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N-determined 2-compact groups. II

by

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Abstract. This is the second part of a paper about the classification of 2-compact groups. In the first part we set up a general classification procedure and applied it to the simple 2-compact groups of the A-family. In this second part we deal with the other simple Lie groups and with the exotic simple 2-compact group DI(4). We show that all simple 2-compact groups are uniquely N-determined and conclude that all connected 2-compact groups are uniquely N-determined. This means that two connected 2-compact groups are isomorphic if their maximal torus normalizers are isomorphic and that the automorphisms of a connected 2-compact group are determined by their effect on a maximal torus. As an application we confirm the conjecture that any connected 2-compact group is the product of a compact Lie group with copies of the exceptional 2-compact group DI(4).

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1. INTRODUCTION

This is the second part of a paper whose aim is to show that connected 2-compact groups are determined by their maximal torus normalizers and that some nonconnected 2-compact groups are determined by their maximal torus normalizers together with information about the group of components. The first part [24] contained a general classification scheme which

(1) reduces the classification problem to the case of a connected, simple 2-compact group with no center, and
(2) deals inductively with connected, simple 2-compact groups with no center.

In the first part we applied this general procedure to the connected 2-compact groups $\mathrm{PGL}(n+1,\mathbf{C})$, $n \geq 1$, of the A-family and showed that they are uniquely N-determined [24, Theorem 1.4]. In this second part we shall apply the same procedure to the D-, B-, and C-families of Lie groups, to the exceptional Lie groups G_2, F_4, E_6, PE_7, and E_8, and to the exotic 2-compact group DI(4) [9]. We show that there do not exist shadow versions of these well-known 2-compact groups. The main results are the following.

1.1. Theorem (The D-, B-, and C-families). *The connected, simple 2-compact groups* $\mathrm{PSL}(2n,\mathbf{R})$, $n \geq 4$, $\mathrm{SL}(2n+1,\mathbf{R})$, $n \geq 2$, *and* $\mathrm{PGL}(n,\mathbf{H})$, $n \geq 3$, *are uniquely N-determined. Their automorphism groups are*

$$\mathrm{Aut}(\mathrm{PSL}(2n,\mathbf{R})) = \begin{cases} \mathbf{Z}^\times \backslash \mathbf{Z}_2^\times \times \Sigma_3, & n = 4, \\ \mathbf{Z}^\times \backslash \mathbf{Z}_2^\times \times \langle c_1 \rangle, & n > 4 \textit{ even}, \\ \mathbf{Z}_2^\times, & n > 4 \textit{ odd}, \end{cases}$$

$$\mathrm{Aut}(\mathrm{SL}(2n+1,\mathbf{R})) = \mathbf{Z}^\times \backslash \mathbf{Z}_2^\times, \quad n \geq 2,$$

$$\mathrm{Aut}(\mathrm{PGL}(n,\mathbf{H})) = \mathbf{Z}^\times \backslash \mathbf{Z}_2^\times, \quad n \geq 3,$$

where $\langle c_1 \rangle$ is a group of order two (generated by conjugation with a matrix $c_1 \in \mathrm{GL}(2n,\mathbf{R})$ of determinant -1).

1.2. THEOREM ([36, 1.3]). *The connected, simple 2-compact group* G_2 *is uniquely N-determined. Its automorphism group is* $\mathrm{Aut}(\mathrm{G}_2) = \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times \times C_2$.

1.3. THEOREM ([35]). *The connected, simple 2-compact group* F_4 *is uniquely N-determined. Its automorphism group is* $\mathrm{Aut}(\mathrm{F}_4) = \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$.

1.4. THEOREM. *The connected, simple 2-compact groups* E_6, PE_7, *and* E_8 *are uniquely N-determined. Their automorphism groups are* $\mathrm{Aut}(\mathrm{E}_6) = \mathbf{Z}_2^\times$, $\mathrm{Aut}(\mathrm{PE}_7) = \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$, *and* $\mathrm{Aut}(\mathrm{E}_8) = \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$.

1.5. THEOREM. *The connected, simple 2-compact group* $\mathrm{DI}(4)$ *is uniquely N-determined. Its automorphism group is* $\mathrm{Aut}(\mathrm{DI}(4)) = \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$.

The method also applies to some nonconnected 2-compact groups and as an example we consider the general linear groups over the field of real numbers.

1.6. COROLLARY. *The 2-compact group* $\mathrm{GL}(n,\mathbf{R})$ *is totally N-determined for all* $n \geq 2$. *Its automorphism group is*

$$\mathrm{Aut}(\mathrm{GL}(n,\mathbf{R})) = \begin{cases} \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times, & n \geq 3 \text{ odd}, \\ \mathbf{Z}_2^\times, & n = 2, \\ \mathbf{Z}_2^\times \times \langle\delta\rangle, & n \equiv 2 \bmod 4,\ n > 2, \\ \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times \times \langle c_1\rangle \times \langle\delta\rangle, & n \equiv 0 \bmod 4, \end{cases}$$

where $\langle\delta\rangle$ *and* $\langle c_1\rangle$ *are subgroups of order two.*

The above results together with the corresponding result for the A-family [24, Theorem 1.4] say that all simple 2-compact groups are uniquely N-determined. Given this information, the general classification procedure shows that in fact all connected 2-compact groups are uniquely N-determined and that some nonconnected 2-compact groups are totally N-determined [24, Theorem 1.1].

2. THE D-FAMILY

Let $\mathrm{GL}(2n,\mathbf{R})$, $n \geq 1$, be the matrix group of $2n$ by $2n$ real matrices and $\mathrm{SL}(2n,\mathbf{R})$ the closed subgroup of matrices with determinant 1. The D-family is the infinite family of matrix groups

$$\mathrm{PSL}(2n,\mathbf{R}) = \frac{\mathrm{SL}(2n,\mathbf{R})}{\langle -E\rangle}, \qquad n \geq 4,$$

with trivial center. Of course, these groups also exist for $n = 1,2,3$; however, $\mathrm{PSL}(2,\mathbf{R}) = \{1\}$ is the trivial group, and $\mathrm{PSL}(4,\mathbf{R}) = \mathrm{PGL}(2,\mathbf{C})^2$, $\mathrm{PSL}(6,\mathbf{R}) = \mathrm{PGL}(4,\mathbf{C})$ will at this stage be known to be uniquely N-determined [24, Theorem 1.4].

The maximal torus and the maximal torus normalizer of the Lie groups $\mathrm{GL}(2n,\mathbf{R})$, $\mathrm{SL}(2n,\mathbf{R})$, and $\mathrm{PSL}(2n,\mathbf{R})$ are

$$\begin{aligned} T(\mathrm{GL}(2n,\mathbf{R})) &= \mathrm{SL}(2,\mathbf{R})^n, & N(\mathrm{GL}(2n,\mathbf{R})) &= \mathrm{GL}(2,\mathbf{R}) \wr \Sigma_n, \\ T(\mathrm{SL}(2n,\mathbf{R})) &= \mathrm{SL}(2,\mathbf{R})^n, & N(\mathrm{SL}(2n,\mathbf{R})) &= \\ &&& \mathrm{SL}(2n,\mathbf{R}) \cap N(\mathrm{GL}(2n,\mathbf{R})), \\ T(\mathrm{PSL}(2n,\mathbf{R})) &= \frac{\mathrm{SL}(2,\mathbf{R})^n}{\langle -E\rangle}, & N(\mathrm{PSL}(2n,\mathbf{R})) &= \frac{N(\mathrm{SL}(2n,\mathbf{R}))}{\langle -E\rangle}. \end{aligned} \tag{2.1}$$

In all three cases, the maximal torus normalizer is the semidirect product for the action of the Weyl group

$$\begin{aligned} &W(\mathrm{GL}(2n,\mathbf{R})) = \Sigma_2 \wr \Sigma_n, \qquad \Sigma_2 = W(\mathrm{GL}(2,\mathbf{R})) = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle, \\ &W(\mathrm{SL}(2n,\mathbf{R})) = A_{2n} \cap (\Sigma_2 \wr \Sigma_n) = W(\mathrm{PSL}(2n,\mathbf{R})) \end{aligned} \tag{2.2}$$

on the maximal torus. It is known that

$$H^0(W;\check{T}) = 0, \qquad H^1(W;\check{T}) = \begin{cases} \mathbf{Z}/2, & n=3, \\ \mathbf{Z}/2 \times \mathbf{Z}/2, & n=4, \\ 0, & n>4, \end{cases} \tag{2.3}$$

for $\mathrm{PSL}(2n,\mathbf{R})$, $n \geq 3$ [6, 16, 21, 22]. (The group of outer Lie automorphisms of the Lie group $\mathrm{PSL}(8,\mathbf{R})$, isomorphic to Σ_3, is faithfully represented in $H^1(W;\check{T})(\mathrm{PSL}(8,\mathbf{R}))$.)

The Lie groups

$$\mathrm{GL}(2n,\mathbf{R}) = \mathrm{SL}(2n,\mathbf{R}) \rtimes \langle D\rangle, \qquad \mathrm{PGL}(2n,\mathbf{R}) = \mathrm{PSL}(2n,\mathbf{R}) \rtimes \langle D\langle -E\rangle\rangle$$

are the semidirect products of their identity components with the order two subgroup generated by the matrix $D = \mathrm{diag}(-1,1,\ldots,1)$ (or any other order two matrix with negative determinant). Conjugation with D induces an outer automorphism of the Lie groups $\mathrm{SL}(2n,\mathbf{R})$ and $\mathrm{PSL}(2n,\mathbf{R})$.

1. The structure of $\mathrm{PSL}(2n,\mathbf{R})$. In this section we investigate the Quillen category $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))$ [24, Definition 2.45] for the 2-compact group $\mathrm{PSL}(2n,\mathbf{R})$ (and the related 2-compact groups $\mathrm{SL}(2n,\mathbf{R})$, $\mathrm{GL}(2n,\mathbf{R})$, and $\mathrm{PGL}(2n,\mathbf{R})$).

Consider the elementary abelian 2-groups

$$\begin{aligned} &t(\mathrm{SL}(2n,\mathbf{R})) = t(\mathrm{GL}(2n,\mathbf{R})) = \langle e_1,\ldots,e_n\rangle \\ &\qquad\qquad \subset \mathrm{SL}(2n,\mathbf{R}) \subset \mathrm{GL}(2n,\mathbf{R}), \\ &\Delta_{2n} = \langle e_1,\ldots,e_n,c_1,\ldots,c_n\rangle = \langle \mathrm{diag}(\pm1,\ldots,\pm1)\rangle \cong (\mathbf{Z}/2)^{2n} \\ &\qquad\qquad \subset \mathrm{GL}(2n,\mathbf{R}), \end{aligned} \tag{2.4}$$

$$
(2.4_{\text{cont.}})\quad
\begin{aligned}
P\Delta_{2n} &= \Delta_{2n}/\langle e_1\cdots e_n\rangle \cong (\mathbf{Z}/2)^{2n-1} \subset \mathrm{PGL}(2n,\mathbf{R}),\\
S\Delta_{2n} &= \langle e_1,\dots,e_n,c_1c_2,\dots,c_1c_n\rangle = \mathrm{SL}(2n,\mathbf{R})\cap\Delta_{2n}\\
&\qquad\cong (\mathbf{Z}/2)^{2n-1}\subset \mathrm{SL}(2n,\mathbf{R}),\\
PS\Delta_{2n} &= S\Delta_{2n}/\langle e_1\cdots e_n\rangle \cong (\mathbf{Z}/2)^{2n-2}\subset \mathrm{PSL}(2n,\mathbf{R}),\\
t(\mathrm{PSL}(2n,\mathbf{R})) &= t(\mathrm{PGL}(2n,\mathbf{R})) = \langle I,e_1,\dots,e_n\rangle/\langle e_1\dots e_n\rangle\\
&\qquad\subset \mathrm{PSL}(2n,\mathbf{R})\subset\mathrm{PGL}(2n,\mathbf{R}),\\
Pt(\mathrm{SL}(2n,\mathbf{R})) &= Pt(\mathrm{GL}(2n,\mathbf{R})) = \langle e_1,\dots,e_n\rangle/\langle e_1\cdots e_n\rangle\\
&\qquad\subset \mathrm{SL}(2n,\mathbf{R})\subset \mathrm{GL}(2n,\mathbf{R}),
\end{aligned}
$$

where

$$
(2.5)\quad
\begin{aligned}
e_j &= \mathrm{diag}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},\dots,\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\dots,\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)\in \mathrm{SL}(2n,\mathbf{R}),\\
I &= \mathrm{diag}\left(\begin{pmatrix}0&-1\\1&0\end{pmatrix},\dots,\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right)\in\mathrm{SL}(2n,\mathbf{R}),\\
c_j &= \mathrm{diag}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},\dots,\begin{pmatrix}-1&0\\0&1\end{pmatrix},\dots,\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)\in\mathrm{GL}(2n,\mathbf{R}).
\end{aligned}
$$

The matrices e_j and c_j, $1\le j\le n$, have order two and commute with each other while $Ie_j = e_jI$, $Ic_j = e_jc_jI$, and $I^2 = e_1\cdots e_n = -E$.

The representation of the Weyl groups

$$(2.6)\qquad W(\mathrm{GL}(2n,\mathbf{R})) = \langle c_1,\dots,c_n\rangle \rtimes \Sigma_n = \Sigma_2\wr\Sigma_n,$$

$$(2.7)\qquad W(\mathrm{SL}(2n,\mathbf{R})) = \langle c_1c_2,\dots,c_1c_n\rangle\rtimes\Sigma_n = A_{2n}\cap(\Sigma_2\wr\Sigma_n)$$

on the maximal toral elementary abelian 2-group $t(\mathrm{GL}(2n,\mathbf{R}))$ is trivial on the subgroup $\langle c_1,\dots,c_n\rangle = \Sigma_2^n$ while $\Sigma_n\subset\mathrm{GL}(n,\mathbf{C})\subset\mathrm{SL}(2n,\mathbf{R})$ permutes the n basis vectors $e_1,\dots,e_n$ of $t(\mathrm{SL}(2n,\mathbf{R})) = t(\mathrm{GL}(2n,\mathbf{R}))$.

Let V be a nontrivial elementary abelian 2-group in $\mathrm{PGL}(2n,\mathbf{R})$ and V^* its inverse image in $\mathrm{GL}(2n,\mathbf{R})$. Let $q\colon V\to\mathbf{F}_2=\{0,1\}$ be the function and $[\ ,\ ]\colon V\times V\to\mathbf{F}_2=\{0,1\}$ the bilinear map given by $v^{*2} = (-E)^{q(v)}$ and $[v_1^*,v_2^*] = (-E)^{[v_1,v_2]}$, where $v^*,v_1^*,v_2^*\in\mathrm{SL}(2n,\mathbf{R})$ are preimages of $v,v_1,v_2\in\mathrm{PSL}(2n,\mathbf{R})$, respectively. The equations

$$[v_1,v_2]=[v_2,v_1],\qquad [v,v]=0,\qquad q(v_1+v_2)=q(v_1)+q(v_2)+[v_1,v_2]$$

show that q is the quadratic function associated to the symplectic bilinear form $[\ ,\ ]$ [17, p. 356]. The bilinear form is the deviation from linearity of the quadratic function. Define $V^\perp\supset R(V)$ to be the subgroups

$$V^\perp = \{v\in V \mid [v,V]=0\}\supset\{v\in V^\perp\mid q(v)=0\} = R(V)$$

of V. Since q is a group homomorphism on $V^\perp$, the subgroup $R(V)$ is either all of $V^\perp$ or a subgroup of index 2.

In the following we write $G \circ H$ for the product of the groups G and H with a common central subgroup amalgamated. The subgroup $\mho_1(V^*)$ is generated by all squares of elements of V^* [17, III.10.4].

2.8. Lemma. *Let V be a nontrivial elementary abelian 2-group in the adjoint group $\mathrm{PGL}(2n, \mathbf{R})$. The preimage, V^*, in $\mathrm{GL}(2n, \mathbf{R})$ of V is*

$$V^* = \begin{cases} C_2 \times V, & q(V) = 0, \\ C_4 \circ V, & [V,V] = 0, \; q(V) \neq 0, \\ P \times R(V), & [V,V] \neq 0, \; q(V^\perp) = 0, \\ (C_4 \circ P) \times R(V), & [V,V] \neq 0, \; q(V^\perp) \neq 0, \end{cases}$$

where $C_2 = \langle -E \rangle \subset C_4 \subset \mathrm{SL}(2n, \mathbf{R})$, $P = 2^{1+2d}_{\pm}$ is extraspecial, $C_4 \circ P$ is generalized extraspecial with center of order 4, and $\mho_1(V^*) \subset \langle -E \rangle$.

Proof. As long as the bilinear form is trivial, $[V,V] = 0$, V^* is abelian and the structure theorem for finitely generated abelian groups applies. Assume that the bilinear form does not completely vanish, $[V,V] \neq 0$. Then V^* is nonabelian with commutator subgroup $[V^*, V^*] = C_2$. Write $V = U \times R(V)$ for some nontrivial subgroup U complementary to $R(V)$. Then $V^\perp = V^\perp \cap (U \times R(V)) = (V^\perp \cap U) \times R(V)$ and $q(V^\perp) = q(V^\perp \cap U)$. If U^* denotes the preimage of U, we have $V^* = U^*(C_2 \times R(V)) = U^* \times R(V)$ as the preimage of $R(V)$, $C_2 \times R(V)$, is central in V^*. The commutator subgroup $[U^*, U^*]$ equals $[U^* R(V), U^* R(V)] = [V^*, V^*] = C_2$ and the center $Z(U^*)$ is the preimage of $V^\perp \cap U$. If $q(V^\perp) = 0$, then $R(V) = V^\perp$ and $V^\perp \cap U = R(V) \cap U$ is trivial so $Z(U^*) = C_2$ and $U^* = P$ is extraspecial. If $q(V^\perp) \neq 0$, $R(V)$ has index 2 in $V^\perp$, $V^\perp \cap U$ has order 2, and $q(V^\perp \cap U) \neq 0$ so that $Z(U^*)$ contains an element of order 4. Therefore $Z(U^*) = C_4$ and U^* is generalized extraspecial. There are two isomorphism classes of such groups but only $U^* = C_4 \circ D_8 \circ \cdots \circ D_8 = C_4 \circ P$ has elementary abelian abelianization [33, Ex. 8, p. 146]. ∎

For instance, the preimage of the maximal toral elementary abelian 2-group $t(\mathrm{PSL}(2n, \mathbf{R})$ of $\mathrm{PSL}(2n, \mathbf{R})$ is the abelian group

$$t(\mathrm{PSL}(2n, \mathbf{R}))^* = \langle I, e_1, \ldots, e_n \rangle, \tag{2.9}$$

generated by I and $t(\mathrm{SL}(2n, \mathbf{R}))$.

2.10. Corollary. *Let V be a nontrivial elementary abelian 2-group in $\mathrm{PSL}(2n, \mathbf{R})$. If*

- *$q(V) = 0$, $[V,V] = 0$ then V is toral in $\mathrm{PSL}(2n, \mathbf{R})$ if and only if $V^* = C_2 \times V$ is toral in $\mathrm{SL}(2n, \mathbf{R})$;*
- *$q(V) \neq 0$, $[V,V] = 0$ then V is toral;*
- *$q(V) \neq 0$, $[V,V] \neq 0$ then V is nontoral.*

Proof. We have

$$V \text{ is toral } \Leftrightarrow\ V \subset t(\mathrm{PSL}(2n,\mathbf{R}))\ \Leftrightarrow\ V^* \subset t(\mathrm{PSL}(2n,\mathbf{R}))^*$$

where the symbol "$\subset$" reads "is subconjugate to". In the first case of the corollary, the preimage V^* contains no elements of order 4 so that

$$V^* \subset t(\mathrm{PSL}(2n,\mathbf{R}))^*\ \Leftrightarrow\ V^* \subset t(\mathrm{SL}(2n,\mathbf{R}))$$

as $t(\mathrm{SL}(2n,\mathbf{R}))$ consists of the elements of order ≤ 2 in $t(\mathrm{PSL}(2n,\mathbf{R}))^*$. In the second case, we have $V^* = C_4 \times R(V)$ so that $R(V) \subset C_{\mathrm{SL}(2n,\mathbf{R})}(I) = \mathrm{GL}(n,\mathbf{C})$. But any complex representation of the elementary abelian 2-group $R(V)$ is toral, so $R(V) \subset t(\mathrm{GL}(n,\mathbf{C})) = t(\mathrm{SL}(2n,\mathbf{R}))$ and $V^* \subset \langle C_4, t(\mathrm{SL}(2n,\mathbf{R}))\rangle = t(\mathrm{PSL}(2n,\mathbf{R}))^*$. In the third case, the nonabelian group V^* cannot be a subgroup of the abelian group $t(\mathrm{PSL}(2n,\mathbf{R}))^*$. ■

2.11. LEMMA. *Let V_1 and V_2 be elementary abelian 2-groups in the adjoint group $\mathrm{PSL}(2n,\mathbf{R})$. Then*

$$V_1 \textit{ and } V_2 \textit{ are conjugate in } \mathrm{PSL}(2n,\mathbf{R}) \Leftrightarrow\ V_1^* \textit{ and } V_2^* \textit{ are conjugate in } \mathrm{SL}(2n,\mathbf{R})$$

where $V_1^*, V_2^* \subset \mathrm{SL}(2n,\mathbf{R})$ are the preimages.

Proof. This is clear. ■

Write $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{R}))^{q=0}$ and $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{R}))^{\leq t,\, q=0}$ for the full subcategories of $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{R}))$ generated by all elementary abelian 2-groups $V \subset \mathrm{PGL}(2n,\mathbf{R})$ with trivial quadratic function q, respectively, all toral elementary abelian 2-groups $V \subset \mathrm{PGL}(2n,\mathbf{R})$ with trivial quadratic function q. Define $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{q=0}$ and $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{\leq t,\, q=0}$ similarly as full subcategories of $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))$.

2.12. LEMMA. *Write* GL *for* $\mathrm{GL}(2n,\mathbf{R})$, SL *for* $\mathrm{SL}(2n,\mathbf{R})$, *and* PSL *for* $\mathrm{PSL}(2n,\mathbf{R})$. *The inclusion functors*

$$\begin{aligned}
&\mathbf{A}(\Sigma_{2n},\Delta_{2n}) \to \mathbf{A}(\mathrm{GL}), && \mathbf{A}(\Sigma_{2n},S\Delta_{2n}) \to \mathbf{A}(\mathrm{SL}),\\
& && \mathbf{A}(W(\mathrm{SL}),t(\mathrm{SL})) \to \mathbf{A}(\mathrm{SL})^{\leq t},\\
&\mathbf{A}(\Sigma_{2n},P\Delta_{2n}) \to \mathbf{A}(\mathrm{PGL})^{q=0}, && \mathbf{A}(\Sigma_{2n},PS\Delta_{2n}) \to \mathbf{A}(\mathrm{PSL})^{q=0},\\
& && \mathbf{A}(W(\mathrm{PSL}),t(\mathrm{PSL})) \to \mathbf{A}(\mathrm{PSL})^{\leq t},\\
& && \mathbf{A}(W(\mathrm{PSL}),Pt(\mathrm{SL})) \to \mathbf{A}(\mathrm{PSL})^{\leq t,\, q=0}
\end{aligned}$$

are equivalences of categories. In particular, $\mathbf{A}(\mathrm{SL})$ *and* $\mathbf{A}(\mathrm{PSL})$ *are full subcategories of* $\mathbf{A}(\mathrm{GL})$ *and* $\mathbf{A}(\mathrm{PGL})$, *respectively.* (*See* [24, Definition 2.68] *for the meaning of* $\mathbf{A}(\Sigma_{2n},\Delta_{2n})$.)

Proof. By real representation theory, any nontrivial elementary abelian 2-group of $\mathrm{GL}(2n,\mathbf{R})$ is conjugate to a subgroup V of Δ_{2n} (2.4) and

$$C_{\mathrm{GL}(2n,\mathbf{R})}(V) = \prod_{\varrho\in V^\vee} \mathrm{GL}(i_\varrho,\mathbf{R})$$

where $i\colon V^\vee \to \mathbf{Z}$ records the multiplicity of $\varrho \in V^\vee$ in the representation $V \subset \Delta_{2n} \subset \mathrm{GL}(2n,\mathbf{R})$. Observe that Δ_{2n} is the maximal elementary abelian 2-group in $C_{\mathrm{GL}(2n,\mathbf{R})}(V)$. (For any $i \geq 1$, $\mathrm{GL}(i,\mathbf{R})$ contains the subgroup Δ_i, consisting of diagonal matrices with ± 1 in the diagonal, as a maximal elementary abelian 2-group.) Therefore, by the standard argument from [7, IV.2.5], used also in [24, Lemma 3.4], any group homomorphism between two nontrivial subgroups of Δ_{2n} induced by conjugation with a matrix from $\mathrm{GL}(2n,\mathbf{R})$ is in fact induced by conjugation with a matrix from $N_{\mathrm{GL}(2n,\mathbf{R})}(\Delta_{2n}) = \Delta_{2n} \rtimes \Sigma_{2n}$ [32, Lemma 3]. Thus the inclusion functor $\mathbf{A}(\Sigma_{2n},\Delta_{2n}) \to \mathbf{A}(\mathrm{GL}(2n,\mathbf{R}))$ is a category equivalence.

Any nontrivial elementary abelian 2-group $V \subset \mathrm{PGL}(2n,\mathbf{R})$ with $q(V)=0$ is conjugate to a subgroup of $P\Delta_{2n}$ since V^*, the preimage in $\mathrm{GL}(2n,\mathbf{R})$, is conjugate to a subgroup of Δ_{2n}. Let V_1, V_2 be two nontrivial subgroups of $P\Delta_{2n}$. From the commutative diagram of morphism sets

$$\begin{array}{ccc}
\mathbf{A}(\Sigma_{2n},\Delta_{2n})(V_1^*,V_2^*) & = & \mathbf{A}(\mathrm{GL}(2n,\mathbf{R}))(V_1^*,V_2^*) \\
\downarrow & & \downarrow \\
\mathbf{A}(\Sigma_{2n},P\Delta_{2n})(V_1,V_2) & \hookrightarrow & \mathbf{A}(\mathrm{PGL}(2n,\mathbf{R}))^{q=0}(V_1,V_2)
\end{array}$$

we see that the bottom horizontal arrow is a bijection. This implies that $\mathbf{A}(\Sigma_{2n},P\Delta_{2n}) \to \mathbf{A}(\mathrm{PGL}(2n,\mathbf{R}))^{q=0}$ is an equivalence of categories.

Any nontrivial elementary abelian 2-group in $\mathrm{SL}(2n,\mathbf{R})$ is conjugate in $\mathrm{GL}(2n,\mathbf{R})$ to a subgroup of $\mathrm{SL}(2n,\mathbf{R})\cap\Delta_{2n} = S\Delta_{2n}$ (2.4). The Quillen category of $\mathrm{SL}(2n,\mathbf{R})$ is a full subcategory of the Quillen category of $\mathrm{GL}(2n,\mathbf{R})$ since $C_{\mathrm{GL}(2n,\mathbf{R})}(V) \not\subset \mathrm{SL}(2n,\mathbf{R})$ for all objects V of $\mathbf{A}(\mathrm{SL}(2n,\mathbf{R}))$. Thus the inclusion functor $\mathbf{A}(\Sigma_{2n},S\Delta_{2n}) \to \mathbf{A}(\mathrm{SL}(2n,\mathbf{R}))$ is an equivalence of categories.

