic to the inverse limits of "two towers of groups that have the same n-type for all n". We trust that the reader sees the obvious analogy and can supply the proper definition for the term in quotes.

The following example shows that even though two spaces X and Y, have the same n-type for all n, it is possible that one group, $Aut(X)$, is infinite, while the other, $Aut(Y)$, is finite!

Example A Let $X = K(\mathbf{Z},3) \times S^3$. Then $Aut(X)$ is isomorphic to the group of upper triangular matrices in $GL(2,\mathbf{Z})$, but there exists a Y in $SNT(X)$ with $Aut(Y) \approx \mathbf{Z}/2$. □

To appreciate this example, it might be helpful to recall a result of Wilkerson; [14], Theorem 2.3. He shows $Aut(Y)$ and $Aut(X)$ are commensurable provided there exists a rational equivalence $f : X \to Y$, and provided both spaces are 1-connected, with finite type, and have only a finite number of nonzero homotopy (or homology) groups. Now, the two spaces in example A are rationally equivalent, although we suspect there are no maps between them, before rationalizing, that induce this equivalence. Therefore, since the automorphism groups in this example are clearly not commensurable, this example shows that Wilkerson's result is, in a sense, best possible.

The proof for example A involves matrix calculations with a lemma that expresses $Aut(Y)$ as a pullback of $Aut(X_o)$ and $Aut(\widehat{X})$. We also use this lemma, together with Mislin's homotopy classification of self-maps of infinite quaternionic projective space to show that $Aut(Y) = \{1\}$ for every Y in the genus of $\mathbf{HP}^\infty$.

Our second example involves the $\varprojlim^1$ term in the short exact sequence above. Following Roitberg, [11], we denote this term by $WI(X)$. Of course, members of this subgroup can be viewed as self maps of X that restrict to the identity map on each finite skeleton. If each of the groups $\pi_1 aut X^{(n)}$ is a finitely generated abelian group, (as would be the case when X is nilpotent with finite type over $\mathbf{Z}$), then the possible values of $WI(X)$ are very limited. In fact, it follows from Jensen, [3], Chapter 2, that this subgroup is either zero or else it is a divisible abelian group of the form

$$WI(X) \approx \mathbf{R} \oplus \sum_{all\ p} (\mathbf{Z}/p^\infty)^{n_p}$$

Here $\mathbf{R}$ is regarded as a rational vector space of rank $2^{\aleph_0}$ and each torsion exponent n_p is either finite or $2^{\aleph_0}$.

Assume for the moment that X and Y are 1-connected, with finite type, and that they have the same n-type for all n. It follows at once that the towers $\{\pi_1 aut X^{(n)}\}$ and $\{\pi_1 aut Y^{(n)}\}$, likewise have the same n-type for all n. It is then easy to check that one tower is Mittag-Leffler if and only if the other is. These will then be towers of countable groups and so by Theorem 2 of [5], it follows that $WI(X)$ is zero if and only if the same is true for $WI(Y)$. Consequently, it follows from Jensen's description, above, that if $WI(X)$ and $WI(Y)$ are not isomorphic, then the only way they can differ is in their torsion summands. We were surprised to find that this can actually happen.

Example B Let $X = \mathbf{CP}^\infty \times \Omega S^3 \times S^3$. Then $WI(X) = \mathbf{R}$ and there exists a Y in $SNT(X)$ with $WI(Y) = \mathbf{R} \oplus \mathbf{Q}/\mathbf{Z}$. □

The verification of these examples takes up the rest of the paper. In the first example, the space Y will be constructed as a homotopy pullback of the rationalization, X_o, and the profinite completion, $\widehat{X}$, over the formal completion, $\bar{X}_o$.

$$\begin{array}{ccc} & & \widehat{X} \\ & & \downarrow \\ X_o & \longrightarrow & \bar{X}_o, \end{array}$$