Any toral elementary abelian 2-group in $\mathrm{SL}(2n,\mathbf{R})$ is conjugate to a subgroup of $t(\mathrm{SL}(2n,\mathbf{R}))$ by its very definition [24, Definition 2.50]. Any morphism between two nontrivial subgroups of $t(\mathrm{SL}(2n,\mathbf{R}))$ induced by conjugation with a matrix from $\mathrm{SL}(2n,\mathbf{R})$ is in fact induced by conjugation with a matrix from $N(\mathrm{SL}(2n,\mathbf{R}))$ and hence from $W(\mathrm{SL}(2n,\mathbf{R}))$ [7, IV.2.5]. Thus $\mathbf{A}(W(\mathrm{SL}),t(\mathrm{SL})) \to \mathbf{A}(\mathrm{SL}(2n,\mathbf{R}))^{\leq t}$ is a category equivalence. The same argument can be used to identify the toral subcategory for $\mathrm{PSL}(2n,\mathbf{R})$ (and it is actually a general fact that the inclusion functor $\mathbf{A}(W(X),t(X)) \to \mathbf{A}(X)^{\leq t}$ is an equivalence of categories, where $t(X)\to X$ is the maximal toral elementary abelian p-group in the connected p-compact group X [28, 2.8]).

Any nontrivial toral elementary abelian 2-group $V \subset \mathrm{PSL}(2n,\mathbf{R})$ with $q(V)=0$ is conjugate to a subgroup of $Pt(\mathrm{SL})$ (2.4) since V^*, the preimage (2.8) in $\mathrm{GL}(2n,\mathbf{R})$, is conjugate to a subgroup of $t(\mathrm{SL}) \subset t(\mathrm{PSL})^*$ (2.9). As $\mathbf{A}(\mathrm{PSL})^{\le t,\, q=0}$ is a full subcategory of $\mathbf{A}(\mathrm{PSL})^{\le t} = \mathbf{A}(W(\mathrm{PSL}), t(\mathrm{PSL}))$, this means that $\mathbf{A}(W(\mathrm{PSL}), Pt(\mathrm{SL})) \to \mathbf{A}(\mathrm{PSL})^{\le t,\, q=0}$ is a category equivalence. ∎

We now specialize to the full subcategory $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{\le t}_{\le 2}$ of toral objects of rank at most two [24, Definition 2.50].

2.13. Proposition. *The chart*

$\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{\le t}_{\le 2}$	*Lines*		*Planes*	
	$q=0$	$q\neq 0$	$q=0$	$q \neq 0$
n *even*	$n/2$	2	$P(n,3)+P(n,4)$	$n/2+[n/4]$
n *odd*	$[n/2]$	1	$P(n,3)+P(n,4)$	$[n/2]$

gives the number of isomorphism classes of toral objects of rank at most two in $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))$.

When n is even, the $n/2$ toral lines with $q=0$ are $L(2i,2n-2i)$, $1 \le i \le n/2$, and the two toral lines with $q \neq 0$ are I and I^D. The toral planes with $q=0$ are the planes $P(2i_0,2i_1,2i_2,0)$ where (i_0,i_1,i_2) is a partition of n into three natural numbers, $P(2i_0,2i_1,2i_2,2i_3)$ where (i_0,i_1,i_2,i_3) is a partition of n into four natural numbers, and the toral planes with $q\neq 0$ are $I\#L(i,n-i)$, $1\le i\le n/2$, and $I\#L(i,n-i)^D$ for even i.

When n is odd, the $[n/2]$ toral lines with $q=0$ are $L(2i,2n-2i)$, $1 \le i \le [n/2]$, and the toral line with $q\neq 0$ is I. The toral planes with $q=0$ are the planes $P(2i_0,2i_1,2i_2,0)$ where (i_0,i_1,i_2) is a partition of n into three natural numbers, $P(2i_0,2i_1,2i_2,2i_3)$ where (i_0,i_1,i_2,i_3) is a partition of n into four natural numbers, and the toral planes with $q \neq 0$ are $I\#L(i,n-i)$, $1 \le i \le [n/2]$.

In (2.14) *and* (2.15) *we list the centralizers of the rank one objects and in* (2.16) *and* (2.17) *the centralizers of the rank two objects.*

Proposition 2.13 is the conclusion of the following considerations.

For any partition $i=(i_0,i_1)$ of $n=i_0+i_1$ into a sum of two positive integers $i_0 \ge i_1 \ge 1$ let $L(i)=L(2i_0,2i_1) \subset t(\mathrm{SL}(2n,\mathbf{R})) \subset \mathrm{SL}(2n,\mathbf{R})$ be the toral subgroup generated by

$$\mathrm{diag}(\overbrace{+E,\dots,+E}^{i_0},\overbrace{-E,\dots,-E}^{i_1}).$$

Then the centralizer (of the image in $\mathrm{PSL}(2n,\mathbf{R})$) of this subgroup is

$$\text{(2.14)}\quad C_{\mathrm{PSL}(2n,\mathbf{R})}L(2i_0,2i_1) = \begin{cases} \dfrac{\mathrm{SL}(2i_0,\mathbf{R})\times \mathrm{SL}(2i_i,\mathbf{R})}{\langle -E\rangle} \rtimes \langle \mathrm{diag}(D_1,D_2)\rangle, & i_0\neq i_1,\\ \dfrac{\mathrm{SL}(2i_0,\mathbf{R})^2}{\langle -E\rangle} \rtimes \left\langle \mathrm{diag}(D_1,D_2), \begin{pmatrix} O & E\\ E & 0\end{pmatrix}\right\rangle, & i_0=i_1,\end{cases}$$

where $D_j = \mathrm{diag}(-1,1,\dots,1)\in \mathrm{GL}(2i_j,\mathbf{R})$ are matrices of determinant -1. The diagonal matrix $\mathrm{diag}(D_1,D_2)$ acts on the identity component of the centralizer by the outer action on both factors. In the second case, which only occurs when $n=2i_0$ is even, the matrix $\left(\begin{smallmatrix} O & E\\ E & 0\end{smallmatrix}\right)$ acts by permuting the factors.

The element $I\in t(\mathrm{PSL}(2n,\mathbf{R}))^*\subset \mathrm{SL}(2n,\mathbf{R})$ of order four generates an order two toral subgroup of $\mathrm{PSL}(2n,\mathbf{R})$ with centralizer [28, 5.11]

$$\text{(2.15)}\qquad C_{\mathrm{PSL}(2n,\mathbf{R})}(I) = \begin{cases} \mathrm{GL}(n,\mathbf{C})/\langle -E\rangle, & n \text{ odd},\\ \mathrm{GL}(n,\mathbf{C})/\langle -E\rangle \rtimes \langle c_1\cdots c_n\rangle, & n\text{ even},\end{cases}$$

where, in the even case, the component group acts on the identity component through the unstable Adams operation ψ^{-1}. The nontrivial outer automorphism of $\mathrm{PSL}(2n,\mathbf{R})$ takes I to I^D where $I\neq I^D$ if and only if n is even [24, Example 5.4(4)].

For any partition $i=(i_0,i_1,i_2,0)$ of $n=i_0+i_1+i_2$ into a sum of three positive integers $i_0\geq i_1\geq i_2>0$ or any partition $i=(i_0,i_1,i_2,i_3)$ of $n=i_0+i_1+i_2+i_3$ into a sum of four positive integers $i_0\geq i_1\geq i_2\geq i_3>0$ let $P(i)=P(2i_0,2i_1,2i_2,2i_3)\subset t(\mathrm{SL}(2n,\mathbf{R}))\subset \mathrm{SL}(2n,\mathbf{R})$ be the subgroup generated by the two elements

$$\mathrm{diag}(\overbrace{+E,\dots,+E}^{i_0},\overbrace{-E,\dots,-E}^{i_1},\overbrace{+E,\dots,+E}^{i_2},\overbrace{-E,\dots,-E}^{i_3}),$$
$$\mathrm{diag}(\overbrace{+E,\dots,+E}^{i_0},\overbrace{+E,\dots,+E}^{i_1},\overbrace{-E,\dots,-E}^{i_2},\overbrace{-E,\dots,-E}^{i_3}).$$

The centralizers in $\mathrm{PSL}(2n,\mathbf{R})$ are

$$\text{(2.16)}\quad C_{\mathrm{PSL}(2n,\mathbf{R})}(P(i)) = \begin{cases} \dfrac{\mathrm{SL}(2i_0,\mathbf{R})^2\times \mathrm{SL}(2i_2,\mathbf{R})^2}{\langle -E,-E,-E,-E\rangle}\rtimes (\ker(C_2^{S(i)}\to C_2)\rtimes \mathbf{Z}/2), & i=(2i_0,2i_0,2i_2,2i_2),\\ \dfrac{\mathrm{SL}(2i_0,\mathbf{R})^4}{\langle -E,-E,-E,-E\rangle}\rtimes (\ker(C_2^{S(i)}\to C_2)\rtimes (\mathbf{Z}/2\times\mathbf{Z}/2)), & i=(2i_0,2i_0,2i_0,2i_0),\\ \dfrac{\prod_{S(i)}\mathrm{SL}(2i_j,\mathbf{R})}{\langle -E\rangle}\rtimes \ker(C_2^{S(i)}\to C_2), & \text{otherwise},\end{cases}$$

where

$$\ker(C_2^{S(i)} \to C_2) \\ = \langle \mathrm{diag}(D_1, D_2, E, E), \mathrm{diag}(D_1, E, D_3, E), \mathrm{diag}(D_1, E, E, D_4)\rangle$$

(when $\#S(i) = 4$) is generated by the diagonal matrices

$$D_j = \mathrm{diag}(-1, 1, \dots, 1) \in \mathrm{GL}(2i_j, \mathbf{R}), \qquad 1 \le j \le 4,$$

and the groups

$$\mathbf{Z}/2 = \left\langle \begin{pmatrix} 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & E \\ 0 & 0 & E & 0 \end{pmatrix} \right\rangle,$$

$$\mathbf{Z}/2 \times \mathbf{Z}/2 = \left\langle \begin{pmatrix} 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & E \\ 0 & 0 & E & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & E & 0 \\ 0 & 0 & 0 & E \\ E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \end{pmatrix} \right\rangle$$

are generated by block permutation matrices. (The component group of the first line is $C_2 \times D_8$; the component group of the second line is extraspecial of order 32 isomorphic to $D_8 \circ D_8$.)

For any partition $i = (i_0, i_1)$ of $n = i_0 + i_1$ into a sum of two positive integers $i_0 \ge i_1 > 0$ let $I\#L(i_0, i_1) \subset \mathrm{PSL}(2n, \mathbf{R})$ be the elementary abelian 2-group that is the quotient of

$$(I\#L(i_0, i_1))^* = \langle I, \mathrm{diag}(\overbrace{+E, \dots, +E}^{i_0}, \overbrace{-E, \dots, -E}^{i_1})\rangle \subset t(\mathrm{PSL}(2n, \mathbf{R}))^*$$

where $t(\mathrm{PSL}(2n, \mathbf{R}))^*$ is the group (2.9). It follows that

$$(2.17) \qquad C_{\mathrm{PSL}(2n,\mathbf{R})} I\#L(i_0, i_1) \\ = \begin{cases} \dfrac{\mathrm{GL}(i_0, \mathbf{C}) \times \mathrm{GL}(i_1, \mathbf{C})}{\langle -E, -E\rangle}, & n \text{ odd}, \\ \dfrac{\mathrm{GL}(i_0, \mathbf{C}) \times \mathrm{GL}(i_1, \mathbf{C})}{\langle -E, -E\rangle} \rtimes \langle c_1 \cdots c_n\rangle, & n \text{ even}, i_0 \ne i_1, \\ \dfrac{\mathrm{GL}(i_0, \mathbf{C}) \times \mathrm{GL}(i_0, \mathbf{C})}{\langle -E, -E\rangle} \rtimes \langle c_1 \cdots c_n, P\rangle, & n \text{ even}, i_0 = i_1, \end{cases}$$

where $P = \left(\begin{smallmatrix} 0 & E \\ E & 0 \end{smallmatrix}\right)$ permutes the two identical factors.

2.18. Proposition. *$I\#L(i, n-i) \ne I\#L(i, n-i)^D$ if and only if n and i are even.*

Proof. The automorphism group of $\langle i\rangle \times \langle \varepsilon\rangle = C_4 \times C_2 = I\#L(i, n-i)^*$ is the dihedral group of order eight

$$\mathrm{Aut}(C_4 \times C_2) = \langle a, b \mid a^4, b^2, bab = a^3\rangle$$

generated by the two automorphisms given by $a(i) = i\varepsilon$, $a(\varepsilon) = i^2\varepsilon$ and $b(i) = i$, $b(\varepsilon) = i^2\varepsilon$. The automorphism $a^2 \in \mathrm{Aut}(C_4) \subset \mathrm{Aut}(C_4 \times C_2)$ is induced by conjugation with the matrix

$$\mathrm{diag}(P, \ldots, P), \qquad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

of determinant $(-1)^n$. Thus

$$\mathbf{A}(\mathrm{SL}(2n, \mathbf{R}))(I\#L(i, n-i)^*) \neq \mathbf{A}(\mathrm{GL}(2n, \mathbf{R}))(I\#L(i, n-i)^*)$$

and $I\#L(i, n-i) = I\#L(i, n-i)^D$ when n is odd [24, Lemma 5.2].

Assume now that n is even. The group of trace preserving automorphisms

$$\mathbf{A}(\mathrm{GL}(2n, \mathbf{R}))(C_4 \times C_2) = \begin{cases} \langle a^2, ba\rangle, & 2i < n, \\ \mathrm{Aut}(C_4 \times C_2), & 2i = n, \end{cases}$$

has index two in general but is actually equal to the full automorphism group in case $i = n/2$. The conjugating matrix for ba is

$$\mathrm{diag}(\overbrace{P, \ldots, P}^{i}, \overbrace{E, \ldots, E}^{n-i}),$$

of determinant $(-1)^i$. Thus $I\#L(i, n-i) = I\#L(i, n-i)^D$ when i is odd. If $n = 2i$ then the conjugating matrices for the automorphisms a and b are

$$\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \mathrm{diag}(\overbrace{P, \ldots, P}^{i}, \overbrace{E, \ldots, E}^{i}) \quad \text{and} \quad \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}.$$

The permutation matrix for b has positive determinant and the matrix for a has determinant $(-1)^i$. Thus $I\#L(i, n-i) = I\#L(i, n-i)^D$ if and only if i is odd. ∎

2. Centralizers of objects of $\mathbf{A}(\mathrm{PSL}(2n, \mathbf{R}))^{\leq t}_{\leq 2}$ are LHS. In this section we check that all toral objects of rank ≤ 2 have LHS [24, 2.26] centralizers.

2.19. Lemma. *The centralizers of the objects of* $\mathbf{A}(\mathrm{PSL}(2n, \mathbf{R}))^{\leq t}_{\leq 2}$,

(1) $\mathrm{GL}(i, \mathbf{C})/\langle -E\rangle \rtimes C_2$, $1 \leq i$ (2.15),
(2) $\mathrm{SL}(2i_0, \mathbf{R}) \circ \mathrm{SL}(2i_1, \mathbf{R}) \rtimes C_2$, $1 \leq i_0 < i_1$ (2.14),
(3) $(\mathrm{SL}(2i, \mathbf{R}) \circ \mathrm{SL}(2i, \mathbf{R})) \rtimes (C_2 \times C_2)$, $1 \leq i$ (2.14),
(4) $C_{\mathrm{PSL}(2n,\mathbf{R})}(V)$, $q(V) = 0$ (2.16),
(5) $C_{\mathrm{PSL}(2n,\mathbf{R})}(V)$, $q(V) \neq 0$ (2.17),

are LHS.

The cases of interest here are summarized in the following charts, obtained by use of a computer, for rank one centralizers with quadratic form $q = 0$ (2.14)

$\mathrm{SL}(2i_0,\mathbf{R})\circ\mathrm{SL}(2i_1,\mathbf{R})$	$\ker\theta$	$\mathrm{Hom}(W,\check{T}^W)$	$H^1(W;\check{T})$	θ
$1 = i_0, 2 = i_1$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^2$	$\mathbf{Z}/2$	0
$1 = i_0, 3 = i_1$	0	$\mathbf{Z}/2$	$(\mathbf{Z}/2)^2$	mono
$1 = i_0, 4 \le i_1$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	iso
$2 = i_0 < i_1$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^3$	$\mathbf{Z}/2$	epi
$3 \le i_0 < i_1$	0	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^2$	iso

$\mathrm{SL}(2i,\mathbf{R})\circ\mathrm{SL}(2i,\mathbf{R})$	$\ker\theta$	$\mathrm{Hom}(W,\check{T}^W)$	$H^1(W;\check{T})$	θ
$i = 2$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^3$	0
$i \ge 3$	0	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^2$	iso

and $q \neq 0$ (2.15)

$\mathrm{GL}(i,\mathbf{C})/\langle -E\rangle$	$\ker\theta$	$\mathrm{Hom}(W;\check{T}^W)$	$H^1(W;\check{T})$	θ
$i = 2$	$\mathbf{Z}/2$	$(\mathbf{Z}/2)^2$	$\mathbf{Z}/2$	epi
$i = 3$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	iso
$i = 4$	0	$\mathbf{Z}/2$	$(\mathbf{Z}/2)^2$	mono
$i > 4$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	iso

and for rank two centralizers with quadratic form $q = 0$ (2.16)

$\mathrm{SL}(2i_0,\mathbf{R})^2\circ\mathrm{SL}(2i_1,\mathbf{R})^2$	$\ker\theta$	$\mathrm{Hom}(W;\check{T}^W)$	$H^1(W;\check{T})$	θ
$1 = i_0, 2 = i_1$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^{12}$	$(\mathbf{Z}/2)^8$	epi
$1 = i_0, 2 < i_1$	0	$(\mathbf{Z}/2)^6$	$(\mathbf{Z}/2)^6$	iso
$2 = i_0 < i_1$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^{18}$	$(\mathbf{Z}/2)^{14}$	epi
$2 < i_0 < i_1$	0	$(\mathbf{Z}/2)^{12}$	$(\mathbf{Z}/2)^{12}$	iso

$\prod_{j=0}^2\mathrm{SL}(2i_j,\mathbf{R})/\langle -E\rangle$	$\ker\theta$	$\mathrm{Hom}(W;\check{T}^W)$	$H^1(W;\check{T})$	θ
$1 = i_0, 2 = i_1 < i_2$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^6$	$(\mathbf{Z}/2)^4$	epi
$1 = i_0, 2 < i_1 < i_2$	0	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^4$	iso
$2 = i_0 < i_1 < i_2$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^8$	$(\mathbf{Z}/2)^6$	epi
$2 < i_0 < i_1 < i_2$	0	$(\mathbf{Z}/2)^6$	$(\mathbf{Z}/2)^6$	iso

$\prod_{j=0}^3\mathrm{SL}(2i_j,\mathbf{R})/\langle -E\rangle$	$\ker\theta$	$\mathrm{Hom}(W;\check{T}^W)$	$H^1(W;\check{T})$	θ
$1 = i_0, 2 = i_1 < i_2 < i_3$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^{12}$	$(\mathbf{Z}/2)^{10}$	epi
$1 = i_0, 2 < i_1 < i_2 < i_3$	0	$(\mathbf{Z}/2)^9$	$(\mathbf{Z}/2)^9$	iso
$2 = i_0 < i_1 < i_2 < i_3$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^{15}$	$(\mathbf{Z}/2)^{13}$	epi
$2 < i_0 < i_1 < i_2 < i_3$	0	$(\mathbf{Z}/2)^{12}$	$(\mathbf{Z}/2)^{12}$	iso

$\mathrm{SL}(2i,\mathbf{R})^4/\langle -E\rangle$	$\ker\theta$	$\mathrm{Hom}(W;\check{T}^W)$	$H^1(W;\check{T})$	θ
$2 = i$	$(\mathbf{Z}/2)^8$	$(\mathbf{Z}/2)^{24}$	$(\mathbf{Z}/2)^{16}$	epi
$3 \le i$	0	$(\mathbf{Z}/2)^{12}$	$(\mathbf{Z}/2)^{12}$	iso

and with quadratic form $q \neq 0$ (2.17)

$\mathrm{GL}(i_0,\mathbf{C}) \circ \mathrm{GL}(i_1,\mathbf{C})$	$\ker\theta$	$\mathrm{Hom}(W;\check{T}^W)$	$H^1(W;\check{T})$	θ
$1 = i_0, 2 = i_1$	$\mathbf{Z}/2$	$(\mathbf{Z}/2)^2$	$\mathbf{Z}/2$	epi
$1 = i_0, 2 < i_1$	0	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^2$	iso
$2 = i_0 < i_1$	$\mathbf{Z}/2$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^3$	epi
$2 < i_0 < i_1$	0	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^4$	iso

$\mathrm{GL}(i,\mathbf{C}) \circ \mathrm{GL}(i,\mathbf{C})$	$\ker\theta$	$\mathrm{Hom}(W;\check{T}^W)$	$H^1(W;\check{T})$	θ
$2 = i$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^3$	
$3 \leq i$	0	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^4$	iso

Observe that the dimension of $H^1(W;\check{T})$ stabilizes within the infinite families of Lie groups included in these tables. Consider for instance the case $X = \prod_{j=0}^3 \mathrm{SL}(2i_j,\mathbf{R})$. The first cohomology $H^1(W;\check{T}) = H^1(W;\check{T})(X)$ group sits in an exact sequence

$$H^1(W;\check{Z}) \to H^1(W;\check{T}) \to H^1(W;\check{T}/\check{Z}) \xrightarrow{\partial} H^2(W;\check{Z})$$

where W is the Weyl group. The kernel of the first homomorphism stabilizes [24, Lemma 2.22]. As X/Z is a product of simple Lie groups, the table from [16, Main Theorem] shows that the dimensions of the $\mathbf{F}_2$-vector spaces $H^1(W;\check{T}/\check{Z})$ stabilize. Also $H^{\leq 2}(\Sigma_n;\mathbf{F}_2)$ stabilize [30, 6.7]. The formula for the cohomology of a wreath product [13, 5.3.1], $H^*(C_2 \wr \Sigma_n;\mathbf{F}_2) \cong H^*(\Sigma_n; H^*(C_2;\mathbf{F}_2)^{\otimes n})$, now shows that the $\mathbf{F}_2$-vector spaces $H^{\leq 2}(C_2 \wr \Sigma_n;\mathbf{F}_2)$ and $H^{\leq 2}(W;\check{Z})$ stabilize. By naturality, the kernel of the homomorphism ∂ stabilizes. We conclude that $H^1(W;\check{T})$ stabilizes.

Proof of Lemma 2.19. (a) Let $X = \mathrm{GL}(i,\mathbf{C})/\langle -E\rangle \rtimes C_2$ for $i \geq 1$. Since the Weyl group for X is a direct product $W = W_0 \times C_2$, X is LHS.

(b) Let $X = (\mathrm{SL}(2i_0,\mathbf{R}) \circ \mathrm{SL}(2i_1,\mathbf{R})) \rtimes C_2$ for $1 \leq i_0 < i_1$. The first problematic case is when $i_0 = 1$ and $i_1 = 2$ or 3. In this case, explicit computer computation results in the chart

X	$H^1(\pi;\check{T}^{W_0})$	$H^1(W;\check{T})$	$H^1(W_0;\check{T})$	$H^1(W_0;\check{T})^\pi$
$i_1 = 2$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$
$i_1 = 3$	0	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^2$

showing that X is LHS. The second problematic case is $2 = i_0 < i_1$ where $\theta(X_0)$ is epimorphic. Since $H^1(W_0;\check{T}) = \mathbf{Z}/2$, also $\theta(X_0)^\pi$ is epimorphic so that X is LHS [24, Lemma 2.28].

(c) Let $X = (\mathrm{SL}(2i,\mathbf{R}) \circ \mathrm{SL}(2i,\mathbf{R})) \rtimes (C_2 \times C_2)$ for $i \geq 1$. Then X is a 2-compact toral group when $i = 1$ and hence obviously LHS. For $i \geq 2$ explicit computer computation gives

X	$H^1(\pi;\check{T}^{W_0})$	$H^1(W;\check{T})$	$H^1(W_0;\check{T})$	$H^1(W_0;\check{T})^\pi$
$i=2$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^3$	$(\mathbf{Z}/2)^2$
$i\geq 3$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^3$	$(\mathbf{Z}/2)^2$	$\mathbf{Z}/2$

so X is manifestly LHS for $i=2$. For $i>2$, $\theta(X_0)$ is bijective so X is LHS [24, Lemma 2.28].

(d) Let $X=(\mathrm{SL}(2i,\mathbf{R})^4/\langle -E\rangle)\rtimes(D_8\circ D_8)$ for $i\geq 1$. When $i=1$, X is a 2-compact toral group which are all LHS. When $i=2$, explicit computer computation gives

X	$H^1(\pi;\check{T}^{W_0})$	$H^1(W;\check{T})$	$H^1(W_0;\check{T})$	$H^1(W_0;\check{T})^\pi$
$i=2$	$(\mathbf{Z}/2)^7$	$(\mathbf{Z}/2)^9$	$(\mathbf{Z}/2)^{16}$	$(\mathbf{Z}/2)^2$

so X is LHS by definition. For $i>2$, $\theta(X_0)$ is bijective.

(e) Let $X=(\mathrm{SL}(2i_0,\mathbf{R})^2\circ\mathrm{SL}(2i_1,\mathbf{R})^2)\rtimes(C_2\times D_8)$ for $1\leq i_0<i_1$. The problematic cases are $i_0=1$, $i_1=2$ and $2=i_0<i_1$ where $\theta(X_0)$ is surjective but not bijective. With the help of computer computations we obtain the table

X	$H^1(\pi;\check{T}^{W_0})$	$H^1(W;\check{T})$	$H^1(W_0;\check{T})$	$H^1(W_0;\check{T})^\pi$
$i_0=1,i_1=2$	$(\mathbf{Z}/2)^5$	$(\mathbf{Z}/2)^7$	$(\mathbf{Z}/2)^8$	$(\mathbf{Z}/2)^2$
$i_0=2,3\leq i_1$	$(\mathbf{Z}/2)^6$	$(\mathbf{Z}/2)^{11}$	$(\mathbf{Z}/2)^{14}$	$(\mathbf{Z}/2)^5$

showing that X is LHS in these cases also.

(f) Let $X=(\prod_{j=0}^2\mathrm{SL}(2i_j,\mathbf{R})/\langle -E\rangle)\rtimes C_2^2$. The problematic cases are $1=i_0$, $2=i_1<i_2$ and $2=i_0<i_1<i_2$. With the help of computer computations we obtain the table

X	$H^1(\pi;\check{T}^{W_0})$	$H^1(W;\check{T})$	$H^1(W_0;\check{T})$	$H^1(W_0;\check{T})^\pi$
$i_0=1,2=i_1<i_2$	$(\mathbf{Z}/2)^3$	$(\mathbf{Z}/2)^6$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^3$
$2=i_0<i_1<i_2$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^9$	$(\mathbf{Z}/2)^6$	$(\mathbf{Z}/2)^5$

showing that X is LHS in these cases also.

(g) Let $X=(\prod_{j=0}^3\mathrm{SL}(2i_j,\mathbf{R})/\langle -E\rangle)\rtimes C_2^3$. The problematic cases are $1=i_0$, $2=i_1<i_2<i_3$ and $2=i_0<i_1<i_2<i_3$. With the help of computer computations we obtain the table

X	$H^1(\pi;\check{T}^{W_0})$	$H^1(W;\check{T})$	$H^1(W_0;\check{T})$	$H^1(W_0;\check{T})^\pi$
$i_0=1,2=i_1<i_2<i_3$	$(\mathbf{Z}/2)^8$	$(\mathbf{Z}/2)^{16}$	$(\mathbf{Z}/2)^{10}$	$(\mathbf{Z}/2)^8$
$2=i_0<i_1<i_2<i_3$	$(\mathbf{Z}/2)^9$	$(\mathbf{Z}/2)^{20}$	$(\mathbf{Z}/2)^{13}$	$(\mathbf{Z}/2)^{11}$

showing that X is LHS in these cases also.

(h) The 2-compact group $(\mathrm{GL}(i,\mathbf{C})\circ\mathrm{GL}(i,\mathbf{C}))\rtimes(\mathbf{Z}/2\times\mathbf{Z}/2)$ is LHS for $i>2$ where θ is bijective. When $i=2$ we find

X	$H^1(\pi;\check{T}^{W_0})$	$H^1(W;\check{T})$	$H^1(W_0;\check{T})$	$H^1(W_0;\check{T})^\pi$
$i=2$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^3$	$(\mathbf{Z}/2)^2$

so X is also LHS in this case.

(i) Let $X = \mathrm{GL}(i_0,\mathbf{C})\circ \mathrm{GL}(i_1,\mathbf{C})\rtimes C_2$, $1\le i_0<i_1$. Since the identity component has surjective θ-homomorphism and the component group $\pi = C_2$ acts trivially on $H^1(W_0;\check{T})$, X is LHS by [24, Lemma 2.28]. The values of the relevant cohomology groups are

X	$H^1(\pi;\check{T}^{W_0})$	$H^1(W;\check{T})$	$H^1(W_0;\check{T})$	$H^1(W_0;\check{T})^\pi$
$1=i_0, 2=i_1$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$
$1=i_0, 2<i_1$	0	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^2$
$2=i_0<i_1$	0	$(\mathbf{Z}/2)^3$	$(\mathbf{Z}/2)^3$	$(\mathbf{Z}/2)^3$
$2<i_0<i_1$	0	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^4$

according to computer computations. ∎

3. The limit of the functor $H^1(W_0;\check{T})^{W/W_0}$ on $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{\le t}_{\le 2}$. Let $H^1(W_0;\check{T})\colon \mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{\le t}_{\le t}\to\mathbf{Ab}$ be the functor that takes the toral elementary abelian 2-group $V\subset t(\mathrm{PSL}(2n,\mathbf{R}))$ to the abelian group $H^1(W_0C_{\mathrm{PSL}(2n,\mathbf{R})}(V);\check{T}))$, and $H^1(W_0;\check{T})^{W/W_0}$ the functor that takes V to the invariants for the action of the component group $\pi_0C_{\mathrm{PSL}(2n,\mathbf{R})}(V)$ on this first cohomology group [24, 2.53].

2.20. Proposition. *The restriction map*

$$H^1(W(\mathrm{PSL}(2n,\mathbf{R}));\check{T})\to \lim{}^0(\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{\le t}_{\le 2}; H^1(W_0;\check{T})^{W/W_0})$$

is an isomorphism for all $n\ge 4$.

Proof. Consider first the case where $n=4$. The 2-compact group $X=\mathrm{PSL}(8,\mathbf{R})$ contains (2.13) the four rank one elementary abelian 2-groups $L(2,6), L(4,4), I, I^D$ with centralizers

$$\mathrm{SL}(2,\mathbf{R})\circ\mathrm{SL}(6,\mathbf{R})\rtimes C_2,\qquad \mathrm{SL}(4,\mathbf{R})\circ\mathrm{SL}(4,\mathbf{R})\rtimes(C_2\times C_2),$$
$$\mathrm{GL}(4,\mathbf{C})/\langle -E\rangle\quad\text{(twice)}.$$

The claim of the proposition follows from the fact, verifiable by computer computations, that in all four cases, the restriction

$$H^1(W;\check{T}(X))\to H^1(W_0(C_X(L));\check{T})^{W/W_0}$$

happens to be an isomorphism.

For $n>4$, the claim is that the limit of the functor $H^1(W_0;\check{T})^{W/W_0}$ is trivial. In fact, even the limit of the functor $H^1(W_0;\check{T})$ is trivial. To see this, recall (2.13) that $\mathrm{PSL}(2n,\mathbf{R})$ contains the toral lines $L(2i,2n-2i)$, $1\le i\le[n/2]$, I, and also I^D when n is even. Computer computations show that the morphisms

$$H^1(W_0;\check{T})(L(2,2n-2)) \hookrightarrow H^1(W_0;\check{T})(I\#L(1,n-1)) \hookleftarrow H^1(W_0;\check{T})(I)$$

are injective and that their images intersect trivially. When $n \geq 6$ is even, also the images of the injective morphisms

$$H^1(W_0;\check{T})(L(4,2n-4)) \hookrightarrow H^1(W_0;\check{T})(I\#L(2,n-2)^D) \hookleftarrow H^1(W_0;\check{T})(I^D)$$

intersect trivially. More computer computations show that, similarly, the morphisms

$$H^1(W_0;\check{T})(L(2i,2n-2i)) \hookrightarrow H^1(W_0;\check{T})(I\#L(i,n-i)) \hookleftarrow H^1(W_0;\check{T})(I)$$

are injective and that their images intersect trivially, $1 \leq i \leq [n/2]$. ■

4. Rank two nontoral objects of $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))$. In this section we take a closer look at the nontoral rank two objects of $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))$ in order to verify the conditions of [24, Lemma 2.63].

Nontoral rank two objects P of $\mathrm{PSL}(2n,\mathbf{R})$ satisfy either $q(P)=0$ or $[P,P]\neq 0$ (2.10) and the latter case only occurs if n is even.

$\underline{q(P)=0}$: For any partition $i_0 \geq i_1 \geq i_2 \geq i_3 \geq 1$ of $2n$, let

$$\begin{aligned} P(i_0,i_1,i_2,i_3)^* &= \langle (+1)^{i_0}(-1)^{i_1}(+1)^{i_2}(-1)^{i_3}, (+1)^{i_0}(+1)^{i_1}(-1)^{i_2}(-1)^{i_3}, -E\rangle \\ &\subset \varDelta_{2n}, \\ P(i_0,i_1,i_2,i_3) &= P(i_0,i_1,i_2,i_3)^*/\langle -E\rangle \subset P\varDelta_{2n}, \end{aligned}$$

where we apply the notation from [24, (3.5), (3.9)]. Then $P(i_0,i_1,i_2,i_3)^* \subset S\varDelta_{2n}$ if and only if i_0, i_1, i_2, and i_3 all have the same parity and $P(i_0,i_1,i_2,i_3)^*$ is nontoral iff this parity is odd. It follows (2.12) that the set of isomorphism classes of nontoral rank two objects of $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{q=0}$ corresponds bijectively to the $P(n+2,4)$ partitions $i=(i_0,i_1,i_2,i_3)$ of $n+2$ into sums of four natural numbers, $n+2=i_0+i_1+i_2+i_3$, $i_0 \geq i_1 \geq i_2 \geq i_3 \geq 1$. The correspondence is via the map

$$i=(i_0,i_1,i_2,i_3) \mapsto P(i) = P(2i_0-1,2i_1-1,2i_2-1,2i_3-1)$$

that to the partition $i=(i_0,i_1,i_2,i_3)$ associates the quotient $P(i) \subset PS\varDelta_{2n}$ of $P(i)^* \subset S\varDelta_{2n}$ generated by the three elements

$$\begin{aligned} v_1 &= \mathrm{diag}(\overbrace{+1,\dots,+1}^{2i_0-1},\overbrace{-1,\dots,-1}^{2i_1-1},\overbrace{+1,\dots,+1}^{2i_2-1},\overbrace{-1,\dots,-1}^{2i_3-1}), \\ v_2 &= \mathrm{diag}(\overbrace{+1,\dots,+1}^{2i_0-1},\overbrace{+1,\dots,+1}^{2i_1-1},\overbrace{-1,\dots,-1}^{2i_2-1},\overbrace{-1,\dots,-1}^{2i_3-1}), \\ v_3 &= \mathrm{diag}(-1,\dots,-1). \end{aligned}$$

The centralizer of $P(i)^*$ in $\mathrm{SL}(2n,\mathbf{R})$ is

$$\begin{aligned} C_{\mathrm{SL}(2n,\mathbf{R})}(P(i)^*) &= \mathrm{SL}(2n,\mathbf{R}) \cap C_{\mathrm{SL}(2n,\mathbf{R})}(P(i)^*) \\ &= \mathrm{SL}(2n,\mathbf{R}) \cap \Big(\prod_{j=0}^{3} \mathrm{GL}(2i_j-1,\mathbf{R}) \Big) \\ &= P(i)^* \times \prod_{j=0}^{3} \mathrm{SL}(2i_j-1,\mathbf{R}) \end{aligned}$$

and the centralizer of $P(i)$ in $\mathrm{PSL}(2n,\mathbf{R})$ is therefore [28, 5.11]

$$C_{\mathrm{PSL}(2n,\mathbf{R})}(P(i)) = P(i) \times \Big(\prod_{j=0}^{3} \mathrm{SL}(2i_j-1,\mathbf{R}) \Big) \rtimes P(i)_i^\vee \tag{2.21}$$

where $P(i)_i^\vee$ is a group of permutation matrices isomorphic to C_2 if $i = (i_0,i_0,i_2,i_2)$, to $C_2 \times C_2$ if $i = (i_0,i_0,i_0,i_0)$, and trivial in all other cases. Note that $P(i)^*$ is contained in $N(\mathrm{SL}(2n,\mathbf{R})) = \mathrm{SL}(2n,\mathbf{R}) \cap \mathrm{GL}(2,\mathbf{R}) \wr \Sigma_n$ because with $R = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ we may write

$$v_1 = \mathrm{diag}(\overbrace{E,\dots,E}^{i_0-1}, R, \overbrace{-E,\dots,-E}^{i_1-1}, \overbrace{E,\dots,E}^{i_2-1}, R, \overbrace{-E,\dots,-E}^{i_3-1}), \tag{2.22}$$

$$v_2 = \mathrm{diag}(\overbrace{E,\dots,E}^{i_0-1}, E, \overbrace{E,\dots,E}^{i_1-1}, \overbrace{-E,\dots,-E}^{i_2-1}, -E, \overbrace{-E,\dots,-E}^{i_3-1}), \tag{2.23}$$

and that the centralizer of $P(i)^*$ there is

$$\begin{aligned} C_{N(\mathrm{SL}(2n,\mathbf{R}))}(P(i)^*) &= \mathrm{SL}(2n,\mathbf{R}) \cap C_{\mathrm{GL}(2,\mathbf{R})\wr\Sigma_n}(v_1) \cap C_{\mathrm{GL}(2,\mathbf{R})\wr\Sigma_n}(v_2) \\ &\overset{[24,\ 5.10]}{=} \mathrm{SL}(2n,\mathbf{R}) \cap C_{\mathrm{GL}(2,\mathbf{R})\wr(\Sigma_{i_0+i_1-1}\times\Sigma_{i_2+i_3-1})}(v_1) \\ &\overset{[24,\ 5.10]}{=} P(i)^* \times \Big(\prod_{j=0}^{3} \mathrm{GL}(2,\mathbf{R}) \wr \Sigma_{i_j-1} \Big) \end{aligned}$$

It follows that the centralizer of $P(i)$ in $N(\mathrm{PSL}(2n,\mathbf{R}))$ is

$$\begin{aligned} C_{N(\mathrm{PSL}(2n,\mathbf{R}))}(P(i)) &= P(i) \times \Big(\prod_{j=0}^{3} \mathrm{GL}(2,\mathbf{R}) \wr \Sigma_{i_j-1} \Big) \rtimes P(i)_i^\vee \\ &= N(C_{\mathrm{PSL}(2n,\mathbf{R})}(P(i))) \end{aligned}$$

For instance, if $i = (i_0,i_0,i_2,i_2)$, then $P(i)_i^\vee$ is the group of order two generated by $\mathrm{diag}(C_0,C_2) \in N(\mathrm{PSL}(2n,\mathbf{R}))$ where C_0 is the $(4i_0-2)\times(4i_0-2)$ matrix

$$\begin{pmatrix} 0 & 0 & E \\ 0 & T & 0 \\ E & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and C_2 is a similar $(4i_2-2)\times(4i_2-2)$ matrix. Thus $P(i)\subset N(\mathrm{PSL}(2n,\mathbf{R}))$ is a preferred lift [27] of $P(i)\subset\mathrm{PSL}(2n,\mathbf{R})$. The other two preferred lifts [26, 6.2] of $P(i)\subset\mathrm{PSL}(2n,\mathbf{R})$ are obtained by composing the inclusion $P(i)\subset N(\mathrm{PSL}(2n,\mathbf{R}))$ with the inner automorphism given by the permutation matrices $(1,2)(2i_0,2n-2i_3+1)$,

$$(\overbrace{+1,+1,\dots,+1}^{2i_0-1},\overbrace{-1,\dots,-1}^{2i_1-1},\overbrace{+1,\dots,+1}^{2i_2-1},\overbrace{-1,\dots,-1}^{2i_3-1})$$

$$(\overbrace{+1,+1,\dots,+1}^{2i_0-1},\overbrace{+1,\dots,+1}^{2i_1-1},\overbrace{-1,\dots,-1}^{2i_2-1},\overbrace{-1,\dots,-1}^{2i_3-1})$$

and $(1,2)(2i_0,2n-2i_3+2)$,

$$(\overbrace{+1,+1,\dots,+1}^{2i_0-1},\overbrace{-1,\dots,-1}^{2i_1-1},\overbrace{+1,\dots,+1}^{2i_2-1},\overbrace{-1,\dots,-1}^{2i_3-1})$$

$$(\overbrace{+1,+1,\dots,+1}^{2i_0-1},\overbrace{+1,\dots,+1}^{2i_1-1},\overbrace{-1,\dots,-1}^{2i_2-1},\overbrace{-1,\dots,-1}^{2i_3-1})$$

taking v_1 and v_2 as in (2.22), (2.23) to

$$v_1=\mathrm{diag}(\overbrace{E,\dots,E}^{i_0-1},E,\overbrace{-E,\dots,-E}^{i_1-1},\overbrace{E,\dots,E}^{i_2-1},-E,\overbrace{-E,\dots,-E}^{i_3-1}),$$
$$v_2=\mathrm{diag}(\overbrace{E,\dots,E}^{i_0-1},R,\overbrace{E,\dots,E}^{i_1-1},\overbrace{-E,\dots,-E}^{i_2-1},R,\overbrace{-E,\dots,-E}^{i_3-1}),$$

respectively to

$$v_1=\mathrm{diag}(\overbrace{E,\dots,E}^{i_0-1},R,\overbrace{-E,\dots,-E}^{i_1-1},\overbrace{E,\dots,E}^{i_2-1},R,\overbrace{-E,\dots,-E}^{i_3-1}),$$
$$v_2=\mathrm{diag}(\overbrace{E,\dots,E}^{i_0-1},R,\overbrace{E,\dots,E}^{i_1-1},\overbrace{-E,\dots,-E}^{i_2-1},R,\overbrace{-E,\dots,-E}^{i_3-1}).$$

In the same way as above, we see that these are really preferred lifts of $P(i)$. The three lifts are not conjugate in $N(\mathrm{PSL}(2n,\mathbf{R}))$ because the intersection with the maximal torus is v_2 in case (2.22), (2.23) but v_1 and v_1+v_2 in the other two cases. Observe that all three preferred lifts of $P(i)$ have the same image in $W(\mathrm{PSL}(2n,\mathbf{R}))=\pi_0N(\mathrm{PSL}(2n,\mathbf{R}))\subset\pi_0\mathrm{GL}(2,\mathbf{R})\wr\Sigma_n$. Observe also that the inclusion $P(i)\times P(i)_i^\vee\to C_{\mathrm{PSL}(2n,\mathbf{R})}(P(i))$ induces an isomorphism on component groups and that the centralizer

$$C_{\mathrm{PSL}(2n,\mathbf{R})}(P(i)\times P(i)_i^\vee) = C_{C_{\mathrm{PSL}(2n,\mathbf{R})}(P(i))}(P(i)_i^\vee)$$
$$= \begin{cases} P(i)\times \mathrm{SL}(2i_0-1,\mathbf{R}), & i=(i_0,i_0,i_0,i_0),\\ P(i)\times \mathrm{SL}(2i_0-1,\mathbf{R})\times \mathrm{SL}(2i_2-1,\mathbf{R}), & i=(i_0,i_0,i_2,i_2),\\ C_{\mathrm{PSL}(2n,\mathbf{R})}(P(i)), & \text{otherwise,}\end{cases}$$

has nontrivial identity component when $n>2$.

$\underline{[P,P]\neq 0}$: $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))$ contains (up to isomorphism) four rank two objects with nontrivial inner product, namely H_+, H_+^D, H_-, and H_-^D, where $H_\pm$ is the image of $2_\pm^{1+2}\subset \mathrm{SL}(2n,\mathbf{R})$ (2.51).

The extraspecial 2-group $2_+^{1+2}\subset \mathrm{SL}(4n,\mathbf{R})$ is described in [24, Example 5.4(6)] or, alternatively, in [24, 5.7] as

$$\left\langle \mathrm{diag}\Big(\overbrace{\begin{pmatrix}E&0\\0&-E\end{pmatrix},\dots,\begin{pmatrix}E&0\\0&-E\end{pmatrix}}^{n}\Big), \mathrm{diag}\Big(\overbrace{\begin{pmatrix}0&E\\E&0\end{pmatrix},\dots,\begin{pmatrix}0&E\\E&0\end{pmatrix}}^{n}\Big)\right\rangle = \langle g_1,g_2\rangle.$$

Note that 2_+^{1+2} is contained in $N(\mathrm{SL}(4n,\mathbf{R}))=\mathrm{SL}(4n,\mathbf{R})\cap(\mathrm{GL}(2,\mathbf{R})\wr\Sigma_{2n})$ where its centralizer is

$$C_{N(\mathrm{SL}(4n,\mathbf{R}))}(2_+^{1+2}) = \mathrm{SL}(4n,\mathbf{R})\cap C_{\mathrm{GL}(4n,\mathbf{R})}(v_1)\cap C_{\mathrm{GL}(2,\mathbf{R})\wr\Sigma_{2n}}(v_2)$$
$$\overset{[24,\ 5.10]}{=} \mathrm{SL}(4n,\mathbf{R})\cap C_{\mathrm{GL}(2,\mathbf{R})^n\rtimes(C_2\wr\Sigma_n)}(v_1) = \mathrm{GL}(2,\mathbf{R})\wr\Sigma_n = N(\mathrm{GL}(2n,\mathbf{R})).$$

It follows, as in 2.51, that the centralizer of H_+ in $N(\mathrm{PSL}(4n,\mathbf{R}))$ is

$$C_{N(\mathrm{SL}(4n,\mathbf{R}))}(H_+) = H_+\times C_{N(\mathrm{SL}(4n,\mathbf{R}))}(2_+^{1+2})/\langle -E\rangle = N(H_+\times \mathrm{PGL}(2n,\mathbf{R})),$$

which means that $H_+\subset N(\mathrm{PSL}(4n,\mathbf{R}))$ is a preferred lift of H_+. Another preferred lift can be obtained by precomposing the inclusion $H_+\subset N(\mathrm{PSL}(4n,\mathbf{R}))$ with the nontrivial automorphism in $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_+) = O^+(2,\mathbf{F}_2)$. The final preferred lift is

$$(2_+^{1+2})^{\mathrm{diag}(\overbrace{B,\dots,B}^{n})} = \langle -(g_1g_2)^{\mathrm{diag}(B,\dots,B)}, g_2^{\mathrm{diag}(B,\dots,B)}\rangle, \qquad B=\frac{1}{\sqrt2}\begin{pmatrix}E&I\\I&E\end{pmatrix},$$

$$-(g_1g_2)^{\mathrm{diag}(B,\dots,B)} = \mathrm{diag}\left(\begin{pmatrix}I&0\\0&-I\end{pmatrix},\dots,\begin{pmatrix}I&0\\0&-I\end{pmatrix}\right), \qquad g_2^B=g_2.$$

Also, this subgroup is actually contained in the maximal torus normalizer with centralizer

$$C_{N(\mathrm{SL}(4n,\mathbf{R}))}(2_+^{1+2})^{\mathrm{diag}(B,\dots,B)}$$
$$= \mathrm{SL}(4n,\mathbf{R})\cap C_{\mathrm{GL}(2,\mathbf{R})\wr\Sigma_{2n}}(-(g_1g_2)^{\mathrm{diag}(B,\dots,B)})\cap C_{\mathrm{GL}(4n,\mathbf{R})}(g_2)$$

$$\overset{[24,\,5.10]}{=} (\mathrm{GL}(1,\mathbf{C})^2 \rtimes C_2) \wr \Sigma_n \cap C_{\mathrm{GL}(4n,\mathbf{R})}(g_2)$$
$$= (\mathrm{GL}(1,\mathbf{C}) \rtimes C_2) \wr \Sigma_n = N(\mathrm{GL}(2n,\mathbf{R}))$$

where

$$C_2 = \left\langle \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix} \right\rangle, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Observe that, for all three preferred lifts of H_+, the image in the Weyl group $W(\mathrm{PSL}(4n,\mathbf{R})) = \pi_0 N(\mathrm{PSL}(4n,\mathbf{R})) \subset \pi_0\mathrm{GL}(2,\mathbf{R}) \wr \Sigma_{2n}$ is the order two subgroup of Σ_{2n} generated by the permutation $(1,2)(3,4)\cdots(2n-1,2n)$. Observe also that the inclusion $H_+\#L(1,2n-1) \to C_{\mathrm{PSL}(4n,\mathbf{R})}(H_+)$ (2.40) induces an isomorphism on component groups and that the centralizer $C_{\mathrm{PSL}(4n,\mathbf{R})}(H_+\#L(1,2n-1))$ has nontrivial identity component (according to the proof of 2.55) when $n \geq 2$.

The extraspecial 2-group $2_-^{1+2} \subset \mathrm{SL}(4n,\mathbf{R})$ is described in [24, Example 5.4(7)] or, alternatively, in [24, 5.7] as

$$\left\langle \mathrm{diag}\Bigl(\overbrace{\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \ldots, \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}}^{n}\Bigr), \mathrm{diag}\Bigl(\overbrace{\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}}^{n}\Bigr) \right\rangle$$
$$= \langle g_1, g_2 \rangle.$$

Note that 2_-^{1+2} is contained in $N(\mathrm{SL}(4n,\mathbf{R})) = \mathrm{SL}(4n,\mathbf{R}) \cap (\mathrm{GL}(2,\mathbf{R}) \wr \Sigma_{2n})$ where its centralizer is

$$C_{N(\mathrm{SL}(4n,\mathbf{R}))}(2_-^{1+2}) = \mathrm{SL}(4n,\mathbf{R}) \cap C_{\mathrm{GL}(2,\mathbf{R})\wr\Sigma_{2n}}(g_1) \cap C_{\mathrm{GL}(4n,\mathbf{R})}(g_2)$$
$$\overset{[24,\,5.10]}{=} (\mathrm{GL}(1,\mathbf{C})^2 \rtimes C_2) \wr \Sigma_n \cap C_{\mathrm{GL}(4n,\mathbf{R})}(g_2)$$
$$= \left\langle \mathrm{GL}(1,\mathbf{C}), \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix} \right\rangle \wr \Sigma_n \overset{(1)}{=} N(\mathrm{GL}(n,\mathbf{H}))$$

It follows, as in 2.51, that the centralizer of H_- in $N(\mathrm{PSL}(4n,\mathbf{R}))$ is

$$C_{N(\mathrm{SL}(4n,\mathbf{R}))}(H_-) = H_- \times C_{N(\mathrm{SL}(4n,\mathbf{R}))}(2_+^{1+2})/\langle -E \rangle = N(H_- \times \mathrm{PGL}(n,\mathbf{H})),$$

which means that $H_- \subset N(\mathrm{PSL}(4n,\mathbf{R}))$ is a preferred lift of H_-. The other two preferred lifts can be obtained by precomposing the inclusion $H_- \subset N(\mathrm{PSL}(4n,\mathbf{R}))$ with the nontrivial automorphisms in

$$\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_-) = O^-(2,\mathbf{F}_2) = \mathrm{Sp}(2,\mathbf{F}_2) = \mathrm{GL}(2,\mathbf{F}_2).$$

Observe that, for all three preferred lifts of H_-, the image in the Weyl group $W(\mathrm{PSL}(4n,\mathbf{R})) = \pi_0 N(\mathrm{PSL}(4n,\mathbf{R})) \subset \pi_0\mathrm{GL}(2,\mathbf{R}) \wr \Sigma_{2n}$ is the order two subgroup of Σ_{2n} generated by $(1,2)(3,4)\cdots(2n-1,2n)$. Observe also that H_- is contained in the rank three subgroup $H_-\#L(1,n-1)$ (2.42) whose centralizer has a nontrivial identity component when $n \geq 2$ (according to the proof of 2.55).

We conclude that for every nontoral rank two object P of $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))$ the identity component $C_{\mathrm{PSL}(2n,\mathbf{R})}(P)_0$ of the centralizer is centerless. By (part of) [25, 5.2], the homomorphism

$$\mathrm{Aut}(C_{\mathrm{PSL}(2n,\mathbf{R})}(P)) \to \mathrm{Aut}(\pi_0 C_{\mathrm{PSL}(2n,\mathbf{R})}(P)) \times \mathrm{Aut}(C_{\mathrm{PSL}(2n,\mathbf{R})}(P)_0),$$

obtained by applying the functors π_0 and $(\ )_0$, is injective. Under the inductive assumption that $C_{\mathrm{PSL}(2n,\mathbf{R})}(P)_0$ (see (2.21) and 2.51) has $\pi_*(N)$-determined automorphisms it then follows from [24, Lemma 2.63, (2.64)] that condition (3) of [24, Theorem 2.51] is satisfied.

5. Limits over the Quillen category of $\mathrm{PSL}(2n,\mathbf{R})$**.** In this section we show that the problem of computing the higher limits of the functors $\pi_i(BZC_{\mathrm{PSL}(2n,\mathbf{R})})$, $i = 1, 2$, [24, (2.47)] is concentrated on objects of the Quillen category with $q \neq 0$.

2.24. Lemma. *Let* $V \subset PS\Delta_{2n}$ (2.4) *be a nontrivial subgroup representing an object of the category* $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{q=0} = \mathbf{A}(\Sigma_{2n}, PS\Delta_{2n})$ (2.12)*. Then*

$$ZC_{\mathrm{PSL}(2n,\mathbf{R})}(V) = PS\Delta_{2n}^{\Sigma_{2n}(V)}$$

where $\Sigma_{2n}(V) \subset \Sigma_{2n}$ *is the pointwise stabilizer subgroup* [24, Definition 2.68].

Proof. Let $\nu^* \colon V \to S\Delta_{2n}$ be a lift to $\mathrm{SL}(2n,\mathbf{R})$ of the inclusion homomorphism of V into $\mathrm{PSL}(2n,\mathbf{R})$. Then

$$C_{\mathrm{SL}(2n,\mathbf{R})}(\nu^* V) = \mathrm{SL}(2n,\mathbf{R}) \cap \prod_{\varrho \in V^\vee} \mathrm{GL}(i_\varrho, \mathbf{R}), \qquad \Sigma_{2n}(\nu^* V) = \prod_{\varrho \in V^\vee} \Sigma_{i_\varrho}$$

where $i\colon V^\vee \to \mathbf{Z}$ records the multiplicity of each $\varrho \in V^\vee$ in the representation ν^*. Write

$$\nu^*(v) = \mathrm{diag}(\overbrace{\varrho_1(v), \dots, \varrho_1(v)}^{i_1}, \dots, \overbrace{\varrho_m(v), \dots, \varrho_m(v)}^{i_m})$$

where $\varrho_1, \dots, \varrho_m \in V^\vee = \mathrm{Hom}(V, C_2)$ are pairwise distinct homomorphisms $V \to C_2 = \langle \pm 1 \rangle$ and $i_1 + \cdots + i_m = 2n$. There is a corresponding decomposition $\{1, \dots, 2n\} = I_1 \cup \cdots \cup I_m$ of the set $\{1, \dots, 2n\}$ into k disjoint subsets I_j containing i_j elements.