The vertical map in this diagram is fixed. It first rationalizes and then identifies $(\widehat{X})_o$ with $\bar{X}_o$. This identification is valid for 1-connected spaces with finite type, which are the sort of spaces that will be considered here. The horizontal map in the diagram will be altered by composition with certain automorphisms of $\bar{X}_o$. Both maps in the diagram are required to induce isomorphisms on $\pi_*(\ )\otimes\mathbf{Q}\otimes\widehat{\mathbf{Z}}$. It is then immediate that the rational homotopy type and the profinite completion of Y are equivalent to those of X. In other words, $[Y] \in \widehat{G}_o(X)$. It is a theorem of Wilkerson that every member of $\widehat{G}_o(X)$ can be obtained this way and that the double coset formula

$$\widehat{G}_o(X) \approx Aut(X_o)\backslash CAut(\bar{X}_o)/Aut(\widehat{X})$$

is valid. The group $CAut(\bar{X}_o)$ in this formula consists of those self maps f for which $\pi_*(f)$ is a $\mathbf{Q} \otimes \widehat{\mathbf{Z}}$ module isomorphism. For more details here, see [14].

Lemma 4 Let X be a 1-connected CW-space with finite type such that

$$H_n(X; \pi_{n+1}(\widehat{X}_o)) = 0, \quad \text{for each } n \geq 0.$$

Let φ be in $CAut(\bar{X}_o)$. If Y is the homotopy pullback of

$$X_o \longrightarrow \bar{X}_o \xrightarrow{\varphi} \bar{X}_o \longleftarrow \widehat{X}$$

then $Aut(Y)$ is isomorphic to the group theoretic pullback of

$$Aut(X_o) \longrightarrow CAut(\bar{X}_o) \xrightarrow{\varphi^*} CAut(\bar{X}_o) \longleftarrow Aut(\widehat{X})$$

where in φ^* denotes conjugation by φ. □

The proof of this lemma will be given later. It depends, in part, on W. Meier's results on pullbacks and phantom maps, [8]. Notice that, in this lemma, if $CAut(\bar{X}_o)$ is abelian, then φ^* is the identity. So, under these special conditions, it follows that $Aut(Y)$ is isomorphic to $Aut(X)$ for every $Y \in \widehat{G}_o(X)$. This is the case, for example, when $X = \mathbf{HP}^\infty$. Since Mislin has shown in [9], that $Aut(\mathbf{HP}^\infty) = \{1\}$, we deduce that the same is true for every Y in the genus of this space.

Proof of Example A. Recall that $X = S^3 \times K(\mathbf{Z}, 3)$. Therefore $\bar{X}_o$ is a $K(V, 3)$ where V is a free module of rank 2 over $\mathbf{Q} \otimes \widehat{Z}$. We will identify $CAut(\bar{X}_o)$ with the adele group $GL(2, \mathbf{Q} \otimes \widehat{\mathbf{Z}})$ and we will represent the self-equivalence φ as follows. First, for each prime p, let B_p be the matrix

$$B_p = \begin{pmatrix} 1 & 0 \\ c_p & 1 \end{pmatrix}$$

where $c_p = 1$ when p is 2, -1 when p is 3 , and 0 when p is 5 or more. Then take φ to be the the map represented by the sequence of matrices $(B_p) = (B_2, B_3, B_5, \ldots)$ in $\Pi_p GL(2, \widehat{\mathbf{Z}}_p) \subseteq GL(2, \mathbf{Q} \otimes \widehat{\mathbf{Z}})$. Use this φ to construct Y as a pullback of X_o and $\widehat{X}$ over $\bar{X}_o$, as in the lemma. As a pullback,

$$Aut(Y) = \{(\rho, \mu) | \rho \in Aut(X_o), \mu \in Aut(\widehat{X}), \text{and} [\rho] = \varphi^*[\mu]\}.$$

Let R and (M_p) be the matrices that represent $[\rho]$ and $[\mu]$ in $GL(2, \mathbf{Q}\otimes\hat{\mathbf{Z}})$. The matrices M_p can be taken to be upper triangular. This follows from a routine analysis of $[X, X]$ together with some special facts; namely the cohomotopy groups $[K, S]$ and $[\Sigma^3 K, S]$ are trivial, where K denotes a $K(\mathbf{Z}, 3)$ and S denotes the p-completion of a 3-sphere. These facts are due to Zabrodsky, [16].