Using [28, 5.11] and [24, Lemma 5.20] we get

$$C_{\mathrm{PSL}(2n,\mathbf{R})}(V) = \frac{C_{\mathrm{SL}(2n,\mathbf{R})}(\nu^* V)}{\langle -E \rangle} \rtimes V^\vee_{\nu^*}, \qquad \Sigma_{2n}(V) = \Sigma_{2n}(\nu^* V) \rtimes V^\vee_{\nu^*}$$

where $V^\vee_{\nu^*} = \{\zeta \in \mathrm{Hom}(V, \mathrm{GL}(1,\mathbf{R})) \mid \forall \varrho \in V^\vee : i_{\zeta\varrho} = i_\varrho\}$. Suppose that ζ is a nontrivial element of $V^\vee_{\nu^*}$. Choose a vector $v \in V$ such that $\zeta(v) = -1$. Then the determinant of $\nu^*(v)$ is $(-1)^n$, for $\nu^*(v)$ consists of an equal number of $+1$ and -1. Thus n is even. Let σ be the permutation associated to ζ that moves the subset I_j monotonically to I_k where $\zeta\varrho_j = \varrho_k$. Then σ is even, for it is a product of n transpositions. In this way we imbed $V_{\nu^*}^\vee$ as a subgroup of the alternating group $A_{2n} \subset \mathrm{PSL}(2n,\mathbf{R})$ to obtain the semidirect products.

The center of the centralizer is

$$\begin{aligned} ZC_{\mathrm{PSL}(2n,\mathbf{R})}(V) &= Z\left(\frac{\prod \mathrm{SL}(2n,\mathbf{R}) \cap \mathrm{GL}(i_\varrho,\mathbf{R})}{\langle -E\rangle} \rtimes V_{\nu^*}^\vee\right) \\ &\overset{[24,\ 5.14]}{=} Z\left(\frac{\prod \mathrm{SL}(2n,\mathbf{R}) \cap \mathrm{GL}(i_\varrho,\mathbf{R})}{\langle -E\rangle}\right)^{V_{\nu^*}^\vee} \\ &\overset{[24,\ 5.18]}{=} \left(\frac{\mathrm{SL}(2n,\mathbf{R}) \cap \prod Z\mathrm{GL}(i_\varrho,\mathbf{R})}{\langle -E\rangle}\right)^{V_{\nu^*}^\vee} \\ &= \left(\frac{S\Delta_{2n}^{\Sigma_{2n}(\nu^*V)}}{\langle -E\rangle}\right)^{V_{\nu^*}^\vee} = (PS\Delta_{2n}^{\Sigma_{2n}(\nu^*V)})^{V_{\nu^*}^\vee} = PS\Delta_{2n}^{\Sigma_{2n}(V)}, \end{aligned}$$

where the penultimate equality sign is justified by observing that the coefficient group homomorphism $H^1(\Sigma_{2n}(\nu^*V);\langle -E\rangle) \to H^1(\Sigma_{2n}(\nu^*V); S\Delta_{2n}) \to H^1(\Sigma_{2n}(\nu^*V);\Delta_{2n})$ is injective. ∎

Let $\pi_i(BZC) = \pi_i(BZC_{\mathrm{PSL}(2n,\mathbf{R})})$ [24, (2.47)].

2.25. Corollary. *$\lim^*(\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{q=0};\pi_i(BZC)) = 0$ for $n \geq 2$ and $i = 1,2$.*

Proof. This is obvious for $i = 2$ as $\pi_2(BZC) = 0$. For $i = 1$, use [24, Lemma 2.69] to compute the limits of the functor $\pi_1(BZC)(V) = PS\Delta_{2n}^{\Sigma_{2n}(V)}$ (2.24). The fixed point group $PSD_{2n}^{\Sigma_{2n}}$ is trivial since PSD_{2n} is an irreducible $\mathbf{F}_2\Sigma_{2n}$-module of dimension $2n-2$ for $n \geq 2$. ∎

2.26. Lemma. *Let $V \subset Pt(\mathrm{SL}) = Pt(\mathrm{SL}(2n,\mathbf{R}))$* (2.4) *be a nontrivial subgroup representing an object of the category $\mathbf{A}(A_{2n} \cap (\Sigma_2 \wr \Sigma_n), Pt(\mathrm{SL})) = \mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{\leq t, q=0}$* (2.12)*. Then*

$$ZC_{\mathrm{PSL}(2n,\mathbf{R})}(V) = Pt(\mathrm{SL})^{(A_{2n}\cap(\Sigma_2\wr\Sigma_n))(V)}$$

where $(A_{2n} \cap (\Sigma_2 \wr \Sigma_n))(V)$ is the pointwise stabilizer subgroup [24, Definition 2.68].

Proof. The pointwise stabilizer subgroups are

$$(2.27)\qquad (A_{2n} \cap (\Sigma_2 \wr \Sigma_n))(V) = A_{2n} \cap \Sigma_{2n}(V), \qquad \Sigma_{2n}(V) = \Sigma_2^n \rtimes \Sigma_n(V).$$

Because these stabilizer subgroups have these particular forms and $PS\Delta_{2n}^{\Sigma_2^n} = Pt(\mathrm{SL})$, we get

$$\begin{aligned} ZC_{\mathrm{PSL}(2n,\mathbf{R})}(V) &= PS\Delta_{2n}^{\Sigma_{2n}(V)} = PS\Delta_{2n}^{\Sigma_2^n \rtimes \Sigma_n(V)} = Pt(\mathrm{SL})^{\Sigma_n(V)} \\ &= Pt(\mathrm{SL})^{A_{2n}\cap(\Sigma_2^n \cap \Sigma_n(V))} \end{aligned}$$

from 2.24. ∎

Lemma 2.26 can also be proved along the lines of [28, 2.8] using [24, Proposition 2.33].

2.28. COROLLARY. $\lim^*(\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{\leq t,\, q=0}; \pi_i(BZC)) = 0$ *for* $n \geq 2$ *and* $i = 1, 2$.

Proof. Similar to 2.25 but based on $H^0(A_{2n} \cap (\Sigma_2 \wr \Sigma_n); Pt(\mathrm{SL})) = H^0(\Sigma_n; Pt(\mathrm{SL})) = 0$. ∎

2.29. LEMMA. *Let* $V \subset t(\mathrm{PSL}) = t(\mathrm{PSL}(2n,\mathbf{R}))$ (2.4) *be a nontrivial subgroup representing an object of the category* $\mathbf{A}(A_{2n} \cap (\Sigma_2 \wr \Sigma_n), t(\mathrm{PSL})) = \mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{\leq t}$ (2.12) *where* $n \geq 32$. *Then*

$$\check{Z}C_{\mathrm{PSL}(2n,\mathbf{R})}(V) = \check{T}^{(A_{2n} \cap (\Sigma_2 \wr \Sigma_n))(V)},$$

where $\check{T} = \check{T}(\mathrm{PSL}(2n,\mathbf{R}))$ *is the discrete approximation* [10, §3] *to the maximal torus of* $\mathrm{PSL}(2n,\mathbf{R})$ *and* $(A_{2n} \cap (\Sigma_2 \wr \Sigma_n))(V)$ *is the pointwise stabilizer subgroup of* V [24, Definition 2.68].

Proof. Consider first the case where $V \subset Pt(\mathrm{SL}) \subset t(\mathrm{PSL})$. One checks that $\check{T}^{A_{2n} \cap \Sigma_2^n} = Pt(\mathrm{SL})$ for $n > 2$. Since $(A_{2n} \cap (\Sigma_2 \wr \Sigma_n))(V) \supset A_{2n} \cap \Sigma_2^n$ we get

$$ZC_{\mathrm{PSL}(2n,\mathbf{R})}(V) \overset{2.26}{=} Pt(\mathrm{SL})^{(A_{2n} \cap (\Sigma_2 \wr \Sigma_n))(V)} = \check{T}^{(A_{2n} \cap (\Sigma_2 \wr \Sigma_n))(V)}$$

in this case.

Consider next the case where V^*, the preimage of V in $\mathrm{SL}(2n,\mathbf{R})$, contains I (2.5) so that $V^* = \langle I, U\rangle$ (2.8) for some (possibly trivial) elementary abelian 2-group $U \subset t(\mathrm{SL}) \subset C_{\mathrm{SL}(2n,\mathbf{R})}(C_4) = \mathrm{GL}(n,\mathbf{C})$. Then

$$C_{\mathrm{SL}(2n,\mathbf{R})}(V^*) = \prod_{\varrho \in U^\vee} \mathrm{GL}(i_\varrho, \mathbf{C}), \qquad (\Sigma_2 \wr \Sigma_n)(V^*) = \Sigma_n(U) \subset A_{2n},$$

where $i\colon U^\vee \to \mathbf{Z}$ records the multiplicity of the linear character $\varrho \in U^\vee$ in the representation $\nu^*\colon U \to \mathrm{GL}(n,\mathbf{C})$ and $\Sigma_n(U)$ is the pointwise stabilizer subgroup for the action of $\Sigma_n = W(\mathrm{GL}(n,\mathbf{C}))$ on $t(\mathrm{SL}) = t(\mathrm{GL}(n,\mathbf{C})) = \langle e_1, \dots e_n\rangle$. It now follows from [28, 5.11] and [24, Lemma 5.20], as in (2.15) and (2.17), that

$$C_{\mathrm{PSL}(2n,\mathbf{R})}(V) = \begin{cases} \dfrac{C_{\mathrm{SL}(2n,\mathbf{R})}(V^*)}{\langle -E\rangle}, & n \text{ odd}, \\[2ex] \dfrac{C_{\mathrm{SL}(2n,\mathbf{R})}(V^*)}{\langle -E\rangle} \rtimes (U^\vee_{\nu^*} \times \langle c_1 \cdots c_n\rangle), & n \text{ even}, \end{cases}$$

$$A_{2n} \supset (\Sigma_2 \wr \Sigma_n)(V) = \begin{cases} \Sigma_n(U), & n \text{ odd}, \\ \Sigma_n(U) \rtimes (U^\vee_{\nu^*} \times \langle c_1 \cdots c_n\rangle), & n \text{ even}, \end{cases}$$

where $U^\vee_{\nu^*} = \{\zeta \in U^\vee = \mathrm{Hom}(U, \langle -E\rangle) \mid \forall \varrho \in U^\vee\colon i_{\zeta\varrho} = i_\varrho\}$ can be realized as a subgroup of Σ_n and $\langle c_1 \cdots c_n\rangle$ is the diagonal order two subgroup of Σ_2^n.

Consequently, if n is odd,

$$\check{Z}C_{\mathrm{PSL}(2n,\mathbf{R})}(V) = \check{Z}\Big(\frac{\prod \mathrm{GL}(i_\varrho, \mathbf{C})}{\langle -E\rangle}\Big) = \frac{\prod \check{Z}\mathrm{GL}(i_\varrho, \mathbf{C})}{\langle -E\rangle}$$
$$= \frac{\check{T}(\mathrm{SL}(2n,\mathbf{R}))^{\Sigma_n(U)}}{\langle -E\rangle} = \check{T}^{\Sigma_n(U)} = \check{T}^{(A_{2n}\cap(\Sigma_2\wr\Sigma_n))(V)},$$

and if n is even,

$$\check{Z}C_{\mathrm{PSL}(2n,\mathbf{R})}(V) = \check{Z}\Big(\frac{\prod \mathrm{GL}(i_\varrho, \mathbf{C})}{\langle -E\rangle} \rtimes (U_{\nu^*}\times\langle c_1\cdots c_n\rangle)\Big)$$
$$\overset{[24,\ 5.14]}{=} \Big(\frac{\prod \check{Z}\mathrm{GL}(i_\varrho, \mathbf{C})}{\langle -E\rangle}\Big)^{U_{\nu^*}\times\langle c_1\cdots c_n\rangle}$$
$$= \Big(\frac{\check{T}(\mathrm{SL}(2n,\mathbf{R}))^{\Sigma_n(U)}}{\langle -E\rangle}\Big)^{U_{\nu^*}\times\langle c_1\cdots c_n\rangle}$$
$$= (\check{T}^{\Sigma_n(U)})^{U_{\nu^*}\times\langle c_1\cdots c_n\rangle} = \check{T}^{U_{\nu^*}\times\langle c_1\cdots c_n\rangle} = \check{T}^{(A_{2n}\cap(\Sigma_2\wr\Sigma_n))(V)}$$

where we use the fact that $H^1(\Sigma_n(U);\langle -E\rangle) \to H^1(\Sigma_n(U);\check{T}(\mathrm{SL}(2n,\mathbf{R})))$ is injective. (In fact, the center of the centralizer, $\check{Z}C_{\mathrm{PSL}(2n,\mathbf{R})}(V)$, is a product, $\check{T}^{\Sigma_n(U)}$, of 2-compact tori when n is odd, and a finite abelian group,

$$\check{T}^{\Sigma_n(U)\rtimes(U^\vee_{\nu^*}\rtimes\langle c_1\cdots c_n\rangle)} = (\check{T}^{\langle c_1\cdots c_n\rangle})^{\Sigma_n(U)\rtimes U^\vee_{\nu^*}} = t(\mathrm{PSL})^{\Sigma_n(U)\rtimes U^\vee_{\nu^*}},$$

when n is even.) ∎

2.30. Corollary. *$\lim^*(\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{\le t};\pi_i(BZC)) = 0$ for $n \ge 3$ and $i = 1, 2$.*

Proof. Similar to that of 2.25 but based on $H^0(W;\check{T})(\mathrm{PSL}(2n,\mathbf{R})) = 0$ for $n \ge 3$ (2.3). ∎

As we shall see next, Corollaries 2.25, 2.28 and 2.30 reduce the problem of computing the graded abelian group $\lim^*(\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}));\pi_i(BZC))$ considerably.

Let $\mathbf{A}$ be a category containing two full subcategories, $\mathbf{A}_j$, $j = 1, 2$, such that any object of $\mathbf{A}$ with a morphism to an object of $\mathbf{A}_j$ is an object of $\mathbf{A}_j$. Write $\mathbf{A}_1\cap\mathbf{A}_2$ for the full subcategory with objects $\mathrm{Ob}(\mathbf{A}_1\cap\mathbf{A}_2) = \mathrm{Ob}(\mathbf{A}_1)\cap \mathrm{Ob}(\mathbf{A}_2)$, and $\mathbf{A}_1\cup\mathbf{A}_2$ for the full subcategory with objects $\mathrm{Ob}(\mathbf{A}_1\cup\mathbf{A}_2) = \mathrm{Ob}(\mathbf{A}_1)\cup\mathrm{Ob}(\mathbf{A}_2)$. Let $M\colon \mathbf{A}\to\mathbf{Ab}$ be a functor taking values in abelian groups. Consider the subfunctor M_{12} of M given by

$$M_{12}(a) = \begin{cases} 0, & a \in \mathrm{Ob}(\mathbf{A}_1\cup\mathbf{A}_2),\\ M(a), & a \notin \mathrm{Ob}(\mathbf{A}_1\cup\mathbf{A}_2).\end{cases}$$

We now state a kind of Mayer–Vietoris sequence argument for cohomology of categories.

2.31. Lemma. *If the graded abelian groups* $\lim^*(\mathbf{A}_1; M)$, $\lim^*(\mathbf{A}_2; M)$, *and* $\lim^*(\mathbf{A}_1 \cap \mathbf{A}_2; M)$ *are trivial, then* $\lim^*(\mathbf{A}; M_{12}) \cong \lim^*(\mathbf{A}; M)$.

Proof. Consider also the subfunctor M_1 of M given by

$$M_1(a) = \begin{cases} 0, & a \in \mathrm{Ob}(\mathbf{A}_1), \\ M(a), & a \notin \mathrm{Ob}(\mathbf{A}_1). \end{cases}$$

Then there are natural transformations $M_{12} \to M_1 \to M$ of functors. The induced long exact sequences imply that it suffices to show $\lim^*(\mathbf{A}; M/M_1) = 0 = \lim^*(\mathbf{A}; M_1/M_{12})$.

The quotient functor M/M_1 vanishes outside $\mathbf{A}_1$ where it agrees with M and therefore [28, 13.12] $\lim^*(\mathbf{A}; M/M_1) \cong \lim(\mathbf{A}_1; M)$, which is trivial by assumption.

The same argument applied to $\mathbf{A}_2$ instead of $\mathbf{A}$ gives

$$\lim^*(\mathbf{A}_2; M/M_1) \cong \lim(\mathbf{A}_1 \cap \mathbf{A}_2; M)$$

Since this abelian group is trivial by assumption, we have

$$\lim^*(\mathbf{A}_2; M_1) \cong \lim^*(\mathbf{A}_2; M)$$

Also this abelian group is trivial by assumption.

The quotient functor M_1/M_{12} vanishes outside $\mathbf{A}_1 \cup \mathbf{A}_2$, where it agrees with M_1 and therefore $\lim^*(\mathbf{A}; M_1/M_{12}) \cong \lim(\mathbf{A}_1 \cup \mathbf{A}_2; M_1)$. Here, the functor M_1 vanishes outside $\mathbf{A}_2$ and hence $\lim(\mathbf{A}_1 \cup \mathbf{A}_2; M_1) \cong \lim^*(\mathbf{A}_2; M_1)$. Since we just showed that this abelian group is trivial, we see that so is the graded group $\lim^*(\mathbf{A}; M_1/M_{12})$. ∎

We conclude that

$$\lim^*(\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R})); \pi_j(BZC_{\mathrm{PSL}(2n,\mathbf{R})})_{12}) = \lim^*(\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R})); \pi_j(BZC_{\mathrm{PSL}(2n,\mathbf{R})})),$$

where $\pi_j(BZC_{\mathrm{PSL}(2n,\mathbf{R})})_{12}$ is the subfunctor of $\pi_j(BZC_{\mathrm{PSL}(2n,\mathbf{R})})$ given by

$$\pi_j(BZC_{\mathrm{PSL}(2n,\mathbf{R})})_{12}(V) = \begin{cases} 0, & V \text{ is toral or } q(V) = 0, \\ \pi_j(BZC_{\mathrm{PSL}(2n,\mathbf{R})}(V)), & V \text{ is nontoral and } q(V) \neq 0. \end{cases}$$

According to 2.10 we have

$$V \text{ is nontoral and } q(V) \neq 0 \;\Leftrightarrow\; [V, V] \neq 0$$

for all elementary abelian 2-groups V in $\mathrm{PSL}(2n,\mathbf{R})$. Thus the problem of computing the higher limits of the functors $\pi_i(BZC_{\mathrm{PSL}(2n,\mathbf{R})})$ is concentrated on the full subcategory $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{[\,,\,]\neq 0}$ of $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))$ generated by all elementary abelian 2-groups $V \subset \mathrm{PSL}(2n,\mathbf{R})$ with nontrivial inner product. Note that if $\mathrm{PSL}(2n, \mathbf{R})$ contains an elementary abelian 2-group V with $[V, V] \neq 0$ then $\mathrm{PSL}(2n, \mathbf{R})$ in particular contains such a subgroup of rank two. The preimage in $\mathrm{SL}(2n, \mathbf{R})$ of rank two $V \subset \mathrm{PSL}(2n, \mathbf{R})$ with nontrivial inner product is an extraspecial 2-group $2_{\pm}^{1+2}$ with central $\mho_1$ (2.8) so that, by real representation theory [24, 5.5], n must be even.

6. Higher limits of the functors $\pi_i(BZC)$ on $\mathbf{A}(\mathrm{PSL}(4n, \mathbf{R}))^{[\,,\,]\neq 0}$. In this section we compute the first higher limits of the center functors $\pi_i BZC_{\mathrm{PSL}(4n,\mathbf{R})}$, $i = 1, 2$ ([24, (2.47)]), using Oliver's cochain complex [31].

2.32. Lemma. *The higher limits of the center functors are*

$$\lim{}^1 \pi_1 BZC_{\mathrm{PSL}(4n,\mathbf{R})} = 0 = \lim{}^2 \pi_1 BZC_{\mathrm{PSL}(4n,\mathbf{R})},$$
$$\lim{}^2 \pi_2 BZC_{\mathrm{PSL}(4n,\mathbf{R})} = 0 = \lim{}^3 \pi_2 BZC_{\mathrm{PSL}(4n,\mathbf{R})}.$$

The case $i = 2$ is easy. Since $\pi_2 BZC_{\mathrm{PSL}(4n,\mathbf{R})}$ has value 0 on all objects of $\mathbf{A}(\mathrm{PSL}(4n, \mathbf{R}))^{[\,,\,]\neq 0}$ of rank ≤ 4 (2.55), it is immediate from Oliver's cochain complex that $\lim^2$ and $\lim^3$ of this functor are trivial.

We shall therefore now concentrate on the case $i = 1$. The claim of the above lemma is that Oliver's cochain complex [31]

$$0 \to \prod_{|P|=2^2} [P] \xrightarrow{d^1} \prod_{|V|=2^3} [V] \xrightarrow{d^2} \prod_{|E|=2^4} [E] \xrightarrow{d^3} \cdots \tag{2.33}$$

is exact at objects of rank ≤ 3. Here, as a matter of notational convention,

$$[E] = \mathrm{Hom}_{\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(E)}(\mathrm{St}(E), E) \tag{2.34}$$

stands for the $\mathbf{F}_2$-vector space of $\mathbf{F}_2\mathbf{A}(\mathrm{PSL}(4n, \mathbf{R}))(E)$-module homomorphisms from the Steinberg module $\mathrm{St}(E)$ to E. The Steinberg module is the $\mathbf{F}_2\mathrm{GL}(E)$-module obtained in the following way.

Let $P = \mathbf{F}_2 e_1 + \mathbf{F}_2 e_2$ be a 2-dimensional vector space over $\mathbf{F}_2$ with basis vectors e_1, e_2. Let $\mathbf{F}_2[0]$ be the 3-dimensional $\mathbf{F}_2$-vector space on length zero flags, $[L]$, of nontrivial and proper subspaces L of P. The Steinberg module $\mathrm{St}(P)$ is the 2-dimensional kernel of the augmentation map $d\colon \mathbf{F}_2[0] \to \mathbf{F}_2$ given by $d[L] = 1$.

Let $V = \mathbf{F}_2 e_1 + \mathbf{F}_2 e_2 + \mathbf{F}_2 e_3$ be a 3-dimensional vector space over $\mathbf{F}_2$ with basis vectors e_1, e_2, e_3. Let $\mathbf{F}_2[1]$ be the 21-dimensional $\mathbf{F}_2$-vector space on length one flags $[P > L]$ of nontrivial and proper subspaces of V, and $\mathbf{F}_2[0]$ the 14-dimensional $\mathbf{F}_2$-vector space on all length 0 flags, $[P]$ or $[L]$, of nontrivial and proper subspaces of V. The Steinberg module $\mathrm{St}(V)$ over $\mathbf{F}_2$ for V is the 2^3-dimensional kernel of the linear map $d\colon \mathbf{F}_2[1] \to \mathbf{F}_2[0]$ given by $d[P > L] = [P] + [L]$.

2.35. Proposition. *$H_+ \neq H_+^D$ and $H_- \neq H_-^D$ in $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))$. The automorphism groups of the objects H_+ and H_- (2.51) are*

$$\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_+) = O^+(2,\mathbf{F}_2) \cong C_2,$$
$$\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_-) = O^-(2,\mathbf{F}_2) \cong \mathrm{GL}(2,\mathbf{F}_2),$$

and the dimensions of the spaces of equivariant maps are

$$\dim[H_+] = 2, \qquad \dim[H_-] = 1.$$

Proof. The first part will be proved in 2.51. The Quillen automorphism group $\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(2_\pm^{1+2})$ equals $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(2_\pm^{1+2}) = \mathrm{Out}(2_\pm^{1+2}) \cong O^\pm(2,\mathbf{F}_2)$ where the isomorphism is induced by the abelianization $2_\pm^{1+2} \to H_\pm$ [24, Example 5.4(2)–(3), 5.5]. ∎

The $\mathbf{F}_2\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_+)$-equivariant maps given by

$$f_+[L] = L, \qquad f_0[L] = \begin{cases} H_+^{\mathbf{A}(H_+)}, & q(L) = 0, \\ 0, & \text{otherwise}, \end{cases} \tag{2.36}$$

form a basis for the 2-dimensional space $[H_+]$. The $\mathbf{F}_2\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_-)$-equivariant map given by

$$f_-[L] = L \tag{2.37}$$

is a basis for the 1-dimensional space $[H_-]$.

The quadratic function [24, 5.5] $q(v_1,v_2,v_3) = v_1^2 + v_2v_3$ on V_0 (2.52) has automorphism group

$$O(q) \cong \mathrm{Sp}(2,\mathbf{F}_2) = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \subset \mathrm{GL}(3,\mathbf{F}_2)$$

of order 6.

2.38. Proposition. *$V_0 \neq V_0^D$ in $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))$. The automorphism group $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(V_0)$ equals $O(q)$ and $\dim[V_0] = 4$.*

Proof. See [24, Example 5.4(5)] for the first part. According to *magma*, $\dim[V_0] = 4$. ∎

The four $\mathbf{F}_2\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(V_0)$-module homomorphisms

$$\{df_+, df_0, df_-, f_0\} \tag{2.39}$$

given by

$$df_+[P > L] = \begin{cases} L, & P = H_+, \\ 0, & \text{otherwise}, \end{cases} \qquad df_0[P > L] = \begin{cases} P^{\mathbf{A}(P)}, & P = H_+, \\ & q(L) = 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$df_-[P > L] = \begin{cases} L, & P = H_-, \\ 0, & \text{otherwise}, \end{cases} \qquad f_0[P > L] = \begin{cases} V_0^{\mathbf{A}(V_0)}, & [P,P] = 0, \\ & q(L) = 0, \\ 0, & \text{otherwise}, \end{cases}$$

form a basis for $[V_0]$.

The quadratic function on $H_+\#L(i, 2n-i) \in \mathrm{Ob}(\mathbf{A}(\mathrm{PSL}(4n, \mathbf{R})))$, $0 \leq i \leq n$, $q(v_1, v_2, v_3) = v_1 v_2$, has automorphism group

$$O(q) = \begin{pmatrix} O^+(2, \mathbf{F}_2) & 0 \\ * & 1 \end{pmatrix} = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \right\rangle$$

of order $|O^+(2, \mathbf{F}_2)| \cdot 2^2 = 8$.

2.40. PROPOSITION. *$H_+\#L(i, 2n-i) \neq (H_+\#L(i, 2n-i))^D$ if and only if i is even. The Quillen automorphism group is*

$$\mathbf{A}(\mathrm{PSL}(4n, \mathbf{R}))(H_+\#L(i, 2n-i)) = \begin{cases} \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \right\rangle, & i \text{ odd}, \\ O(q), & i \text{ even}, \end{cases}$$

and the dimension of the space of equivariant maps is

$$\dim[H_+\#L(i, 2n-i)] = \begin{cases} 6, & i \text{ odd}, \\ 3, & i \text{ even}. \end{cases}$$

Proof. $H_+\#L(i, 2n-i) \subset \mathrm{PSL}(4n, \mathbf{R})$ is (2.52) the quotient of

$$G = \langle \mathrm{diag}(R, \ldots, R), \mathrm{diag}(T, \ldots, T), \mathrm{diag}(\overbrace{-E, \ldots, -E}^{i}, \overbrace{E, \ldots, E}^{2n-i}) \rangle = \langle g_1, g_2, g_3 \rangle \subset \mathrm{SL}(4n, \mathbf{R}).$$

The centralizer of G in $\mathrm{GL}(4n, \mathbf{R})$ is contained in the centralizer of its subgroup 2_+^{1+2},which is contained in $\mathrm{SL}(4n, \mathbf{R})$ [24, Example 5.4(6)]. Observe that

- R and T are conjugate in $\mathrm{GL}(2, \mathbf{R})$.
- Conjugation with $\mathrm{diag}(\overbrace{T, \ldots, T}^{i}, \overbrace{E, \ldots, E}^{2n-i})$ induces $(g_1, g_2, g_3) \overset{\phi_1}{\mapsto} (g_1 g_3, g_2, g_3)$.

- Conjugation with $\mathrm{diag}(\overbrace{R,\dots,R}^{i},\overbrace{E,\dots,E}^{2n-i})$ induces $(g_1,g_2,g_3)\overset{\phi_2}{\mapsto}(g_1,g_2g_3,g_3)$.
- When $i=n$, conjugation with $\left(\begin{smallmatrix}0&E\\E&0\end{smallmatrix}\right)$ induces $(g_1,g_2,g_3)\overset{\phi}{\mapsto}(g_1,g_2,-g_3)$.

Consider the automorphism groups

$$\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G)\subset\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)\subset\mathrm{Out}(G)\to O(q)\\ \subset\mathrm{Aut}(H_+\#L(i,2n-i))$$

where the outer automorphism group has order 16. Note that the automorphism ϕ is in the kernel of the homomorphism $\mathrm{Out}(G)\to O(q)$ induced by the abelianization $G\to H_+\#L(i,2n-i)$. Using the above observations we see that $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$, even $\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G)$ for even i, maps onto $O(q)$. Thus the Quillen automorphism group $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$ has order 8 or 16. When $i=n$ the automorphism ϕ is in $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$, even in $\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G)$, and when $i\neq n$, $\phi\notin\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$ as it does not preserve trace. Thus

$$|\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)|=\begin{cases}16, & i=n,\\ 8, & i\neq n.\end{cases}$$

In any case the group $\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G)$ equals the group $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$ if and only if i is even. When i is odd, the automorphism ϕ_1 is induced from a matrix of negative determinant so that $N_{\mathrm{GL}(4n,\mathbf{R})}(G)\not\subset\mathrm{SL}(4n,\mathbf{R})$. According to *magma*, $\dim[H_+\#L(i,2n-i)]$ is 3 when i is even and 6 when i is odd. ∎

The six $\mathbf{F}_2\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_+\#L(i,2n-i))$-linear maps

$$\{df_+,df_0,f_0,df_+^D,df_0^D,f_0^D\} \tag{2.41}$$

given by

$$df_+[P>L]=\begin{cases}L, & P=H_+,\\ 0, & \text{otherwise},\end{cases}$$
$$df_0[P>L]=\begin{cases}P^{\mathbf{A}(P)}, & P=H_+,\ q(L)=0,\\ 0, & \text{otherwise},\end{cases}$$
$$f_0[P>L]=\begin{cases}v_1, & [P,P]=0,\ q(L)=0,\\ 0, & \text{otherwise},\end{cases}$$

from a basis for the 6-dimensional $\mathbf{F}_2$-vector space $[H_+\#L(i,2n-i)]$ for i odd and $[H_+\#L(i,2n-i)]\times[(H_+\#L(i,2n-i))^D]$ for i even. Here, v_1 is one of the two nonzero vectors of $V^{\mathbf{A}(V)}$ that are not D-invariant when i is odd and the nonzero vector of $V^{\mathbf{A}(V)}$ when i is even, where $V=H_+\#L(i,2n-i)$.

The quadratic function on $H_-\#L(i,n-i) \in \mathrm{Ob}(\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R})))$, $1 \leq i \leq [n/2]$, $q(v_1,v_2,v_3) = v_1^2 + v_1v_2 + v_2^2$, has automorphism group

$$O(q) = \begin{pmatrix} O^-(2,\mathbf{F}_2) & * \\ 0 & 1 \end{pmatrix}$$

of order $|O^-(2,\mathbf{F}_2)| \cdot 2^2 = 24$.

2.42. Proposition. *$H_-\#L(i,n-i) \neq (H_-\#L(i,n-i))^D$ for all $n \geq 2$. The Quillen automorphism group $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_-\#L(i,n-i)) = O(q)$ has order 24 and the dimension of the space of equivariant maps is* $\dim[H_-\#L(i,n-i)] = 1$.

Proof. $H_-\#L(i,n-i) \subset \mathrm{PSL}(4n,\mathbf{R})$ is the quotient of

$$\begin{aligned} G &= 2_-^{1+2} \times 2 \\ &= \Big\langle \mathrm{diag}\Big(\begin{pmatrix} 0 & -R \\ R & 0\end{pmatrix},\dots,\begin{pmatrix} 0 & -R \\ R & 0\end{pmatrix}\Big), \\ &\qquad \mathrm{diag}\Big(\begin{pmatrix} 0 & -T \\ T & 0\end{pmatrix},\dots,\begin{pmatrix} 0 & -T \\ T & 0\end{pmatrix}\Big), \\ &\qquad \mathrm{diag}\Big(\overbrace{\begin{pmatrix} -E & 0 \\ 0 & -E\end{pmatrix},\dots,\begin{pmatrix} -E & 0 \\ 0 & -E\end{pmatrix}}^{i},\overbrace{\begin{pmatrix} E & 0 \\ 0 & E\end{pmatrix},\dots,\begin{pmatrix} E & 0 \\ 0 & E\end{pmatrix}}^{n-i}\Big)\Big\rangle \\ &= \langle g_1,g_2,g_3\rangle \subset \mathrm{SL}(4n,\mathbf{R}). \end{aligned}$$

The centralizer of G in $\mathrm{GL}(4n,\mathbf{R})$ is contained in the centralizer of its subgroup 2_-^{1+2}, which is contained in $\mathrm{SL}(4n,\mathbf{R})$ [24, Example 5.4.(7)]. Observe that:

- $\mathbf{A}(\mathrm{SL}(4,\mathbf{R}))(2_-^{1+2}) \cong O(q)$.
- Conjugation with

$$\mathrm{diag}\Big(\overbrace{\begin{pmatrix} T & 0 \\ 0 & T\end{pmatrix},\dots,\begin{pmatrix} T & 0 \\ 0 & T\end{pmatrix}}^{i},\overbrace{\begin{pmatrix} E & 0 \\ 0 & E\end{pmatrix},\dots,\begin{pmatrix} E & 0 \\ 0 & E\end{pmatrix}}^{n-i}\Big)$$

  induces the automorphism $(g_1,g_2,g_3) \overset{\phi_1}{\mapsto} (g_1g_3,g_2,g_3)$.
- Conjugation with

$$\mathrm{diag}\Big(\overbrace{\begin{pmatrix} R & 0 \\ 0 & R\end{pmatrix},\dots,\begin{pmatrix} R & 0 \\ 0 & R\end{pmatrix}}^{i},\overbrace{\begin{pmatrix} E & 0 \\ 0 & E\end{pmatrix},\dots,\begin{pmatrix} E & 0 \\ 0 & E\end{pmatrix}}^{n-i}\Big)$$

  induces the automorphism $(g_1,g_2,g_3) \overset{\phi_2}{\mapsto} (g_1,g_2g_3,g_3)$.

- When $i = n/2$, conjugation with $\left(\begin{smallmatrix} 0 & E \\ E & 0 \end{smallmatrix}\right)$ induces the automorphism $(g_1, g_2, g_3) \overset{\phi}{\mapsto} (g_1, g_2, -g_3)$.

Consider the automorphism groups

$$\begin{aligned}\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G) \subset \mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G) \subset \mathrm{Out}(G) \to O(q) \\ \subset \mathrm{Aut}(H_-\#L(i,n-i))\end{aligned}$$

where the outer automorphism group has order 48. Note that the automorphism ϕ is in the kernel of the homomorphism $\mathrm{Out}(G) \to O(q)$ induced by the abelianization $G \to H_-\#L(i,n-i)$. Using the above observations we see that $\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G)$ maps onto $O(q)$. Thus the Quillen automorphism group $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$ has order 48 or 24. When n is even and $i = n/2$, the automorphism ϕ is in $\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G)$, and when $i < n/2$, ϕ is not in $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$ as it does not preserve trace. Thus

$$|\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)| = \begin{cases} 48, & i = n/2, \\ 24, & i < n/2. \end{cases}$$

In any case, the group $\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G)$ equals $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$ so that $H_-\#L(i,n-i) \neq (H_-\#L(i,n-i))^D$ [24, Lemma 5.2]. According to *magma*, the dimension $\dim[H_-\#L(i,n-i)]$ equals 1. ∎

The $\mathbf{F}_2\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_-\#L(i,n-i))$-linear map $\{df_-\}$ given by

$$df_-[P > L] = \begin{cases} L, & P = H_-, \\ 0, & \text{otherwise}, \end{cases} \tag{2.43}$$

is a basis for the 1-dimensional $\mathbf{F}_2$-vector space $[H_-\#L(i,n-i)]$.

The quadratic function $q(v_1, v_2, v_3, v_4) = v_1^2 + v_2v_3$ has automorphism group

$$O(q) = \begin{pmatrix} \mathrm{Sp}(2,\mathbf{F}_2) & * \\ 0 & 1 \end{pmatrix},$$

$$\mathrm{Sp}(2,\mathbf{F}_2) \cong \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \subset \mathrm{GL}(3,\mathbf{F}_2),$$

of order 48.

2.44. Proposition. *The 4-dimensional object $V_0\#L(i,n-i)$, $1 \leq i \leq [n/2]$, of the category $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))$ satisfies*

$$V_0\#L(i,n-i) \neq (V_0\#L(i,n-i))^D.$$

It contains the objects V_0, $H_+\#L(2i,2n-2i)$, and $H_-\#L(i,n-i)$. The automorphism group $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(V_0\#L(i,n-i))$ equals $O(q)$ and the dimension of the space of equivariant maps is $\dim[V_0\#L(i,n-i)] = 5$.

Proof. $V_0\#L(i,n-i)\subset \mathrm{PSL}(4n,\mathbf{R})$ is [24, 5.7] the quotient of

$$G=2^{1+2}_{\pm}\circ 4\times 2=\Big\langle \operatorname{diag}\Big(\begin{pmatrix}0&-E\\E&0\end{pmatrix},\dots,\begin{pmatrix}0&-E\\E&0\end{pmatrix}\Big),$$

$$\operatorname{diag}\Big(\begin{pmatrix}R&0\\0&R\end{pmatrix},\dots,\begin{pmatrix}R&0\\0&R\end{pmatrix}\Big),\operatorname{diag}\Big(\begin{pmatrix}T&0\\0&T\end{pmatrix},\dots,\begin{pmatrix}T&0\\0&T\end{pmatrix}\Big),$$

$$\operatorname{diag}\Big(\overbrace{\begin{pmatrix}-E&0\\0&-E\end{pmatrix},\dots,\begin{pmatrix}-E&0\\0&-E\end{pmatrix}}^{i},\overbrace{\begin{pmatrix}E&0\\0&E\end{pmatrix},\dots\begin{pmatrix}E&0\\0&E\end{pmatrix}}^{n-i}\Big)\Big\rangle$$

$$=\langle g_1,g_2,g_3,g_4\rangle\subset \mathrm{SL}(4n,\mathbf{R}).$$

The centralizer of G in $\mathrm{GL}(4n,\mathbf{R})$ is contained in the centralizer of its subgroup 2^{1+2}_{-}, which is contained in $\mathrm{SL}(4n,\mathbf{R})$ [24, Example 5.4.(7)]. Observe that:

- $\mathbf{A}(\mathrm{SL}(4,\mathbf{R}))(2^{1+2}_{\pm}\circ 4)=\mathrm{Out}(G)\cong\mathrm{Out}(C_4)\times\mathrm{Sp}(2,\mathbf{F}_2)$ [24, Example 5.4(5)].
- Conjugation with

$$\operatorname{diag}\Big(\overbrace{\begin{pmatrix}0&E\\E&0\end{pmatrix},\dots,\begin{pmatrix}0&E\\E&0\end{pmatrix}}^{i},\overbrace{\begin{pmatrix}E&0\\0&E\end{pmatrix},\dots,\begin{pmatrix}E&0\\0&E\end{pmatrix}}^{n-i}\Big)$$

  induces the automorphism $(g_1,g_2,g_3,g_4)\overset{\phi_1}{\mapsto}(g_1g_4,g_2,g_3,g_4)$.
- Conjugation with

$$\operatorname{diag}\Big(\overbrace{\begin{pmatrix}T&0\\0&T\end{pmatrix},\dots,\begin{pmatrix}T&0\\0&T\end{pmatrix}}^{i},\overbrace{\begin{pmatrix}E&0\\0&E\end{pmatrix},\dots,\begin{pmatrix}E&0\\0&E\end{pmatrix}}^{n-i}\Big)$$

  induces the automorphism $(g_1,g_2,g_3,g_4)\overset{\phi_2}{\mapsto}(g_1,g_2g_4,g_3,g_4)$.
- Conjugation with

$$\operatorname{diag}\Big(\overbrace{\begin{pmatrix}R&0\\0&R\end{pmatrix},\dots,\begin{pmatrix}R&0\\0&R\end{pmatrix}}^{i},\overbrace{\begin{pmatrix}E&0\\0&E\end{pmatrix},\dots,\begin{pmatrix}E&0\\0&E\end{pmatrix}}^{n-i}\Big)$$

  induces the automorphism $(g_1,g_2,g_3,g_4)\overset{\phi_3}{\mapsto}(g_1,g_2,g_3g_4,g_4)$.
- Conjugation with $\operatorname{diag}\big(\left(\begin{smallmatrix}0&E\\E&0\end{smallmatrix}\right),\dots,\left(\begin{smallmatrix}0&E\\E&0\end{smallmatrix}\right)\big)$ induces the automorphism $(g_1,g_2,g_3,g_4)\overset{\phi_4}{\mapsto}(-g_1,g_2,g_3g_4)$.

- When $i = n/2$, conjugation with $\left(\begin{smallmatrix} 0 & E \\ E & 0 \end{smallmatrix}\right) \in \mathrm{SL}(4n, \mathbf{R})$ induces the automorphism $(g_1, g_2, g_3, g_4) \overset{\phi_5}{\mapsto} (g_1, g_2, g_3, -g_4)$.

Consider the automorphism groups

$$\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G) \subset \mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G) \subset \mathrm{Out}(G) \to O(q) \subset \mathrm{Aut}(V_0\#L(i,n-i)),$$

where the outer automorphism group has order 196 and $O(q)$ has order 48. Note that the automorphism ϕ_4 of order 2 is in the kernel of the homomorphism $\mathrm{Out}(G) \to O(q)$ induced by the abelianization $G \to V_0\#L(i,n-i)$. Using the above observations we see that $\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G)$ maps onto $O(q)$ with a kernel of order at least 2. Thus the Quillen automorphism group $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$ has order 192 or 96. When n is even and $i = n/2$, the automorphism ϕ_5 is in $\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G)$, and when $i < n/2$, ϕ_5 is not in $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$ as it does not preserve trace. Thus

$$|\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)| = \begin{cases} 192, & i = n/2, \\ 96, & i < n/2. \end{cases}$$

In any case, the group $\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G)$ equals $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$ so that $V_0\#L(i,n-i) \neq (V_0\#L(i,n-i))^D$ [24, Lemma 5.2]. According to *magma*, $\dim[V_0\#L(i,n-i)] = 5$. ∎

The five $\mathbf{F}_2\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(V_0\#L(i,n-i))$-linear maps

$$\{ddf_{+L(2i,2n-2i)}, ddf_{0L(2i,2n-2i)}, df_{0L(2i,2n-2i)}, ddf_{-L(i,n-i)}, df_{0V_0}\} \tag{2.45}$$

given by

$$ddf_{+L(2i,2n-2i)}[V > P > L] = \begin{cases} L, & V = H_+\#L(2i,2n-2i), P = H_+, \\ 0, & \text{otherwise}, \end{cases}$$

$$ddf_{0L(2i,2n-2i)}[V > P > L] = \begin{cases} P^{\mathbf{A}(P)}, & V = H_+\#L(2i,2n-2i),\ P = H_+, q(L) = 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$df_{0L(2i,2n-2i)}[V > P > L] = \begin{cases} V^{\mathbf{A}(V)}, & V = H_+\#L(2i,2n-2i),\ [P,P] = 0,\ q(L) = 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$ddf_{-L(i,n-i)}[V > P > L] = \begin{cases} L, & V = H_-\#L(i,n-i), P = H_-, \\ 0, & \text{otherwise}, \end{cases}$$

$$df_{0V_0}[V > P > L] = \begin{cases} V^{\mathbf{A}(V)}, & V = V_0, [P,P] = 0, q(L) = 0, \\ 0, & \text{otherwise} \end{cases}$$

constitute a basis for $[V_0\#L(i,n-i)]$.

2.46. LEMMA. *The* 4*-dimensional object* $H_+\#P(1,i-1,2n-i,0)$, $2 < i \le n$, *of the category* $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))$, $n > 2$, *satisfies* $H_+\#P(1,i-1,2n-i,0) = (H_+\#P(1,i-1,2n-i,0))^D$. *It contains the* 3*-dimensional objects*

$$H_+\#\begin{cases} L(1,2n-1), L(i-1,2n-i+1), L(i-1,2n-i+1)^D, L(i,2n-i), & i \text{ odd},\\ L(1,2n-1), L(i-1,2n-i+1), L(i,2n-i), L(i,2n-i)^D, & i \text{ even}.\end{cases}$$

Its Quillen automorphism group is

$$\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_+\#P(1,i-1,2n-i,0))$$
$$= \begin{cases} \left\langle \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix} 1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix} 1&0&1&1\\0&1&1&1\\0&0&1&0\\0&0&0&1\end{pmatrix} \right\rangle, & i > 2 \text{ odd},\\ \left\langle \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix} 1&0&0&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix} 1&0&1&1\\0&1&1&1\\0&0&1&0\\0&0&0&1\end{pmatrix} \right\rangle, & i > 2 \text{ even},\end{cases}$$

of order 16*. The space of equivariant maps has dimension*

$$\dim[H_+\#P(1,i-1,2n-i,0)] = 16.$$

Proof. $H_+\#P(1,i-1,2n-i) \subset \mathrm{PSL}(4n,\mathbf{R})$ is (2.53) the quotient of

$$\begin{aligned} G = \langle &\mathrm{diag}(R,\ldots,R), \mathrm{diag}(T,\ldots,T),\\ &\mathrm{diag}(E,\overbrace{-E,\ldots,-E}^{i-1},\overbrace{E,\ldots,E}^{2n-i}), \mathrm{diag}(E,\overbrace{E,\ldots,E}^{i-1},\overbrace{-E,\ldots,-E}^{2n-i})\rangle\\ = \langle &g_1,g_2,g_3,g_4\rangle \subset \mathrm{SL}(4n,\mathbf{R}).\end{aligned}$$

The centralizer of G in $\mathrm{GL}(4n,\mathbf{R})$ is contained in the centralizer of its subgroup 2_+^{1+2}, which is contained in $\mathrm{SL}(4n,\mathbf{R})$ [24, Example 5.4(6)]. This means [24, (5.3)] that the elements of the automorphism groups $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$ and $\mathbf{A}(\mathrm{PGL}(4n,\mathbf{R}))(H_+\#P(1,i-1,2n-i,0))$ have a well-defined sign. The Quillen automorphism group is contained in the group $\left(\begin{smallmatrix} O^+(2,\mathbf{F}_2) & *\\ 0 & E\end{smallmatrix}\right)$ of order $2^5 = 32$. Observe that

- R and T are conjugate in $\mathrm{GL}(2,\mathbf{R})$ so that the automorphism $(g_1,g_2,g_3,g_4) \overset{\phi_1}{\mapsto} (g_2,g_1,g_3,g_4)$ is in the Quillen automorphism group and has sign $+1$.

- Conjugation with $\mathrm{diag}(E,\overbrace{E,\dots,E}^{i-1},\overbrace{T,\dots,T}^{2n-i})$ induces the automorphism $(g_1,g_2,g_3,g_4)\overset{\phi_2}{\mapsto}(g_1g_4,g_2,g_3,g_4)$ of sign $(-1)^i$.
- Conjugation with $\mathrm{diag}(E,\overbrace{T,\dots,T}^{i-1},\overbrace{E,\dots,E}^{2n-i})$ induces the automorphism $(g_1,g_2,g_3,g_4)\overset{\phi_3}{\mapsto}(g_1g_3,g_2,g_3,g_4)$ of sign $-(-1)^i$.
- Conjugation with $\mathrm{diag}(E,\overbrace{E,\dots,E}^{i-1},\overbrace{R,\dots,R}^{2n-i})$ induces the automorphism $(g_1,g_2,g_3,g_4)\overset{\phi_4}{\mapsto}(g_1,g_2g_4,g_3,g_4)$ of sign $(-1)^i$.
- Conjugation with $\mathrm{diag}(E,\overbrace{R,\dots,R}^{i-1},\overbrace{E,\dots,E}^{2n-i})$ induces the automorphism $(g_1g_3,g_2,g_3,g_4)\overset{\phi_5}{\mapsto}(g_1,g_2g_3,g_3,g_4)$ of sign $-(-1)^i$.
- Conjugation with $\mathrm{diag}(E,\overbrace{RT,\dots,RT}^{i-1},\overbrace{RT,\dots,RT}^{2n-i})$ induces the automorphism given by $(g_1,g_2,g_3,g_4)\overset{\phi_6}{\mapsto}(g_1g_3g_4,g_2g_3g_4,g_3,g_4)$ of sign $+1$.

It follows that $N_{\mathrm{GL}(4n,\mathbf{R})}(G)\not\subset \mathrm{SL}(4n,\mathbf{R})$ as this normalizer contains elements of negative determinant regardless of the parity of i. Also, the automorphism group $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_+\#P(1,i-1,2n-i,0))$ is generated by (the automorphisms induced by) ϕ_1, ϕ_2, ϕ_4, and ϕ_6 when i is even, and ϕ_1, ϕ_3, ϕ_5, and ϕ_6 when i is odd. ■

The fourteen $\mathbf{F}_2\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_+\#P(1,i-1,2n-1))$-linear maps

$$\begin{aligned}\{&ddf_{+L(i-1,2n-i+1)}, ddf^D_{+L(i-1,2n-i+1)}, ddf_{0L(i-1,2n-i+1)},\\ &ddf^D_{0L(i-1,2n-i+10)}, df_{0L(i-1,2n-i+1)}, df^D_{0L(i-1,2n-i+1)},\\ &ddf_{+L(i,2n-i)}, ddf^D_{+L(i,2n-i)}, ddf_{0L(i,2n-i)}, ddf^D_{0L(i,2n-i)},\\ &df_{0L(i,2n-i)}, df^D_{0L(i,2n-i)}, df_{0L(1,2n-1)}, df^D_{0L(1,2n-1)}\}\end{aligned}\tag{2.47}$$

form a partial basis for the 16-dimensional vector space $[H_+\#P(1,i-1,2n-1)]$, $2<i\le n$. For $1<i\le n$ and i odd,

$$ddf_{+L(i,2n-i)}[V>P>L]=\begin{cases}L, & V=H_+\#L(i,2n-i),\ P=H_+,\\ 0, & \text{otherwise,}\end{cases}$$

$$ddf^D_{+L(i,2n-i)}[V>P>L]=\begin{cases}L, & V=H_+\#L(i,2n-i),\ P=H^D_+,\\ 0, & \text{otherwise,}\end{cases}$$

$$ddf_{0L(i,2n-i)}[V>P>L]\\ =\begin{cases}P^{\mathbf{A}(P)}, & V=H_+\#L(i,2n-i),\ P=H_+,\ q(L)=0,\\ 0, & \text{otherwise,}\end{cases}$$

$$ddf^D_{0L(i,2n-i)}[V > P > L] = \begin{cases} P^{\mathbf{A}(P)}, & V = H_+\#L(i,2n-i),\ P = H_+^D,\ q(L) = 0,\\ 0, & \text{otherwise,}\end{cases}$$

$$df_{01L(i,2n-i)}[V > P > L] = \begin{cases} V \cap O_1, & V = H_+\#L(i,2n-i),\ [P,P] = 0,\ q(L) = 0,\\ 0, & \text{otherwise,}\end{cases}$$

$$df^D_{01L(i,2n-i)}[V > P > L] = \begin{cases} V \cap O_2, & V = H_+\#L(i,2n-i),\ [P,P] = 0,\ q(L) = 0,\\ 0, & \text{otherwise,}\end{cases}$$

where (in the last two formulas), O_1 and O_2 are the two orbits of length 2 for the action of $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_+\#P(1,i-1,2n-i,0))$ on $H_+\#P(1,i-1, 2n-i,0)$. Each of the hyperplanes isomorphic to $V = H_+\#L(i,2n-i)$ contains precisely one vector v_1 from O_1 and one vector v_2 from O_2 and $\{v_1,v_2\}$ is a basis for the fixed point group $V^{\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(V)}$. For $1 < i \le n$ and i even,

$$ddf_{+L(i,2n-i)}[V > P > L] = \begin{cases} L, & V = H_+\#L(i,2n-i),\ P = H_+,\\ 0, & \text{otherwise,}\end{cases}$$

$$ddf^D_{+L(i,2n-i)}[V > P > L] = \begin{cases} L, & V = (H_+\#L(i,2n-i))^D,\ P = (H_+)^D,\\ 0, & \text{otherwise,}\end{cases}$$

$$ddf_{0L(i,2n-i)}[V > P > L] = \begin{cases} P^{\mathbf{A}(P)}, & V = H_+\#L(i,2n-i),\ P = H_+,\ q(L) = 0,\\ 0, & \text{otherwise,}\end{cases}$$

$$ddf^D_{0L(i,2n-i)}[V > P > L] = \begin{cases} P^{\mathbf{A}(P)}, & V = (H_+\#L(i,2n-i))^D,\ P = (H_+)^D,\ q(L) = 0,\\ 0, & \text{otherwise,}\end{cases}$$

$$df_{0L(i,2n-i)}[V > P > L] = \begin{cases} V^{\mathbf{A}(V)}, & V = H_+\#L(i,2n-i),\ [P,P] = 0,\ q(L) = 0,\\ 0, & \text{otherwise,}\end{cases}$$

$$df^D_{0L(i,2n-i)}[V > P > L] = \begin{cases} V^{\mathbf{A}(V)}, & V = (H_+\#L(i,2n-i))^D,\ [P,P] = 0,\ q(L) = 0,\\ 0, & \text{otherwise.}\end{cases}$$

2.48. Lemma. *The 4-dimensional object $H_+\#P(1,1,2,0)$ of the category $\mathbf{A}(\mathrm{PSL}(8,\mathbf{R}))$ satisfies $H_+\#P(1,1,2,0) = (H_+\#P(1,1,2,0))^D$. It contains the 3-dimensional objects*

$$H_+\#L(1,3), \quad H_+\#L(2,2), \quad (H_+\#L(2,2))^D.$$

Its Quillen automorphism group is

$$\mathbf{A}(\mathrm{PSL}(8,\mathbf{R}))(H_+\#P(1,1,2,0)) = \left\langle \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&1&1\\0&1&1&1\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix} \right\rangle$$

of order 32, *and* $\dim[H_+\#P(1,i-1,2n-i,0)] = 8$.