From the description just given of $Aut(Y)$, it follows that for each prime p there is a matrix equation

$$B_p R = M_p B_p$$

We claim that these equations force R to be $\pm I$. Since B_5 is the identity, it is clear that R is at least upper triangular. Therefore,

$$\text{if } R = \begin{pmatrix} r & s \\ 0 & t \end{pmatrix} \text{ and if } M_2 = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix},$$

then the $p = 2$ equation,

$$\begin{pmatrix} r & s \\ r & s+t \end{pmatrix} = B_2 R = M_2 B_2 = \begin{pmatrix} x+y & y \\ z & z \end{pmatrix}$$

implies that $r = s + t$. Similarly, the $p = 3$ equation,

$$\begin{pmatrix} r & s \\ -r & -s+t \end{pmatrix} = B_3 R = M_3 B_3 = \begin{pmatrix} u-v & v \\ -w & w \end{pmatrix}$$

implies that $r = -s + t$. It follows that $s = 0$ and $r = t = \pm 1$. Hence $R = \pm I = M_p$, for each prime p. Since the inclusions of $Aut(X_o)$ and of each $Aut(\widehat{X}_p)$ in $CAut(\bar{X}_o)$ are injective, it follows that the pullback group $Aut(Y) \approx \mathbf{Z}/2$, as claimed.

It remains to be shown that Y is in $SNT(X)$. To this end, we first establish that $Y \in G(X)$, the localization genus of X. This follows because of the rational entries in each of the matrices B_p. Indeed, it is easy to see that $Y_{(p)}$ is homotopy equivalent to the pullback

$$\begin{array}{ccccc} & & & & \widehat{X}_p \\ & & & & \downarrow \\ X_o & \to & (\widehat{X}_p)_o & \xrightarrow{\varphi_p} & (\widehat{X}_p)_o \end{array}$$

where φ_p is represented by B_p. Such pullbacks are classified by a double coset formula, as noted earlier. The rational nature of B_p places φ_p in the image of $Aut(X_o)$, which, in turn, is in the double coset of the identity. Thus the induced pullback $Y_{(p)}$, is equivalent to $X_{(p)}$, and so Y and X are in the same genus. Consequently, $Y^{(n)} \in G(X^{(n)})$ for each positive integer n. However, each $G(X^{(n)})$ has only one member; namely $X^{(n)}$. This is an easy consequence of Zabrodsky's genus theorem, [15], and elementary facts about self-maps of X. Therefore $Y^{(n)} \simeq X^{(n)}$ for each n, as was to be shown.

Proof of the lemma The rational homology condition on X, together with the finite type hypothesis, ensures that the group $WI(X) = 0$. Notice that if X satisfies these conditions, then so does every member of $\widehat{G}_o(X)$. Moreover, this ensures that if φ is the identity, then $Aut(X)$ is isomorphic to this pullback, by [8], Theorem 4. Now let Y be in $\widehat{G}_o(X)$ and choose equivalences

$$Y_o \xrightarrow{f} X_o \quad \text{and} \quad \widehat{Y} \xrightarrow{g} \widehat{X}.$$

Then $Y = X_\varphi$, the pullback determined by the equivalence $\varphi = g_o(\bar{f})^{-1}$, by [14], Theorem 3.8. We have isomorphisms

$$Aut(X_o) \xrightarrow{\rho \mapsto f^{-1}\rho f} Aut(Y_o) \quad \text{and} \quad Aut(\widehat{X}) \xrightarrow{\mu \mapsto g^{-1}\mu g} Aut(\widehat{Y})$$

Hence $Aut(Y) \subseteq Aut(Y_o) \times Aut(\widehat{Y}) \approx Aut(X_o) \times Aut(\widehat{X})$, and consists of pairs (ρ, μ) for which $\overline{(f^{-1}\rho f)} = (g^{-1}\mu g)_o$. Equivalently, this is all pairs (ρ, μ) such that $\varphi[\rho]\varphi^{-1} = [\mu]$. □