Proof. The proof is similar to that of 2.46. The elementary abelian 2-group $H_+\#P(1,1,2,0) \subset \mathrm{PSL}(8,\mathbf{R})$ is the quotient of the group

$$G = \langle \mathrm{diag}(R,R,R,R), \mathrm{diag}(T,T,T,T), \mathrm{diag}(E,-E,E,E), \mathrm{diag}(E,E,-E,-E)\rangle \subset \mathrm{SL}(8,\mathbf{R})$$

The extra element of $\mathbf{A}(\mathrm{PSL}(8,\mathbf{R}))(H_+\#P(1,1,2,0))$ is induced by conjugation with the matrix $\mathrm{diag}\left(\left(\begin{smallmatrix}0&E\\E&0\end{smallmatrix}\right),\left(\begin{smallmatrix}E&0\\0&E\end{smallmatrix}\right)\right) \in \mathrm{SL}(8,\mathbf{R})$. According to *magma*, $\dim[H_+\#P(1,1,2,0)] = 8$. ■

The eight $\mathbf{F}_2\mathbf{A}(\mathrm{PSL}(8,\mathbf{R}))(H_+\#P(1,1,2,0))$-linear maps

$$\{ddf_{+L(2,2)}, ddf^D_{+L(2,2)}, ddf_{0L(2,2)}, ddf^D_{0L(2,2)}, df_{0L(2,2)}, df^D_{0L(2,2)}, df_{01L(1,3)}, df^D_{01L(1,3)}\} \tag{2.49}$$

from a basis for the vector space $[H_+\#P(1,1,2,0)]$.

We are now ready to describe the differentials d^1 and d^2 in Oliver's cochain complex (2.33) for computing the higher limits of the functor

$$\pi_1(BZC_{\mathrm{PSL}(4n,\mathbf{R})}(V)) = V$$

on the category $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))$. The $6\times(6n+2[n/2]+8)$ matrix for d^1 is of the following form (shown here for $n=3$):

	$[H_+\#L(1,5)]$	$[H_+\#L(2,4)]\times[H_+\#L(2,4)]^D$	$H_+\#L(3,3)$
$[H_+]$	$(A\quad 0)$	$(A\quad 0)$	$(A\quad 0)$
$[H_+]^D$	$(0\quad A)$	$(0\quad A)$	$(0\quad A)$
$[H_-]$			
$[H_-]^D$			

$[H_-\#L(1,1)]\times[H_-\#L(1,1)]^D$	$[V_0]\times[V_0]^D$	
	$(H\quad 0)$	$[H_+]$
	$(0\quad H)$	$[H_+]^D$
$(1\quad 0)$	$(B\quad 0)$	$[H_-]$
$(0\quad 1)$	$(0\quad B)$	$[H_-]^D$

where

$$A=\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix},\qquad H=\begin{pmatrix}1&0&0&0\\0&1&0&0\end{pmatrix},\qquad B=\begin{pmatrix}0&0&1&0\end{pmatrix},$$

is injective so $\lim^1=0$. Exactness is thus equivalent to

$$\dim(\operatorname{im} d^2)\geq 6n+2[n/2]+2.$$

We shall show this by mapping the $n+[n/2]+2[n/2]+2$ objects of dimension 3,

$$H_+\#L(i,2n-i),(H_+\#L(i,2n-i))^D\quad (i\text{ even}),\qquad 1\leq i\leq n,$$
$$H_-\#L(i,n-i),(H_-\#L(i,n-i))^D,\quad 1\leq i\leq [n/2],\qquad V_0,\ V_0^D,$$

of $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))$ to the $n-2+2[n/2]$ objects of dimension 4,

$$H_+\#P(1,i-1,2n-i,0),\qquad 2<i\leq n,$$
$$V_0\#L(n-i,i),\ (V_0\#L(n-i,i))^D,\qquad 1\leq i\leq [n/2],$$

for $n>2$, and to

$$H_+\#P(1,1,2,0),\ V_0\#L(1,1),\ (V_0\#L(1,1))^D$$

when $n=2$. The $(6n+2[n/2]+8)\times(16(n-2)+10[n/2])$ matrix for d^2 (shown here for $n=5$) is

	$[H_+\#P(1,2,7)]$	$[H_+\#P(1,3,6)]$	$[H_+\#P(1,4,5)]$
$[H_+\#L(1,9)]$	$(A\ \ A\ \ B)$	$(A\ \ A\ \ B)$	$(A\ \ A\ \ B)$
$[H_+\#L(2,8)]\times[H_+\#L(2,8)]^D$	$(E\ \ 0\ \ 0)$		
$[H_+\#L(3,7)]$	$(0\ \ E\ \ 0)$	$(E\ \ 0\ \ 0)$	
$[H_+\#L(4,6)]\times[H_+\#L(4,6)]^D$		$(0\ \ E\ \ 0)$	$(E\ \ 0\ \ 0)$
$[H_+\#L(5,5)]$			$(0\ \ E\ \ 0)$
$[H_-\#L(1,4)]\times[H_-\#L(1,4)]^D$			
$[H_-\#L(2,3)]\times[H_-\#L(2,3)]^D$			
$[V_0]\times[V_0]^D$			

$[V_0\#L(1,4)]$	$[V_0\#L(1,4)]^D$	$[V_0\#L(2,3)]$	$[V_0\#L(2,3)]^D$	
				$[H_+\#L(1,9)]$
$\begin{pmatrix}H\\0\end{pmatrix}$	$\begin{pmatrix}0\\H\end{pmatrix}$			$[H_+\#L(2,8)]\times[H_+\#L(2,8)]^D$
				$[H_+\#L(3,7)]$
		$\begin{pmatrix}H\\0\end{pmatrix}$	$\begin{pmatrix}0\\H\end{pmatrix}$	$[H_+\#L(4,6)]\times[H_+\#L(4,6)]^D$
				$[H_+\#L(5,5)]$
$\begin{pmatrix}L\\0\end{pmatrix}$	$\begin{pmatrix}0\\L\end{pmatrix}$			$[H_-\#L(1,4)]\times[H_-\#L(1,4)]^D$
	$\begin{pmatrix}L\\0\end{pmatrix}$	$\begin{pmatrix}0\\L\end{pmatrix}$		$[H_-\#L(2,3)]\times[H_-\#L(2,3)]^D$
$\begin{pmatrix}K\\0\end{pmatrix}$	$\begin{pmatrix}0\\K\end{pmatrix}$	$\begin{pmatrix}K\\0\end{pmatrix}$	$\begin{pmatrix}0\\K\end{pmatrix}$	$[V_0]\times[V_0]^D$

where

$$A=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\quad B=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0\end{pmatrix},\quad K=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},$$

$$H=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\end{pmatrix},\quad L=\begin{pmatrix}0&0&0&1&0\end{pmatrix},$$

while E is the 6×6 unit matrix and 0 the zero matrix. These matrices are given with respect to the bases (2.39), (2.41), (2.43), (2.47), (2.45).

The case $n=2$ of $\mathrm{PSL}(8,\mathbf{R})$ is special. Part of the matrix for d^2 is the 22×18 matrix

	$[H_+\#P(1,1,2,0)]$	$[V_0\#L(1,1)]$	$[V_0\#L(1,1)]^D$
$[H_+\#L(1,3)]$	$\begin{pmatrix}A & B\end{pmatrix}$		
$[H_+\#L(2,2)]\times[H_+\#L(2,2)]^D$	$\begin{pmatrix}E & 0\end{pmatrix}$	$\begin{pmatrix}H\\0\end{pmatrix}$	$\begin{pmatrix}0\\H\end{pmatrix}$
$[H_-\#L(1,1)]\times[H_-\#L(1,1)]^D$		$\begin{pmatrix}L\\0\end{pmatrix}$	$\begin{pmatrix}0\\L\end{pmatrix}$
$[V_0]\times[V_0]^D$		$\begin{pmatrix}K\\0\end{pmatrix}$	$\begin{pmatrix}0\\K\end{pmatrix}$

where now

$$B=\begin{pmatrix}0&0\\0&0\\0&0\\0&0\\1&0\\0&1\end{pmatrix},$$

while E is the 6×6 unit matrix and 0 the zero matrix. As (partial) bases we use the ordered sets (2.41), (2.43), (2.39), (2.49), (2.45). This matrix has rank 16.

2.50. Corollary. *The partial differential*

$$\prod_{\substack{1\le i\le n\\ i\ odd}}[H_+\#L(i,2n-i)]\times\prod_{\substack{1\le i\le n\\ i\ even}}[H_+\#L(i,2n-i)]\times[H_+\#L(i,2n-i)]^D$$

$$\times \prod_{1\le i\le[n/2]} [H_-\#L(i,n-i)] \times [H_-\#L(i,n-i)]^D \times [V_0]\times[V_0]^D$$
$$\xrightarrow{d^2} \prod_{2<i\le n}[H_+\#P(1,i-1,2n-i,0)] \times \prod_{1\le i\le[n/2]}[V_0\#L(i,n-i)]$$

has rank $6n+2[n/2]+2$.

Proof. By now we know a matrix for this linear map so we simply check its rank. ∎

Proof of Lemma 2.32. For π_2 use the fact that it is trivial on the objects with $[\,,\,]\neq 0$. ∎

7. The category $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))^{[\,,\,]\neq 0}_{\le 4}$. We shall need information about all objects of $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))^{[\,,\,]\neq 0}$ of rank ≤ 3 and some objects of rank 4. If $V\subset \mathrm{PSL}(4n,\mathbf{R})$ is a nontoral elementary abelian 2-group with nontrivial inner product then its preimage $V^*\subset \mathrm{SL}(4n,\mathbf{R})$ is $P\times R(V)$ or $(C_4\circ P)\times R(V)$, where P is an extraspecial 2-group, $C_4\circ P$ a generalized extraspecial 2-group, and $\mho_1(V^*)=\langle -E\rangle$ (2.8). We manufacture all oriented real representations of these product groups as direct sums of tensor products of irreducible representations of the factors [24, 5.6].

2.51. *Rank two objects with nontrivial inner product.* The category $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))$ contains up to isomorphism four rank two objects with nontrivial inner product, $H_\pm$ and $H_\pm^D$. The elementary abelian 2-group $H_\pm\subset \mathrm{PSL}(4n,\mathbf{R})$ is the quotient of the extraspecial 2-group $2^{1+2}_\pm\subset \mathrm{SL}(4n,\mathbf{R})$ with $\mho_1(2^{1+2}_\pm)=\langle -E\rangle$ described in [24, Example 5.4(6)–(7)]. Their centralizers [32, Proposition 4] in $\mathrm{SL}(4n,\mathbf{R})$ and $\mathrm{PSL}(4n,\mathbf{R})$ are

$$C_{\mathrm{SL}(4n,\mathbf{R})}(2^{1+2}_+)=\mathrm{GL}(2n,\mathbf{R}),\quad C_{\mathrm{PSL}(4n,\mathbf{R})}(H_+)=H_+\times \mathrm{PGL}(2n,\mathbf{R}),$$
$$C_{\mathrm{SL}(4n,\mathbf{R})}(2^{1+2}_-)=\mathrm{GL}(n,\mathbf{H}),\quad C_{\mathrm{PSL}(4n,\mathbf{R})}(H_-)=H_-\times \mathrm{PGL}(n,\mathbf{H}),$$

where H_+ and H_- are hyperbolic planes with quadratic functions $q_+(v_1,v_2)=v_1v_2$ and $q_-(v_1,v_2)=v_1^2+v_1v_2+v_2^2$ [24, 5.5], respectively. In the first case, for instance, the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{PGL}(2n,\mathbf{R}) & \longrightarrow & C_{\mathrm{PSL}(4n,\mathbf{R})}(H_+) & \longrightarrow & H_+^\vee & \longrightarrow & 0\\ & & & & \uparrow & \overset{\cong}{\nearrow}{\scriptstyle [\cdot,\cdot]} & & & \\ & & & & H_+ & & & & \end{array}$$

gives a central section of the short exact sequence from [28, 5.11]. Alternatively, $C_{\mathrm{PSL}(4n,\mathbf{R})}(H_+)=H_+\times \mathrm{PGL}(2n,\mathbf{R})=V_+\times(\mathrm{PSL}(2n,\mathbf{R})\rtimes C_2)$.

2.52. *Rank three objects with nontrivial inner product.* Let V be a rank three object of $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))$ with nontrivial inner product. Then V or V^D

is isomorphic to $H_+\#L(i,2n-i)$ $(1\leq i\leq n)$, $H_-\#L(i,n-i)$ $(1\leq i\leq [n/2])$ or V_0. Furthermore, $H_+\#L(i,2n-i)\subset \mathrm{PSL}(4n,\mathbf{R})$ is defined to be the quotient of

$$\langle \mathrm{diag}(R,\dots,R), \mathrm{diag}(T,\dots,T), \mathrm{diag}(\overbrace{-E,\dots,-E}^{i},\overbrace{E,\dots,E}^{2n-i})\rangle,$$
$$R=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad T=\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

isomorphic to $2^{1+2}_+\times C_2\subset \mathrm{SL}(4n,\mathbf{R})$, and $H_-\#L(i,n-i)\subset \mathrm{PSL}(4n,\mathbf{R})$ to be the quotient of

$$\Big\langle \mathrm{diag}\Big(\begin{pmatrix}0&-R\\R&0\end{pmatrix},\dots,\begin{pmatrix}0&-R\\R&0\end{pmatrix}\Big), \mathrm{diag}\Big(\begin{pmatrix}0&-T\\T&0\end{pmatrix},\dots,\begin{pmatrix}0&-T\\T&0\end{pmatrix}\Big),$$
$$\mathrm{diag}\Big(\overbrace{\begin{pmatrix}-E&0\\0&-E\end{pmatrix},\dots,\begin{pmatrix}-E&0\\0&-E\end{pmatrix}}^{i},\overbrace{\begin{pmatrix}E&0\\0&E\end{pmatrix},\dots,\begin{pmatrix}E&0\\0&E\end{pmatrix}}^{n-i}\Big)\Big\rangle,$$

isomorphic to $2^{1+2}_-\times C_2\subset \mathrm{SL}(4n,\mathbf{R})$. The elementary abelian 2-group $V_0\subset \mathrm{PSL}(4n,\mathbf{R})$ is the quotient of

$$\Big\langle \mathrm{diag}\Big(\begin{pmatrix}R&0\\0&R\end{pmatrix},\dots,\begin{pmatrix}R&0\\0&R\end{pmatrix}\Big), \mathrm{diag}\Big(\begin{pmatrix}T&0\\0&T\end{pmatrix},\dots,\begin{pmatrix}T&0\\0&T\end{pmatrix}\Big),$$
$$\mathrm{diag}\Big(\begin{pmatrix}0&-E\\E&0\end{pmatrix},\dots,\begin{pmatrix}0&-E\\E&0\end{pmatrix}\Big)\Big\rangle,$$

isomorphic to the generalized extraspecial 2-group $C_4\circ 2^{1+2}_\pm\subset \mathrm{SL}(4n,\mathbf{R})$ as described in [24, Example 5.4(5)].

2.53. *Rank four objects with nontrivial inner product.* The following partial census of rank four objects with nontrivial inner product suffices for our purposes. Define the elementary abelian 2-group $H_+\#P(1,i-1,2n-i)\subset \mathrm{PSL}(4n,\mathbf{R})$, $2\leq i\leq n$, to be the quotient of

$$\langle \mathrm{diag}(R,\dots,R), \mathrm{diag}(T,\dots,T),$$
$$\mathrm{diag}(E,\overbrace{-E,\dots,-E}^{i-1},\overbrace{E,\dots,E}^{2n-i}), \mathrm{diag}(E,\overbrace{E,\dots,E}^{i-1},\overbrace{-E,\dots,-E}^{2n-i})\rangle$$
$$\subset \mathrm{SL}(4n,\mathbf{R}).$$

Define $V_0\#L(i,n-i)\subset \mathrm{PSL}(4n,\mathbf{R})$, $1\leq i\leq [n/2]$, to be the quotient of

$$\Big\langle \mathrm{diag}\Big(\begin{pmatrix}0&-E\\E&0\end{pmatrix},\dots,\begin{pmatrix}0&-E\\E&0\end{pmatrix}\Big), \mathrm{diag}\Big(\begin{pmatrix}R&0\\0&R\end{pmatrix},\dots,\begin{pmatrix}R&0\\0&R\end{pmatrix}\Big),$$

$$\mathrm{diag}\left(\begin{pmatrix} T & 0 \\ 0 & T\end{pmatrix}, \dots, \begin{pmatrix} T & 0 \\ 0 & T\end{pmatrix}\right),$$

$$\mathrm{diag}\Bigl(\overbrace{\begin{pmatrix} -E & 0 \\ 0 & -E\end{pmatrix}, \dots, \begin{pmatrix} -E & 0 \\ 0 & -E\end{pmatrix}}^{i}, \overbrace{\begin{pmatrix} E & 0 \\ 0 & E\end{pmatrix}, \dots \begin{pmatrix} E & 0 \\ 0 & E\end{pmatrix}}^{n-i}\Bigr)\Bigr\rangle$$

isomorphic to $C_4 \circ 2^{1+2}_{\pm} \times C_2 \subset \mathrm{SL}(4n, \mathbf{R})$.

2.54. *Centers of centralizers.* For the computations in §6 we need to know the centers of the centralizers for some of the low-dimensional objects of $\mathbf{A}(\mathrm{PSL}(4n, \mathbf{R}))^{[\,,\,]\neq 0}$.

2.55. Proposition. *Let* $V \in \mathrm{Ob}(\mathbf{A}(\mathrm{PSL}(4n, \mathbf{R}))^{[\,,\,]\neq 0})$ *be one of the objects*

- H_+, H_-,
- $H_+\#L(i, 2n-i)$ $(1 \le i \le n)$, $H_-\#L(i, n-i)$ $(1 \le i \le [n/2])$, V_0,
- $H_+\#P(1, i-1, 2n-i, 0)$ $(1 < i \le n)$, $V_0\#L(i, n-i)$ $(1 \le i \le [n/2])$

introduced in 2.51–2.53. *Then* $ZC_{\mathrm{PSL}(4n,\mathbf{R})}(V) = V$.

Proof. The proof is a case-by-case checking.

H_+ and H_-: Since the centralizers of the rank two objects H_+ and H_- are $C_{\mathrm{PSL}(4n,\mathbf{R})}(H_+) = H_+ \times \mathrm{PGL}(2n, \mathbf{R})$ and $C_{\mathrm{PSL}(4n,\mathbf{R})}(H_-) = H_- \times \mathrm{PGL}(n, \mathbf{H})$, the assertion is immediate in this case.

$H_+\#L(i, 2n-i)$ $(1 \le i \le n)$ and $H_+\#P(1, i-1, 2n-i, 0)$ $(1 < i \le n)$: We shall only prove the 2-dimensional case since the 3-dimensional case is similar. The centralizer of $H_+\#L(i, 2n-i)$ is isomorphic to the product of H_+ with the centralizer of $L = L(i, 2n-i)$ in $\mathrm{PGL}(2n, \mathbf{R})$. There is [28, 5.11] a short exact sequence

$$1 \to \frac{\mathrm{GL}(i, \mathbf{R}) \times \mathrm{GL}(2n-i, \mathbf{R})}{\langle -E \rangle} \to C_{\mathrm{PGL}(2n,\mathbf{R})}(L) \to \mathrm{Hom}(L, \langle -E \rangle)_{\varrho} \to 1$$

where the rightmost group consists of all homomorphisms $\phi\colon L \to \langle -E \rangle$ such that ϱ and $\phi \cdot \varrho$ are conjugate representations in $\mathrm{GL}(2n, \mathbf{R})$. By trace considerations, this group is trivial if $i < n$ and of order two if $i = n$. Hence

$$C_{\mathrm{PGL}(2n,\mathbf{R})}(L) = \begin{cases} \dfrac{\mathrm{GL}(i, \mathbf{R}) \times \mathrm{GL}(2n-i, \mathbf{R})}{\langle -E \rangle}, & i < n, \\ \dfrac{\mathrm{GL}(n, \mathbf{R})^2}{\langle -E \rangle} \rtimes \langle C_1 \rangle, & i = n, \end{cases}$$

where $C_1 = \left(\begin{smallmatrix} 0 & E \\ E & 0 \end{smallmatrix}\right)$ is the $2n \times 2n$ matrix that interchanges the two $\mathrm{GL}(n, \mathbf{R})$ factors. In case $i < n$, use [24, Lemma 5.18]. In case $i = n$, the center is [24, Lemma 5.13] the pull-back of the group homomorphisms

$$\frac{\mathrm{GL}(n,\mathbf{R})\times\langle(E,-E)\rangle}{\langle -E\rangle} = \left(\frac{\mathrm{GL}(n,\mathbf{R})^2}{\langle -E\rangle}\right)^{\langle C_1\rangle} \to \mathrm{Aut}\left(\frac{\mathrm{GL}(n,\mathbf{R})^2}{\langle -E\rangle}\right) \leftarrow \langle C_1\rangle,$$

which is $\frac{\mathrm{GL}(1,\mathbf{R})\times\langle(-E,E)\rangle}{\langle -E\rangle} = L$ again.

V_0 and $V_0\#L(i,n-i)$: The object $V_0\subset\mathrm{PSL}(4n,\mathbf{R})$ is the quotient of $G = 4\circ 2^{1+2}_{\pm}\subset\mathrm{SL}(4n,\mathbf{R})$ as described in [24, Example 5.4(5)]. As this representation $\varrho = n(\chi+\overline{\chi})$ is the n-fold sum of an irreducible representation of complex type there are exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \frac{\mathrm{GL}(n,\mathbf{C})}{\langle -E\rangle} & \longrightarrow & C_{\mathrm{PSL}(4n,\mathbf{R})}(V_0) & \longrightarrow & \mathrm{Hom}(G,\langle -E\rangle)_\varrho & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & Z(G)/G' & \longrightarrow & G/G' & \longrightarrow & G/Z(G) & \longrightarrow & 1
\end{array}$$

where the top row is [28, 5.11]. The abelian group $\mathrm{Hom}(G,\langle -E\rangle)_\varrho$, consisting of all homomorphisms $\phi\colon G\to\langle -E\rangle$ such that ϱ and $\phi\cdot\varrho$ are conjugate in $\mathrm{SL}(4n,\mathbf{R})$, equals all of $\mathrm{Hom}(G,\langle -E\rangle) = 2^3$ since conjugation with the first two of the generators from 2.52 and with

$$C_2 = \mathrm{diag}\left(\begin{pmatrix} E & 0 \\ 0 & -E\end{pmatrix},\dots,\begin{pmatrix} E & 0 \\ 0 & -E\end{pmatrix}\right)$$

induces three independent generators. Hence

$$C_{\mathrm{PSL}(4n,\mathbf{R})}(V_0) = \left(\frac{\mathrm{GL}(n,\mathbf{C})}{\langle -E\rangle}\times V_0/V_0^\perp\right)\rtimes\langle C_2\rangle.$$

Note that conjugation with the matrix C_2 induces complex conjugation on $\mathrm{GL}(n,\mathbf{C})$. The center of this semidirect product is [24, Lemma 5.13] the pull-back of the group homomorphisms

$$\frac{\mathrm{GL}(n,\mathbf{R})\circ\langle i\rangle}{\langle -E\rangle}\times V_0/V_0^\perp = \left(\frac{\mathrm{GL}(n,\mathbf{C})}{\langle -E\rangle}\times V_0/V_0^\perp\right)^{\langle C_2\rangle}$$
$$\to \mathrm{Aut}\left(\frac{\mathrm{GL}(n,\mathbf{C})}{\langle -E\rangle}\times V_0/V_0^\perp\right)\leftarrow\langle C_2\rangle,$$

which is

$$\frac{\mathrm{GL}(1,\mathbf{R})\circ\langle i\rangle}{\langle -E\rangle}\times V_0/V_0^\perp = \frac{\langle i\rangle}{\langle -E\rangle}\times V_0/V_0^\perp = V_0.$$

The case of $V_0\#L(i,n-i)$, $1\le i<[n/2]$, is quite similar. The centralizer is

$$C_{\mathrm{PSL}(4n,\mathbf{R})}(V_0\#L(i,n-i)) = \left(\frac{\mathrm{GL}(i,\mathbf{C})\times\mathrm{GL}(n-i,\mathbf{C})}{\langle -E\rangle}\times V_0/V_0^\perp\right)\rtimes\langle C_2\rangle$$

and its center is the pull-back of the homomorphisms

$$\frac{(\mathrm{GL}(i,\mathbf{R})\times\mathrm{GL}(n-i,\mathbf{R}))\circ\langle i\rangle}{\langle -E\rangle}\times V_0/V_0^{\perp}$$
$$= \left(\frac{\mathrm{GL}(i,\mathbf{C})\times\mathrm{GL}(n-i,\mathbf{C})}{\langle -E\rangle}\times V_0/V_0^{\perp}\right)^{\langle C_2\rangle}$$
$$\to \mathrm{Aut}\left(\frac{\mathrm{GL}(i,\mathbf{C})\times\mathrm{GL}(n-i,\mathbf{C})}{\langle -E\rangle}\times V_0/V_0^{\perp}\right)\leftarrow\langle C_2\rangle,$$

which is

$$ZC_{\mathrm{PSL}(4n,\mathbf{R})}(V_0\# L(i,n-i)) = \frac{(\mathrm{GL}(1,\mathbf{R})\times\mathrm{GL}(1,\mathbf{R}))\circ\langle i\rangle}{\langle -E\rangle}\times V_0/V_0^{\perp}$$
$$= 2^2\times V_0/V_0^{\perp} = V_0\times L.$$

If n is even and $i=n/2$, there is a short exact sequence

$$1\to\frac{\mathrm{GL}(n,\mathbf{C})^2}{\langle -E\rangle}\to C_{\mathrm{PSL}(4n,\mathbf{R})}(V_0\times L)\to\mathrm{Hom}(G\times L,\langle -E\rangle)_{\varrho}\to 1$$

where the elementary abelian group on the right is all of $\mathrm{Hom}(G\times L,\langle -E\rangle) = 2^4$. Hence the centralizer satisfies

$$C_{\mathrm{PSL}(4n,\mathbf{R})}(V_0\times L) = \left(\frac{\mathrm{GL}(n,\mathbf{C})^2}{\langle -E\rangle}\times V_0/V_0^{\perp}\right)\rtimes\langle C_1,C_2\rangle$$

where C_2 is as above and C_1 is the $4n\times 4n$ matrix $\left(\begin{smallmatrix}0 & E\\ E & 0\end{smallmatrix}\right)$. The matrix C_2 commutes with $V_0/V_0^{\perp}$ and acts as complex conjugation on $\mathrm{GL}(n,\mathbf{C})^2/\langle -E\rangle$. The matrix C_1 commutes with $V_0/V_0^{\perp}$ and switches the two factors of $\mathrm{GL}(n,\mathbf{C})^2$. The center of the centralizer is the pull-back of the group homomorphisms

$$\frac{\mathrm{GL}(n,\mathbf{R})\circ\langle i\rangle\times\langle (E,-E)\rangle}{\langle -E\rangle}\times V_0/V_0^{\perp} = \left(\frac{\mathrm{GL}(n,\mathbf{C})^2}{\langle -E\rangle}\times V_0/V_0^{\perp}\right)^{\langle C_1,C_2\rangle}$$
$$\to\mathrm{Aut}\left(\frac{\mathrm{GL}(n,\mathbf{C})^2}{\langle -E\rangle}\times V_0/V_0^{\perp}\right)\leftarrow\langle C_1,C_2\rangle,$$

which is

$$ZC_{\mathrm{PSL}(4n,\mathbf{R})}(V_0\times L) = \frac{\mathrm{GL}(1,\mathbf{R})\circ\langle i\rangle\times\langle (E,-E)\rangle}{\langle -E\rangle}\times V_0/V_0^{\perp}$$
$$= 2^2\times V_0/V_0^{\perp} = V_0\times L.$$

$\underline{H_-\# L(i,n-i)}$: As above, we have

$$C_{\mathrm{PSL}(4n,\mathbf{R})}(H_-\times L) = \begin{cases}\dfrac{\mathrm{GL}(i,\mathbf{H})\times\mathrm{GL}(n-i,\mathbf{H})}{\langle -E\rangle}\times H_-, & i<[n/2],\\[2ex] \dfrac{\mathrm{GL}(i,\mathbf{H})^2}{\langle -E\rangle}\rtimes\langle C_1\rangle\times H_-, & n \text{ even and } i=n/2,\end{cases}$$

with center

$$ZC_{\mathrm{PSL}(4n,\mathbf{R})}(H_- \times L) = \frac{\mathrm{GL}(1,\mathbf{R}) \times \mathrm{GL}(1,\mathbf{R})}{\langle -E \rangle} = 2 \times H_- = H_- \times L$$

in case $i \neq n-i$. If n is even and $i = n/2$, then the center is the pull-back of the group homomorphisms

$$\frac{\mathrm{GL}(i,\mathbf{H}) \times \langle (-E,E) \rangle}{\langle -E \rangle} \times H_- = \frac{\mathrm{GL}(i,\mathbf{H})^2}{\langle -E \rangle} \times H_- \\ \to \mathrm{Aut}\left(\frac{\mathrm{GL}(i,\mathbf{H})^2}{\langle -E \rangle} \times H_- \right) \leftarrow \langle C_1 \rangle,$$

which is

$$ZC_{\mathrm{PSL}(4n,\mathbf{R})}(H_- \times L) = \frac{\mathrm{GL}(1,\mathbf{R}) \times \langle (-E,E) \rangle}{\langle -E \rangle} \times H_- = 2 \times H_- = H_- \times L. \ \blacksquare$$

3. THE B-FAMILY

The B-family consists of the matrix groups

$$\mathrm{SL}(2n+1,\mathbf{R}), \qquad n \geq 2,$$

of $2n+1$ by $2n+1$ real matrices of determinant $+1$. When $n=1$ we obtain the 2-compact group $\mathrm{SL}(3,\mathbf{R}) = \mathrm{PGL}(2,\mathbf{C})$ considered in [24, Chapter 3]. The embedding

$$\mathrm{GL}(2n,\mathbf{R}) \to \mathrm{SL}(2n+1,\mathbf{R})\colon A \to \begin{pmatrix} A & 0 \\ 0 & \det A \end{pmatrix}$$

permits us to consider the general linear group $\mathrm{GL}(2n,\mathbf{R})$ as a maximal rank subgroup of $\mathrm{SL}(2n+1,\mathbf{R})$. The maximal torus normalizer for the subgroup $\mathrm{GL}(2n,\mathbf{R})$ is therefore also the maximal torus normalizer for $\mathrm{SL}(2n+1,\mathbf{R})$, $N(\mathrm{SL}(2n+1,\mathbf{R})) = N(\mathrm{GL}(2n,\mathbf{R})$ (2.1), so that these two Lie groups have the same Weyl group, $W(\mathrm{SL}(2n+1,\mathbf{R})) = W(\mathrm{GL}(2n,\mathbf{R})) = \Sigma_2 \wr \Sigma_n$ (2.2).

It is known [22, 1.6], [16, Main Theorem] that

$$H^0(W;\check{T}) = \mathbf{Z}/2, \qquad H^1(W;\check{T}) = \begin{cases} \mathbf{Z}/2, & n=2, \\ \mathbf{Z}/2 \times \mathbf{Z}/2, & n>2, \end{cases}$$

for these groups.

The full general linear group $\mathrm{GL}(2n+1,\mathbf{R}) = \mathrm{SL}(2n+1,\mathbf{R}) \times \langle -E \rangle$ is the direct product of $\mathrm{SL}(2n+1,\mathbf{R})$ with the opposite of the identity matrix, so that $\mathrm{PGL}(2n+1,\mathbf{R}) = \mathrm{SL}(2n+1,\mathbf{R})$.

1. The structure of $\mathrm{SL}(2n+1,\mathbf{R})$. Consider the elementary abelian 2-groups

$$\Delta_{2n+1} = \langle \mathrm{diag}(\pm 1, \dots, \pm 1) \rangle \subset \mathrm{GL}(2n+1,\mathbf{R}),$$

$$
\begin{aligned}
S\Delta_{2n+1} &= \mathrm{SL}(2n+1,\mathbf{R}) \cap \Delta_{2n+1} \subset \mathrm{SL}(2n+1,\mathbf{R}),\\
&\qquad t = t(\mathrm{SL}(2n+1,\mathbf{R})) = \Delta_{2n+1} \cap T(\mathrm{SL}(2n+1,\mathbf{R})) = \langle e_1,\ldots,e_n\rangle\\
&\qquad\quad \subset T(\mathrm{SL}(2n+1,\mathbf{R}))
\end{aligned}
$$

in $\mathrm{GL}(2n+1,\mathbf{R})$ and $\mathrm{SL}(2n+1,\mathbf{R})$.

3.1. Lemma. *The inclusion functors*

$$
\begin{aligned}
&\mathbf{A}(\Sigma_{2n+1},\Delta_{2n+1}) \to \mathbf{A}(\mathrm{GL}(2n+1,\mathbf{R})),\\
&\mathbf{A}(\Sigma_{2n+1},S\Delta_{2n+1}) \to \mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R})),\\
&\mathbf{A}(\Sigma_2 \wr \Sigma_n, t) \to \mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))^{\le t}
\end{aligned}
$$

are equivalences of categories. (*See* [24, Definition 2.68] *for the meaning of* $\mathbf{A}(\Sigma_{2n+1},\Delta_{2n+1})$.)

Proof. Similar to 2.12. $\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R})$ is a full subcategory of the category $\mathbf{A}(\mathrm{GL}(2n+1,\mathbf{R})$ since conjugation with the central element $-E$ of negative determinant is the identity. ∎

(Note that the Quillen categories $\mathbf{A}(\mathrm{GL}(2n,\mathbf{R})) = \mathbf{A}(\Sigma_{2n},\Delta_{2n})$ and $\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R})) = \mathbf{A}(\Sigma_{2n+1},\Delta_{2n+1})$ (2.12, 3.1) are not equivalent.)

For any partition $i=(i_0,i_1)$, $i_0\ge 0$, $i_1>0$, of $2n+1$, let $L(i_0,i_1)\subset \Delta_{2n+1}$ be the subgroup generated by

$$
\mathrm{diag}(\overbrace{+1,\ldots,+1}^{i_0},\overbrace{-1,\ldots,-1}^{i_1}) = (i_0\varrho_0 + i_1\varrho_1)(e_1).
$$

For any partition (i_0,i_1,i_2,i_3) of $2n+1$ where at least two of i_1,i_2,i_3 are positive, let $P(i_0,i_1,i_2,i_3)\subset \Delta_{2n+1}$ be the subgroup generated by

$$
\begin{aligned}
&\mathrm{diag}(\overbrace{+1,\ldots,+1}^{i_0},\overbrace{-1,\ldots,-1}^{i_1},\overbrace{+1,\ldots,+1}^{i_2},\overbrace{-1,\ldots,-1}^{i_3})\\
&\qquad\qquad\qquad\qquad = (i_0\varrho_0 + i_1\varrho_1 + i_2\varrho_2 + i_3\varrho_3)(e_1),\\
&\mathrm{diag}(\overbrace{+1,\ldots,+1}^{i_0},\overbrace{+1,\ldots,+1}^{i_1},\overbrace{-1,\ldots,-1}^{i_2},\overbrace{-1,\ldots,-1}^{i_3})\\
&\qquad\qquad\qquad\qquad = (i_0\varrho_0 + i_1\varrho_1 + i_2\varrho_2 + i_3\varrho_3)(e_2).
\end{aligned}
$$

Note that $L(i_0,i_1)$ is a subgroup of $S\Delta_{2n+1}$ if and only if i_1 is even, and that $P(i_0,i_1,i_2,i_3)$ is a subgroup of $S\Delta_{2n+1}$ if and only of i_1,i_2,i_3 have the same parity, the opposite parity of i_0.

Let $P(k,r)$ denote the number of partitions of $k=i_0+\cdots+i_{r-1}$ into sums of r positive integers $1\le i_0\le\cdots\le i_{r-1}$. From the above discussion we conclude

3.2. Proposition. *The category* $\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))$ *contains precisely*:

- n *isomorphism classes of rank one objects represented by the lines* $L(2i_0+1,2i_1)$ *where* $0\le i_0\le n-1$ *and* $i_1=n-i_0$.

- $\sum_{j=2}^{n} P(j,2)+\sum_{j=3}^{n} P(j,3)$ *isomorphism classes of toral rank two objects. They are represented by the subgroups* $P(2i_0+1,2i_1,2i_2,0)$, *where* $0 \le i_0 \le n-2$ *and* (i_1,i_2) *is a partition of* $n-i_0$, *together with the subgroups* $P(2i_0+1,2i_1,2i_2,2i_3)$, *where* $0 \le i_0 \le n-3$ *and* (i_1,i_2,i_3) *is a partition of* $n-i_0$.
- $\sum_{j=3}^{n+2} P(j,3)$ *isomorphism classes of nontoral rank two objects represented by the subgroups* $P(2i_0,2i_1-1,2i_2-1,2i_3-1)$, *where* $0 \le i_0 \le n-1$ *and* (i_1,i_2,i_3) *is a partition of* $n-i_0+2$.

The centralizers of these objects are

$$\begin{aligned} C_{\mathrm{SL}(2n+1,\mathbf{R})}L(2i_0+1,2i_1) &= \mathrm{SL}(2n+1,\mathbf{R}) \cap (\mathrm{GL}(2i_0+1,\mathbf{R}) \times \mathrm{GL}(2i_1,\mathbf{R})) \\ &= \mathrm{SL}(2i_0+1,\mathbf{R}) \times \mathrm{GL}(2i_1,\mathbf{R}), \end{aligned} \tag{3.3}$$

$$C_{\mathrm{SL}(2n+1,\mathbf{R})}P(i) = \mathrm{SL}(2n+1,\mathbf{R}) \cap \prod \mathrm{GL}(i_j,\mathbf{R}) \tag{3.4}$$

$$= \begin{cases} \mathrm{SL}(2i_0+1,\mathbf{R}) \times \mathrm{GL}(2i_1,\mathbf{R}) \times \mathrm{GL}(2i_2,\mathbf{R}) \times \mathrm{GL}(2i_3,\mathbf{R}), \\ \qquad\qquad P(2i_0+1,2i_1,2i_2,2i_3) \text{ toral}, \\ \mathrm{GL}(2i_0,\mathbf{R}) \times \mathrm{GL}(2i_1-1,\mathbf{R}) \times \mathrm{GL}(2i_2-1,\mathbf{R}) \times \mathrm{SL}(2i_3-1,\mathbf{R}), \\ \qquad\qquad P(2i_0,2i_1-1,2i_2-1,2i_3-1) \text{ nontoral}, \end{cases}$$

as, for instance,

$$\begin{aligned} &\mathrm{SL}(2n+1,\mathbf{R}) \cap (\mathrm{GL}(2i_0+1,\mathbf{R}) \times \mathrm{GL}(2i_1,\mathbf{R})) \\ &\quad = \mathrm{SL}(2n+1,\mathbf{R}) \cap (\mathrm{SL}(2i_0+1,\mathbf{R}) \times \langle -E \rangle \times \mathrm{SL}(2i_1,\mathbf{R}) \rtimes \langle D \rangle) \\ &\quad = \mathrm{SL}(2i_0+1,\mathbf{R}) \times \mathrm{SL}(2i_1,\mathbf{R}) \rtimes \langle -D \rangle = \mathrm{SL}(2i_0+1,\mathbf{R}) \times \mathrm{GL}(2i_1,\mathbf{R}), \end{aligned}$$

and the centers of the centralizers are

$$ZC_{\mathrm{SL}(2n+1,\mathbf{R})}L(2i_0+1,2i_1) = L(2i_0+1,2i_1), \tag{3.5}$$

$$\begin{aligned} ZC_{\mathrm{SL}(2n+1,\mathbf{R})}P(i) &= \mathrm{SL}(2n+1,\mathbf{R}) \cap \prod_{i_j>0} Z\mathrm{GL}(i_j,\mathbf{R}) \\ &= \begin{cases} P(i), & \#\{j \mid i_j>0\}=3, \\ P(i) \times \mathbf{Z}/2, & \#\{j \mid i_j>0\}=4. \end{cases} \end{aligned} \tag{3.6}$$

3.7. Lemma. *For any nontrivial subgroup* $V \subset S\Delta_{2n+1}$ *there is a natural isomorphism*

$$ZC_{\mathrm{SL}(2n+1,\mathbf{R})}(V) = H^0(\Sigma_{2n+1}(V); S\Delta_{2n+1})$$

where $\Sigma_{2n+1}(V)$ *is the pointwise stabilizer subgroup* [24, Definition 2.68].

Proof. Let $V \subset S\Delta_{2n+1}$ be any nontrivial subgroup of rank r. Then $V=V(i)$ is the image of the representation $\sum_{\varrho \in V^\vee} i_\varrho \varrho$ for some function

$i\colon \mathrm{Hom}((\mathbf{Z}/2)^r, \mathbf{R}^\times) \to \mathbf{Z}$ where $\sum_{\varrho\in V^\vee} i_\varrho = 2n+1$ and

$$ZC_{\mathrm{SL}(2n+1,\mathbf{R})}V(i) = Z\Big(\mathrm{SL}(2n+1,\mathbf{R}) \cap \prod_{i_\varrho>0} \mathrm{GL}(i_\varrho,\mathbf{R})\Big)$$
$$= \mathrm{SL}(2n+1,\mathbf{R}) \cap \prod_{i_\varrho>0} Z\mathrm{GL}(i_\varrho,\mathbf{R}) = S\Delta_{2n+1} \cap \Delta_{2n+1}^{\prod \Sigma_{i_\varrho}} = S\Delta_{2n+1}^{\Sigma_{2n+1}(V(i))};$$

here the second equality can be proved by recalling that $C_{\mathrm{GL}(i,\mathbf{R})}\mathrm{SL}(i,\mathbf{R}) = Z\mathrm{GL}(i,\mathbf{R})$ and the final equality follows from the observation that the stabilizer subgroup $\Sigma_{2n+1}(V(i))$ equals $\prod_{i_\varrho>0} \Sigma_{i_\varrho}$. ∎

3.8. Corollary. $\lim^i(\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}); \pi_1(BZC_{\mathrm{SL}(2n+1,\mathbf{R})})) = 0$ *for all* $i > 0$.

Proof. Immediate from the general exactness theorem [24, Lemma 2.69] for functors of the form as in 3.7. ∎

3.9. Proposition. *Centralizers of objects of* $\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))^{\le t}_{\le 2}$ *are LHS.*

Proof. Let X_1 and X_2 be connected Lie groups and π_1 and π_2 finite 2-groups acting on them. Suppose that the homomorphisms $\theta(X_1)^{\pi_1}$ and $\theta(X_1)^{\pi_1}$ [24, (2.20)] are surjective. Then also $\theta(X_1\times X_2)^{\pi_1\times\pi_2}$ is surjective and so the product $X_1 \rtimes \pi_1 \times X_2 \rtimes \pi_2$ is LHS [24, Lemma 2.28]. This observation applies to the products (3.3), (3.4) since the θ-homomorphisms are surjective [16, 5.6], [24, Example 2.29(5)] for $\mathrm{SL}(2i+1,\mathbf{R})$, $i \ge 0$, and $\mathrm{SL}(2i,\mathbf{R})$, $i \ge 1$. ∎

2. The limit of the functor $H^1(W;\check{T})/H^1(\pi_0;\check{Z}(\ )_0)$. In this subsection we check, using a modification of [24, 2.53], that conditions (1) and (2) of [24, Theorem 2.51] with $X = \mathrm{SL}(2n+1,\mathbf{R})$ are satisfied under the inductive assumptions that the connected 2-compact groups $\mathrm{SL}(2i+1,\mathbf{R})$, $0 \le i < n$, and $\mathrm{SL}(2i,\mathbf{R})$, $1 \le i \le n$, are uniquely N-determined.

The objects $V \subset \mathrm{SL}(2n+1,\mathbf{R})$ of the category $\mathbf{A}(\mathrm{PSL}(2n+1,\mathbf{R}))^{\le t}_{\le 2}$ are the rank one objects $L(i_0,i_1)$ and the rank two objects $P(2i_0+1, 2i_1, 2i_2, 0)$ and $P(2i_0+1, 2i_1, 2i_2, 2i_3)$ as described in 3.2. The rank two object $P(2i_0+1, 2i_1, 2i_2, 2i_3)$, $i_3 \ge 0$, contains the three lines $L(2i_0+2i_1+1, 2i_2+2i_3)$, $L(2i_0+2i_2+1, 2i_1+2i_3)$, and $L(2i_0+2i_3+1, 2i_1+2i_2)$. Their centralizers are described in (3.3) and (3.4). Note that there are functorial isomorphisms

$$\check{T}^{W_0(C_{\mathrm{SL}(2n+1,\mathbf{R})}(V))} = (\mathbf{Z}/2)^{\min\{i_0,1\}} \times \check{Z}(C_{\mathrm{SL}(2n+1,\mathbf{R})}(V)_0) \tag{3.10}$$

as modules over $\pi_0 C_{\mathrm{SL}(2n+1,\mathbf{R})}(V)$.

Condition (1) of [24, Theorem 2.51] is satisfied as the centralizer $C_X(V)$ has N-determined automorphisms and is N-determined for general reasons

[24, 2.39, 2.35, 2.40]. This means that there are isomorphisms, α_V and f_V, such that the diagrams

$$\begin{array}{ccc} C_N(V) & \xrightarrow[\cong]{\alpha_V} & C_N(V) \\ \downarrow & & \downarrow \\ C_X(V) & \xrightarrow[f_V]{\cong} & C_{X'}(V) \end{array}$$

commute and $\alpha_V \in H^1(W;\check{T})(C_X(V))$. There may be more than one choice for α_V but for each α_V there is just one possibility for f_V [24, Lemma 2.14(2)]. The set of possible α_V for a given V is a $H^1(\pi_0;\check{Z}((\ )_0))(C_X(V))$-coset in $H^1(W;\check{T})(C_X(V))$ [24, Lemma 2.37]. The collection of the α_V for various V represents an element of the inverse limit

$$\lim{}^0\left(\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))^{\leq t}_{\leq 2}; \frac{H^1(W;\check{T})}{H^1(\pi_0;\check{Z}((\ )_0))}\right) \tag{3.11}$$

of the quotient functor over the category $\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))^{\leq t}_{\leq 2}$.

Condition (2) of [24, Theorem 2.51] is satisfied if the restriction map from the abelian group $H^1(W;\check{T})(\mathrm{SL}(2n+1,\mathbf{R}))$ to (3.11) is surjective. Because of the natural splitting (3.10) and because the centralizers $C_{\mathrm{SL}(2n+1,\mathbf{R})}(V)$ are LHS, there is a short exact sequence

$$0 \to \mathrm{Hom}(\pi_0, (\mathbf{Z}/2)^{\min\{i_0,1\}}) \to \frac{H^1(W;\check{T})}{H^1(\pi_0;\check{Z}((\ )_0))} \to H^1(W_0;\check{T})^{\pi_0} \to 0$$

of functors on $\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))^{\leq t}_{\leq 2}$. If we now apply the functor given by $\mathrm{Hom}(\pi_0, (\mathbf{Z}/2)^{\min\{i_0,1\}})$ to the morphisms

$$L(2i_0+1, 2i_1+2i_2) \to P(2i_0+1, 2i_1, 2i_2, 0) \leftarrow L(2i_0+2i_1+1, 2i_2) \tag{3.12}$$

we see that the induced morphisms are injective and that their images intersect trivially. Thus the inverse limit of this functor is trivial and from the above short exact sequence we obtain an injective map

$$\lim{}^0\left(\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))^{\leq t}_{\leq 2}; \frac{H^1(W;\check{T})}{H^1(\pi_0;\check{Z}((\ )_0))}\right) \to \lim{}^0(\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))^{\leq t}_{\leq 2}; H^1(W_0;\check{T})^{\pi_0})$$

between the inverse limits. As the inverse limit on the right is a subgroup of the inverse limit of the functor $H^1(W_0;\check{T})$, we conclude that if the restriction map

$$H^1(W_0;\check{T})(\mathrm{SL}(2n+1,\mathbf{R})) \to \lim{}^0(\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))^{\leq t}_{\leq 2}; H^1(W_0;\check{T})) \tag{3.13}$$

is surjective, then condition (2) of [24, Theorem 2.51] is satisfied.

3.14. LEMMA. *The restriction homomorphism* (3.13) *is an isomorphism for all* $n \geq 2$.

Proof. For $n = 2$, the image under the functor $H^1(W_0; \check{T})$ of the category $L(1,4) \to P(1,2,2,0) \leftarrow L(3,2)$ is $0 \to 0 \leftarrow \mathbf{Z}/2$ so that the limit of the functor $H^1(W_0; \check{T})$ is $\mathbf{Z}/2$. Since $\mathrm{SL}(3, \mathbf{R}) \times \mathrm{SL}(2; \mathbf{R}) \to \mathrm{SL}(5, \mathbf{R})$ turns out to induce an isomorphism on $H^1(W_0; \check{T})$ the claim follows in this case.

For n=3, taking into account only the planes of type $P(2i_0-1, 2i_1, 2i_2, 0)$, we should compute the limit of the diagram

$$\begin{array}{ccc}
H^1(W_0 C_{\mathrm{SL}(7,\mathbf{R})} L(1,6)) & & \\
 & \searrow & \\
 & & H^1(W_0 C_{\mathrm{SL}(7,\mathbf{R})} P(1,4,2,0)) \\
 & \nearrow & \\
H^1(W_0 C_{\mathrm{SL}(7,\mathbf{R})} L(3,4)) & & \\
 & \searrow & \\
 & & H^1(W_0 C_{\mathrm{SL}(7,\mathbf{R})} P(3,2,2,0)) \\
 & \nearrow & \\
H^1(W_0 C_{\mathrm{SL}(7,\mathbf{R})} L(5,2)) & &
\end{array}$$

$$H^1(W_0 C_{\mathrm{SL}(7,\mathbf{R})} L(5,2)) \to H^1(W_0 C_{\mathrm{SL}(7,\mathbf{R})} P(1,4,2,0))$$

of $\mathbf{F}_2$-vector spaces. For each of the planes P take the intersections of the images in the cohomology groups $H^1(W_0 C_{\mathrm{SL}(7,\mathbf{R})} P; \check{T})$ of $H^1(W_0 C_{\mathrm{SL}(7,\mathbf{R})} L; \check{T})$ for each line $L \subset P$. Take the intersection of the preimages in each $H^1(W_0 C_{\mathrm{SL}(7,\mathbf{R})} L; \check{T})$ of these subspaces of $H^1(W_0 C_{\mathrm{SL}(7,\mathbf{R})} P; \check{T})$. Using the computer program *magma* one may see that these subspaces have dimensions $1, 2, 2$ for $L = L(1,6), L(3,4), L(5,2)$, respectively, and that they equal the image of the restriction maps from $H^1(W_0; \check{T})(\mathrm{SL}(7, \mathbf{R}))$. This shows that the lemma is true in this case.

In general, the above-mentioned subspaces of $H^1(W_0 C_{\mathrm{SL}(7,\mathbf{R})} L; \check{T})$ have dimension one for $L = L(1, 2n)$ and dimension two for the lines $L = L(2i+1, 2n-2i)$ with $1 \leq i \leq n-1$ and these subspaces equal the image of the restriction maps from $H^1(W_0; \check{T})(\mathrm{SL}(2n+1, \mathbf{R}))$. ∎

3. Rank two nontoral objects of $\mathbf{A}(\mathrm{SL}(2n+1, \mathbf{R}))$. The nontoral rank two objects of $\mathbf{A}(\mathrm{SL}(2n+1, \mathbf{R}))$ are represented by the subgroups $P(i) \subset S\varDelta_{2n+1}$ generated by the elements

$$e_1 = \mathrm{diag}(\overbrace{+1, \ldots, +1}^{2i_0}, \overbrace{-1, \ldots, -1}^{2i_1-1}, \overbrace{+1, \ldots, +1}^{2i_2-1}, \overbrace{-1, \ldots, -1}^{2i_3-1}),$$
$$e_2 = \mathrm{diag}(\overbrace{+1, \ldots, +1}^{2i_0}, \overbrace{+1, \ldots, +1}^{2i_1-1}, \overbrace{-1, \ldots, -1}^{2i_2-1}, \overbrace{-1, \ldots, -1}^{2i_3-1}),$$

where $i = (2i_0, 2i_1 - 1, 2i_2 - 1, 2i_3 - 1)$, $0 \leq i_0 \leq n-1$, and (i_1, i_2, i_3) is a partition of $n + 2 - i_0$ (3.2). The generators of $P(i)$ may also be written as

$$e_1 = \mathrm{diag}(\overbrace{E,\dots,E}^{i_0-1}, E, \overbrace{-E,\dots,-E}^{i_1-1}, -R, \overbrace{E,\dots,E}^{i_2-1}, \overbrace{-E,\dots,-E}^{i_3-1}, -1), \tag{3.15}$$

$$e_2 = \mathrm{diag}(\overbrace{E,\dots,E}^{i_0-1}, E, \overbrace{E,\dots,E}^{i_1-1}, R, \overbrace{-E,\dots,-E}^{i_2-1}, \overbrace{-E,\dots,-E}^{i_3-1}, -1) \tag{3.16}$$

where

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The centralizer of $P(i)$ is

$$\begin{aligned} &C_{\mathrm{SL}(2n+1,\mathbf{R})}(P(i)) \\ &\quad = \mathrm{SL}(2n+1,\mathbf{R}) \\ &\qquad \cap (\mathrm{GL}(2i_0,\mathbf{R}) \times \mathrm{GL}(2i_1-1,\mathbf{R}) \times \mathrm{GL}(2i_2-1,\mathbf{R}) \times \mathrm{GL}(2i_3-1,\mathbf{R})) \\ &\quad = \mathrm{GL}(2i_0,\mathbf{R}) \times \mathrm{GL}(2i_1-1,\mathbf{R}) \times \mathrm{GL}(2i_2-1,\mathbf{R}) \times \mathrm{SL}(2i_3-1,\mathbf{R}). \end{aligned}$$

Let us observe that $P(i)$ is contained in the maximal torus normalizer $N(\mathrm{SL}(2n+1,\mathbf{R})) = \mathrm{GL}(2,\mathbf{R}) \wr \Sigma_n$. Since the centralizer of $P(i)$ in the maximal torus normalizer,

$$\begin{aligned} C_{\mathrm{GL}(2,\mathbf{R})\wr\Sigma_n}(P(i)) = {}& \mathrm{GL}(2,\mathbf{R}) \wr \Sigma_{i_0} \times \mathrm{GL}(2,\mathbf{R}) \wr \Sigma_{i_1-1} \times \mathrm{GL}(1,\mathbf{R}) \\ & \times \mathrm{GL}(1,\mathbf{R}) \times \mathrm{GL}(2,\mathbf{R}) \wr \Sigma_{i_2-1} \times \mathrm{GL}(2,\mathbf{R}) \wr \Sigma_{i_3-1}, \end{aligned}$$

is the maximal torus normalizer for the centralizer of $P(i)$, the lift $P(i) \subset N(\mathrm{SL}(2n+1,\mathbf{R}))$ is a preferred lift of $P(i) \subset \mathrm{SL}(2n+1,\mathbf{R})$ [27]. The other two preferred lifts are given by composing with the permutation matrices for the permutations $(1,2)(i_0+i_1, 2n+1)$ and $(1,2)(i_0+i_1+1, 2n+1)$ (assuming $i_0 > 0$) resulting in the lifts given by

$$\begin{aligned} e_1 &= \mathrm{diag}(\overbrace{E,\dots,E}^{i_0-1}, E, \overbrace{-E,\dots,-E}^{i_1-1}, -E, \overbrace{E,\dots,E}^{i_2-1}, \overbrace{-E,\dots,-E}^{i_3-1}, -1), \\ e_2 &= \mathrm{diag}(\overbrace{E,\dots,E}^{i_0-1}, E, \overbrace{E,\dots,E}^{i_1-1}, R, \overbrace{-E,\dots,-E}^{i_2-1}, \overbrace{-E,\dots,-E}^{i_3-1}, -1) \end{aligned}$$

and

$$\begin{aligned} e_1 &= \mathrm{diag}(\overbrace{E,\dots,E}^{i_0-1}, E, \overbrace{-E,\dots,-E}^{i_1-1}, R, \overbrace{E,\dots,E}^{i_2-1}, \overbrace{-E,\dots,-E}^{i_3-1}, -1), \\ e_2 &= \mathrm{diag}(\overbrace{E,\dots,E}^{i_0-1}, E, \overbrace{E,\dots,E}^{i_1-1}, -E, \overbrace{-E,\dots,-E}^{i_2-1}, \overbrace{-E,\dots,-E}^{i_3-1}, -1), \end{aligned}$$

respectively. These two lifts are also preferred lifts of $P(i) \subset \mathrm{SL}(2n+1,\mathbf{R})$. The three preferred lifts are not conjugate in $N(\mathrm{SL}(2n+1,\mathbf{R}))$ because the intersection with the maximal torus is generated by e_1+e_2 in the first case and by e_1, respectively e_2, in the other two cases. Note that all three preferred lifts have the same maximal torus, $\mathrm{SL}(2,\mathbf{R})^{i_0} \times \mathrm{SL}(2,\mathbf{R})^{i_1-1} \times \mathrm{SL}(2,\mathbf{R})^{i_2-1} \times \mathrm{SL}(2,\mathbf{R})^{i_3-1}$.