Example B. Let $X = L \times P \times S$ where L denotes the loop space ΩS^3, P denotes $\mathbf{CP}^\infty$, and S denotes the 3-sphere. Since X is an H-space, there are group isomorphisms,

$$\begin{aligned} WI(X) &\approx \Theta(X,X) && \text{by Roitberg [11], Theorem 3.1} \\ &\approx \varprojlim{}^1[\Sigma X, X^{(n)}] \end{aligned}$$

Here $\Theta(X,X)$ denotes the group of phantom self-maps of X. Notice that

$$\Sigma X \simeq \Sigma(L \vee P \vee S) \vee \Sigma W$$

where ΣW is 4-connected. Of course, for each n, we also have

$$X^{(n)} \simeq L^{(n)} \times P \times S^{(n)}$$

With a bouquet of 4 spaces as a domain, and a product of three spaces for a target, one gets a decomposition of the tower $\{[\Sigma X, X^{(n)}]\}$ as a direct sum of 12 towers. Ten of these are towers of finite groups (this is an easy rational calculation) and consequently their $\varprojlim^1$ terms are zero. The remaining two towers are $\{[\Sigma P, S^{(n)}]\}$ and $\{[\Sigma L, S^{(n)}]\}$. Since ΣL has the homotopy type of a bouquet of spheres, the maps in the last tower are surjections. This forces its $\varprojlim^1$ term to be zero. The other tower, $\{[\Sigma P, S^{(n)}]\}$, is not Mittag-Leffler. Its $\lim^1$ term is shown in [8] to be a rational vector space of rank $2^{\aleph_0}$. Thus we have established

$$WI(X) \approx \Theta(\mathbf{CP}^\infty, S^3) \approx \mathbf{R}$$

We will now describe Y. First, let $\{A, B\}$ be a partition of the set of all prime numbers into two proper subsets, and let Z denote a space that is homotopy equivalent to L at all primes in A and homotopy equivalent to P at all primes in B. In other words, Z is a Zabrodsky mixture of L and P. It can be constructed as a pullback

$$\begin{array}{ccc} & & \mathbf{CP}^\infty_B \\ & & \downarrow \\ \Omega S^3_A & \longrightarrow & K(\mathbf{Q},2) \end{array}$$

wherein the maps are rational equivalences. Let Z' be a second mixture of L and P obtained by reversing the roles of A and B. Then define Y to be $Z \times Z' \times S$. The same argument that was used to calculate $WI(X)$, shows that

$$WI(Y) \approx \Theta(Y,Y) \approx \Theta(Z,S) \oplus \Theta(Z',S)$$

By a $\varprojlim^1$ calculation similar to the one Meier makes in [8], page 480, it follows that

$$\Theta(Z,S) \approx \mathbf{R} \oplus \sum_{p \in A} \mathbf{Z}/p^\infty$$

And for Z' in place of Z, the result is almost the same - just replace A with B in the description of the torsion summand. Therefore, since $\mathbf{R} \approx \mathbf{R} \oplus \mathbf{R}$, as rational vector spaces, and the torsion summands add up to $\mathbf{Q}/\mathbf{Z}$, we get

$$WI(Y) \approx \mathbf{R} \oplus \mathbf{Q}/\mathbf{Z},$$

as claimed.

The verification that Y is in $SNT(X)$, is identical to the proof given for Example A. We conclude, as before, that Y is in $G(X) \cap SNT(X)$. Incidently, we refer to such a space as a *clone* of X. In other words, two spaces (or two maps) are said to be clones of each other if they are locally equivalent at each prime and their Postnikov approximations are equivalent. The construction that was used above, $Z \times Z'$, provides, by taking different partitions, uncountably many different clones of $\Omega S^3 \times \mathbf{CP}^\infty$. The discovery of nontrivial homotopy clones came as a rude surprise to us, especially after we had expended so much effort trying to rule them out! They are studied in detail in [7].