Let $U = \langle e_1, e_2, e_3 \rangle$ be the elementary abelian 2-group generated by e_1 and e_2 as in (3.15), (3.16) together with

$$e_3 = \mathrm{diag}(\overbrace{E,\dots,E}^{i_0-1}, R, \overbrace{E,\dots,E}^{i_1-1}, E, \overbrace{E,\dots,E}^{i_2-1}, \overbrace{E,\dots,E}^{i_3-1}, -1),$$

Note that the centralizer of U has a nontrivial identity component, and that the inclusion $U \subset C_{\mathrm{SL}(2n+1,\mathbf{R})}(P(i))$ induces an isomorphism on π_0.

Under the inductive assumption that $\mathrm{SL}(2i,\mathbf{R})$, $1 \le i \le n-1$, and $\mathrm{SL}(2i-1,\mathbf{R})$, $1 \le i \le n$, have $\pi_*(N)$-determined automorphisms (or using [19]) we conclude from [24, Lemma 2.63, (2.64)] and (part of) [25, 5.2] that condition (3) of [24, Theorem 2.51] is satisfied for $\mathrm{SL}(2n+1,\mathbf{R})$. (Namely, [24, Lemma 2.63(1)] says that ν'_L does not depend on the choice of $L < V$. The difference $f^{-1}_{\nu,L_2} \circ f_{\nu,L_1}$ between any two of the maps $f_{\nu,L}$ from [24, Theorem 2.51(3)] is an automorphism of $C_{\mathrm{SL}(2n+1,\mathbf{R})}(P(i))$ that, by [24, Lemma 2.63(2)], is the identity on the identity component and by the commutative diagram [24, (2.64)]

$$\begin{array}{ccc} & U & \\ \swarrow & & \searrow \\ C_{\mathrm{SL}(2n+1,\mathbf{R})}(P(i)) & \xrightarrow{f^{-1}_{\nu,L_2} \circ f_{\nu,L_1}} & C_{\mathrm{SL}(2n+1,\mathbf{R})}(P(i)) \end{array} \tag{3.17}$$

also the identity on $\pi_0 C_{\mathrm{SL}(2n+1,\mathbf{R})}(P(i))$. Any such automorphism of the centralizer $C_{\mathrm{SL}(2n+1,\mathbf{R})}(P(i))$ has [25, 5.2] the form $A \to \varphi(A)A$ where

$$\begin{aligned} \varphi\colon \mathrm{GL}(2i_0,\mathbf{R}) \times \mathrm{GL}(2i_1-1,\mathbf{R}) \times \mathrm{GL}(2i_2-1,\mathbf{R}) \times \mathrm{SL}(2i_3-1,\mathbf{R}) \\ \to \pi_0(\mathrm{GL}(2i_0,\mathbf{R}) \times \mathrm{GL}(2i_1-1,\mathbf{R}) \times \mathrm{GL}(2i_2-1,\mathbf{R}) \times \mathrm{SL}(2i_3-1,\mathbf{R})) \\ \to Z\mathrm{GL}(2i_0,\mathbf{R}) \end{aligned}$$

is some homomorphism. Diagram (3.17) thus implies that the inclusion $U \to \mathrm{SL}(2n+1,\mathbf{R})$ and the monomorphism given by $e_i \mapsto \varphi(e_i)e_i$, $1 \le i \le 3$, are conjugate. Since the trace of e_i, $1 \le i \le 3$, is odd (nonzero), φ must be trivial. Thus f_{ν,L_1} and f_{ν,L_2} are identical isomorphisms.)

4. THE C-FAMILY

Let $\mathbf{H} = \{a + bj \mid a, b \in \mathbf{C}\}$, where $j^2 = -1$ and $ja = \overline{a}j$ for $a \in \mathbf{C}$, be the quaternion algebra. The C-family consists of the matrix groups

$$\mathrm{PGL}(n,\mathbf{H}) = \mathrm{GL}(n,\mathbf{H})/\langle -E \rangle, \qquad n \ge 3,$$

of quaternion projective $n \times n$ matrices. (These 2-compact groups also exist for $n = 1$ or $n = 2$. However, $\mathrm{PGL}(1,\mathbf{H}) = \mathrm{SL}(3,\mathbf{R}) = \mathrm{PGL}(2,\mathbf{C})$ and $\mathrm{PGL}(2,\mathbf{H}) = \mathrm{SL}(5,\mathbf{R})$ [24, 5.24] are already covered.)

The maximal torus normalizer for $\mathrm{GL}(1,\mathbf{H}) = \mathbf{H}^\times$, generated by the maximal torus $\mathrm{GL}(1,\mathbf{C}) = \mathbf{C}^\times$ and the element j, sits in the nonsplit extension

$$1 \to \mathrm{GL}(1,\mathbf{C}) \to N(\mathrm{GL}(1,\mathbf{H})) \to \langle j\rangle/\langle -1\rangle \to 1$$

of Σ_2 by $\mathrm{GL}(1,\mathbf{C}) = \mathbf{C}^\times$. The maximal torus normalizer for $\mathrm{GL}(n,\mathbf{H})$ is the subgroup

$$N(\mathrm{GL}(n,\mathbf{H})) = N(\mathrm{GL}(1,\mathbf{H})) \wr \Sigma_n,$$

generated by $N(\mathrm{GL}(1,\mathbf{H}))^n \subset \mathrm{GL}(n,\mathbf{H})$ and the permutation matrices. The maximal torus normalizer for $\mathrm{PGL}(n,\mathbf{H})$, the quotient $N(\mathrm{GL}(n,\mathbf{H}))$ by the order two group $\langle -E\rangle$, sits in the extension

$$1 \to \frac{\mathrm{GL}(1,\mathbf{C})^n}{\langle -E\rangle} \to \frac{N(\mathrm{GL}(1,\mathbf{H}))^n}{\langle -E\rangle} \to \frac{N(\mathrm{GL}(1,\mathbf{H}))}{\mathrm{GL}(1,\mathbf{C})} \wr \Sigma_n \to 1,$$

which does not split (for $n \geq 3$).

It is known that [22, 1.6], [16, Main Theorem]

$$H^0(W;\check{T})(\mathrm{PGL}(n,\mathbf{H})) = 0, \quad H^1(W;\check{T})(\mathrm{PGL}(n,\mathbf{H})) = \begin{cases} \mathbf{Z}/2, & n = 3,4, \\ 0, & n > 4, \end{cases}$$

for the projective groups.

1. The structure of $\mathrm{PGL}(n,\mathbf{H})$**.** Let

$$\Delta_n = t(\mathrm{GL}(n,\mathbf{H})) = \langle \mathrm{diag}(\pm 1,\dots,\pm 1)\rangle \subset \mathrm{GL}(n,\mathbf{H})$$

be the maximal elementary abelian 2-group in $\mathrm{GL}(n,\mathbf{H})$, and $C_4 = \langle I\rangle \subset \mathrm{GL}(n,\mathbf{H})$ the cyclic order four group generated by $I = \mathrm{diag}(i,\dots,i)$. The maximal elementary abelian 2-group in $\mathrm{PGL}(n,\mathbf{H})$ is the quotient

$$t(\mathrm{PGL}(n,\mathbf{H})) = \frac{t(\mathrm{PGL}(n,\mathbf{H}))^*}{\langle -E\rangle}, \quad t(\mathrm{PGL}(n,\mathbf{H}))^* = C_4 \circ t(\mathrm{GL}(n,\mathbf{H})),$$

so that the toral part of the Quillen category,

$$\mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))^{\leq t} = \mathbf{A}\left(C_2 \wr \Sigma_n, \frac{C_4 \circ \langle \mathrm{diag}(\pm 1,\dots,\pm 1)\rangle}{\langle -E\rangle}\right),$$

is equivalent to the category whose objects are nontrivial subgroups of $t(\mathrm{PGL}(n,\mathbf{H}))$ and whose morphisms are induced from the action of the Weyl group [24, Definition 2.68].

For any partition $i = (i_0, i_1)$ of $n = i_0 + i_1$ into a sum of two positive integers $i_0 \geq i_1 \geq 1 > 0$ let $L(i) = L(i_0,i_1) \subset \mathrm{GL}(n,\mathbf{H})$ be the subgroup generated by

$$\mathrm{diag}(\overbrace{+1,\dots,+1}^{i_0},\overbrace{-1,\dots,-1}^{i_1}).$$

Then the centralizer is

$$
(4.1)\qquad C_{\mathrm{PGL}(n,\mathbf{H})}L(i_0,i_1)=\begin{cases}\dfrac{\mathrm{GL}(i_0,\mathbf{H})\times\mathrm{GL}(i_1,\mathbf{H})}{\langle -E\rangle}, & i_0\neq i_1,\\[2ex] \dfrac{\mathrm{GL}(i_0,\mathbf{H})^2}{\langle -E\rangle}\rtimes\left\langle\begin{pmatrix}0&E\\E&0\end{pmatrix}\langle -E\rangle\right\rangle, & i_0=i_1,\end{cases}
$$

so that $ZC_{\mathrm{PGL}(n,\mathbf{H})}L(i_0,i_1)=L(i_0,i_1)$ as in the proof of 2.55 and [24, Lemma 5.18].

Let (also) $I\in\mathrm{PGL}(n,\mathbf{H})$ denote the order two element that is the image of the order four element $i\in\mathrm{GL}(n,\mathbf{H})$. Then

$$
(4.2)\qquad C_{\mathrm{PGL}(n,\mathbf{H})}(I)=\frac{\mathrm{GL}(n,\mathbf{C})}{\langle -E\rangle}\rtimes\langle j\langle -E\rangle\rangle
$$

so that $ZC_{\mathrm{PGL}(n,\mathbf{H})}(I)=\langle I\rangle$ as shown in the proof of 2.55.

For any partition $(i_0,i_1,i_2,0)$ of $n=i_0+i_1+i_2$ into a sum of three positive integers $i_0\geq i_1\geq i_2>0$ or any partition (i_0,i_1,i_2,i_3) of $n=i_0+i_1+i_2+i_3$ into a sum of four positive integers $i_0\geq i_1\geq i_2\geq i_3>0$ let $P(i_0,i_1,i_2,i_3)\subset\mathit{\Delta}_{2n+1}$ be the subgroup generated by the two elements

$$
\mathrm{diag}(\overbrace{+1,\dots,+1}^{i_0},\overbrace{-1,\dots,-1}^{i_1},\overbrace{+1,\dots,+1}^{i_2},\overbrace{-1,\dots,-1}^{i_3}),
$$
$$
\mathrm{diag}(\overbrace{+1,\dots,+1}^{i_0},\overbrace{+1,\dots,+1}^{i_1},\overbrace{-1,\dots,-1}^{i_2},\overbrace{-1,\dots,-1}^{i_3}).
$$

Then the centralizer is

$$
(4.3)\qquad C_{\mathrm{PGL}(n,\mathbf{H})}(P(i))
=\begin{cases}\dfrac{\mathrm{GL}(i_0,\mathbf{H})^4}{\langle -E\rangle}\rtimes(C_2\times C_2), & i=(i_0,i_0,i_0,i_0),\\[2ex] \dfrac{\mathrm{GL}(i_0,\mathbf{H})^2\times\mathrm{GL}(i_2,\mathbf{H})^2}{\langle -E\rangle}\rtimes C_2, & i=(i_0,i_0,i_2,i_2)\ ,\\[2ex] \dfrac{\mathrm{GL}(i_0,\mathbf{H})\times\mathrm{GL}(i_1,\mathbf{H})\times\mathrm{GL}(i_2,\mathbf{H})\times\mathrm{GL}(i_3,\mathbf{H})}{\langle -E\rangle}, & \#i=4,\end{cases}
$$

where the groups C_2 are generated by permutation matrices.

For any partition $i=(i_0,i_1)$ of $n=i_0+i_1$ into a sum of two positive integers $i_0\geq i_1>0$ let $I\#L(i_0,i_1)\subset\mathrm{PGL}(n,\mathbf{H})$ be the elementary abelian 2-group that is the quotient of

$$
(I\#L(i_0,i_1))^*=\langle I,\mathrm{diag}(\overbrace{+1,\dots,+1}^{i_0},\overbrace{-1,\dots,-1}^{i_1})\rangle.
$$

Then the centralizer is

$$(4.4)\quad C_{\mathrm{PGL}(n,\mathbf{H})}(I\#L(i_0,i_1)) = \begin{cases} \dfrac{\mathrm{GL}(i_0,\mathbf{C})\times\mathrm{GL}(i_1,\mathbf{C})}{\langle -E\rangle} \rtimes \langle j\langle -E\rangle\rangle, & i_0\neq i_1,\\[2ex] \dfrac{\mathrm{GL}(i_0,\mathbf{C})^2}{\langle -E\rangle} \rtimes \left\langle j\langle -E\rangle, \begin{pmatrix} 0 & E\\ E & 0\end{pmatrix}\langle -E\rangle \right\rangle, & i_0 = i_1.\end{cases}$$

4.5. Proposition. *The category* $\mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))$ *contains exactly*

- $[n/2]+1$ *rank one toral objects represented by the lines* $L(i,n-i)$, $1\leq i\leq [n/2]$ *(with* $q=0$*), and by the line* I *(with* $q\neq 0$*).*
- $P(n,3)+P(n,4)+[n/2]$ *rank two toral objects represented by the* $P(n,3)$ *planes* $P(i_0,i_1,i_2,0)$ *(with* $q=0$*), and the* $P(n,4)$ *planes* $P(i_0,i_1,i_2,i_3)$ *(with* $q=0$*), and the* $[n/2]$ *planes* $I\#L(i,n-i)$, $1\leq i\leq [n/2]$ *(with* $q\neq 0$*).*

4.6. Proposition. *Let* $V\subset \mathrm{PGL}(n,\mathbf{H})$ *be a nontrivial elementary abelian* 2*-group. Then*

$$V \text{ is toral } \Leftrightarrow [V,V]\neq 0.$$

Proof. The proof is similar to 2.10 with the extra input that all elementary abelian 2-groups in $\mathrm{GL}(n,\mathbf{H})$ are toral by quaternion representation theory [1]. ∎

4.7. Proposition. *Centralizers of objects of* $\mathbf{A}(\mathrm{GL}(n,\mathbf{H}))^{\leq t}_{\leq 2}$ *are LHS.*

Proof. The centralizers $C=C_0\rtimes\pi$ in question are the nonconnected centralizers listed in (4.1), (4.2), (4.3), and (4.4). In fact, we only need to deal with

$$\frac{\mathrm{GL}(i,\mathbf{H})^2}{\langle -E\rangle}\rtimes C_2,\qquad \frac{\mathrm{GL}(i_0,\mathbf{H})^2\times\mathrm{GL}(i_1,\mathbf{H})^2}{\langle -E\rangle}\rtimes C_2,\qquad \frac{\mathrm{GL}(i,\mathbf{H})^4}{\langle -E\rangle}\rtimes (C_2\times C_2)$$

as the other cases are covered by 2.19. It suffices to show that $\theta(C_0)^\pi$ is surjective [24, Lemma 2.28, (2.20)].

Computations with the program *magma* result in the table

$\frac{\mathrm{GL}(i,\mathbf{H})^2}{\langle -E\rangle}\rtimes C_2$	$\ker\theta$	$\mathrm{Hom}(W,\check{T}^W)$	$H^1(W;\check{T})$	θ	$H^1(W;\check{T})^\pi$
$1=i$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^2$	0	epi	0
$2=i$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^3$		$(\mathbf{Z}/2)^2$
$2<i$	0	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^4$	iso	$(\mathbf{Z}/2)^2$

From the table we see that θ^π is surjective unless $i=2$. In that exceptional case, more computations show that $H^1(\pi;\check{T}^W)=\mathbf{Z}/2$ and $H^1(W\rtimes C_2;\check{T})=(\mathbf{Z}/2)^3$, which means that also $(\mathrm{GL}(2,\mathbf{H})^2/\langle -E\rangle)\rtimes C_2$ is LHS.

Computations with the program *magma* result in the table

$\frac{\mathrm{GL}(i_0,\mathbf{H})^2\times \mathrm{GL}(i_1,\mathbf{H})^2}{\langle -E\rangle}$	$\ker\theta$	$\mathrm{Hom}(W,\check{T}^W)$	$H^1(W;\check{T})$	θ	$H^1(W;\check{T})^\pi$
$1=i_0, 2=i_1$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^{18}$	$(\mathbf{Z}/2)^{14}$	epi	$(\mathbf{Z}/2)^7$
$1=i_0, 2<i_1$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^{18}$	$(\mathbf{Z}/2)^{16}$	epi	$(\mathbf{Z}/2)^8$
$2=i_0<i_1$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^{24}$	$(\mathbf{Z}/2)^{22}$	epi	$(\mathbf{Z}/2)^{11}$
$3<i_0<i_1$	0	$(\mathbf{Z}/2)^{24}$	$(\mathbf{Z}/2)^{24}$	iso	$(\mathbf{Z}/2)^{12}$

Since θ is surjective and $H^{>0}(\pi;\ker\theta)=0$ because the action of π on $\ker\theta$ is induced from the trivial subgroup, θ^π is surjective.

Computations with the program *magma* result in the table

$\frac{\mathrm{GL}(i,\mathbf{H})^4}{\langle -E\rangle}\rtimes(C_2\times C_2)$	$\ker\theta$	$\mathrm{Hom}(W,\check{T}^W)$	$H^1(W;\check{T})$	θ	$H^1(W;\check{T})^\pi$
$1=i$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^{12}$	$(\mathbf{Z}/2)^{8}$	epi	$(\mathbf{Z}/2)^2$
$2=i$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^{24}$	$(\mathbf{Z}/2)^{20}$	epi	$(\mathbf{Z}/2)^5$
$2<i$	0	$(\mathbf{Z}/2)^{24}$	$(\mathbf{Z}/2)^{24}$	iso	$(\mathbf{Z}/2)^6$

Since θ is surjective and $H^{>0}(\pi;\ker\theta)=0$ because the action of π on $\ker\theta$ is induced from the trivial subgroup, θ^π is surjective. ∎

2. The limit of the functor $H^1(W_0;\check{T})^{W/W_0}$ on $\mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))^{\le t}_{\le 2}$. Let $H^1(W_0;\check{T})\colon \mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))^{\le t}_{\le 2}\to\mathbf{Ab}$ be the functor that takes the toral elementary abelian 2-group $V\subset t(\mathrm{PGL}(n,\mathbf{H}))$ to the abelian group $H^1(W_0C_{\mathrm{PGL}(n,\mathbf{H})}(V);\check{T})$, and $H^1(W_0;\check{T})^{W/W_0}$ the functor that takes V to the invariants for the action of the component group $\pi_0C_{\mathrm{PGL}(n,\mathbf{H})}(V)$ on this first cohomology group.

4.8. Proposition. *The restriction map*

$$H^1(W(\mathrm{PGL}(n,\mathbf{H}));\check{T})\to\lim{}^0(\mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))^{\le t}_{\le 2};H^1(W_0;\check{T})^{W/W_0})$$

is an isomorphism for all $n>3$.

Proof.

PGL$(4,\mathbf{H})$: Computer computations show that the intersection of the images of the morphisms

$$H^1(W_0;\check{T})^{W/W_0}(L(1,3))\to H^1(W_0;\check{T})^{W/W_0}(I\#L(1,3)) \xleftarrow{\cong} H^1(W_0;\check{T})^{W/W_0}(I)$$

is 1-dimensional and that its preimage in $H^1(W_0;\check{T})^{W/W_0}(I)$ equals the image of the restriction map from $H^1(W,\check{T})(\mathrm{PGL}(4,\mathbf{H}))$. Similarly, the images of the monomorphisms

$$H^1(W_0;\check{T})^{W/W_0}(L(1,3))\hookrightarrow H^1(W_0;\check{T})^{W/W_0}(P(1,1,2,0)) \hookleftarrow H^1(W_0;\check{T})^{W/W_0}(L(2,2))$$

meet in a 1-dimensional subspace whose inverse images in the cohomology groups to the right and to the left agree with the images of the restriction maps from $H^1(W,\check{T})(\mathrm{PGL}(4,\mathbf{H}))$.

$\underline{\mathrm{PGL}(n,\mathbf{H}),\ n>4}$: Computer computations show that the images of the morphisms

$$H^1(W_0;\check{T})^{W/W_0}(L(1,n-1)) \to H^1(W_0;\check{T})^{W/W_0}(I\#L(1,n-1)) \xleftarrow{\cong} H^1(W_0;\check{T})^{W/W_0}(I)$$

intersect trivially and that the arrow pointing left is an isomorphism. Similarly, the images of the injective morphisms

$$H^1(W_0;\check{T})^{W/W_0}(L(i,n-i)) \hookrightarrow H^1(W_0;\check{T})^{W/W_0}(P(i,1,n-i-1,0)) \hookleftarrow H^1(W_0;\check{T})^{W/W_0}(L(i+1,n-i-1)), \quad 1\le i<[n/2],$$

intersect trivially. These observations imply that

$$\lim{}^0(\mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))^{\le t}_{\le 2}; H^1(W_0;\check{T})^{W/W_0}) = 0$$

and also $H^1(W,\check{T})(\mathrm{PGL}(n,\mathbf{H}))=0$ as $n>4$. ∎

3. The category $\mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))^{[\,,\,]\neq 0}_{\le 4}$. We shall need information about all nontoral objects of $\mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))$ of rank ≤ 3 and some objects of rank 4. If $V\subset \mathrm{PGL}(n,\mathbf{H})$ is an elementary abelian 2-group with nontrivial inner product then its preimage $V^*\subset \mathrm{GL}(n,\mathbf{H})$ is $P\times R(V)$ or $(C_4\circ P)\times R(V)$, where P is an extraspecial 2-group, $C_4\circ P$ a generalized extraspecial 2-group, and $\mho_1(V^*)=\langle -E\rangle$ (2.8). We manufacture all oriented quaternion representations of these product groups as direct sums of tensor products of irreducible representations of the factors [24, 5.6] as described in [1, 3.7, 3.65].

Note that the degrees of the faithful irreducible representations over $\mathbf{H}$ for the groups 2^{1+2}_+ and $C_4\circ 2^{1+2}_\pm$ are even, and that the quaternion group 2^{1+2}_- has a faithful irreducible representation over $\mathbf{H}$, namely the defining representation.

4.9. *The category* $\mathbf{A}(\mathrm{PGL}(2n+1,\mathbf{H}))^{[\,,\,]\neq 0}_{\le 4}$. The Quillen category $\mathbf{A}(\mathrm{PGL}(2n+1,\mathbf{H}))$ contains up to isomorphism just one nontoral rank two object, H_-, whose inverse image in $\mathrm{GL}(2n+1,\mathbf{H})$ is

$$Q_8 = 2^{1+2}_- = \langle \mathrm{diag}(i,\dots,i), \mathrm{diag}(j,\dots,j)\rangle.$$

As in 2.51, the centralizers [32, Proposition 4] of 2^{1+2}_- and H_- are

$$C_{\mathrm{GL}(2n+1,\mathbf{H})}(2^{1+2}_-) = \mathrm{GL}(2n+1,\mathbf{R}),$$
$$C_{\mathrm{PGL}(2n+1,\mathbf{H})}(H_-) = H_-\times \mathrm{SL}(2n+1,\mathbf{R}),$$

so that $ZC_{\mathrm{PGL}(2n+1,\mathbf{H})}(H_-)=H_-$.

There are n nontoral objects of rank three, $H_-\#L(i,2n+1-i)$, $1\le i\le n$. The inverse image in $\mathrm{GL}(2n+1,\mathbf{H})$ of $H_-\#L(i,2n+1-i)$ is

$$\langle \mathrm{diag}(i,\dots,i),\mathrm{diag}(j,\dots,j),\mathrm{diag}(\overbrace{+1,\dots,+1}^{i},\overbrace{-1,\dots,-1}^{2n+1-i})\rangle$$

and the center of the centralizer, $C_{\mathrm{PGL}(2n+1,\mathbf{H})}(H_-\#L(i,2n+1-i)) = H_-\times C_{\mathrm{SL}(2n+1,\mathbf{R})}(L(i,n-1))$, is $ZC_{\mathrm{PGL}(2n+1,\mathbf{H})}(H_-\#L(i,2n+1-i)) = H_-\#L(i,2n+1-i)$ according to (3.5).

The objects $H_-\#P(i_0,i_1,i_2,i_3)$, where $P(i_0,i_1,i_2,i_3)$ is as in (4.3), are rank four nontoral objects.

We need to know that the nontoral object H_- satisfies condition (3) of [24, Theorem 2.51]. Note that the conditions of [24, Lemma 2.63] are satisfied because the identity component of $C_{\mathrm{PGL}(2n+1,\mathbf{H})}(H_-)$ is nontrivial and because the Quillen automorphism group $\mathbf{A}(\mathrm{PGL}(2n+1,\mathbf{H}))(H_-) = \mathrm{GL}(2,\mathbf{F}_2)$ acts transitively on the set of preferred lifts $H_-\subset N(\mathrm{PGL}(2n+1,\mathbf{H}))$ of $H_-\subset\mathrm{PGL}(2n+1,\mathbf{H})$. Under the inductive assumption that $\mathrm{SL}(2n+1,\mathbf{R})$ has $\pi_*(N)$-determined automorphisms (or using [19]) we conclude from [24, Lemma 2.63, (2.64)] and (part of) [25, 5.2] that condition (3) of [24, Theorem 2.51] is satisfied for the nontoral rank two object H_-. (Namely, [24, Lemma 2.63(1)] says that ν'_L does not depend on the choice of $L<V$. The difference $f^{-1}_{\nu,L_2}\circ f_{\nu,L_1}$ between any two of the maps $f_{\nu,L}$ from [24, Theorem 2.51(3)] is an automorphism of $C_{\mathrm{PGL}(2n+1,\mathbf{H})}(H_-)$ that, by [24, Lemma 2.63(2)], is the identity on the identity component and by the commutative diagram [24, (2.64)]

(4.10)
$$\begin{array}{ccc} & H_- & \\ \swarrow & & \searrow \\ C_{\mathrm{PGL}(2n+1,\mathbf{H})}(H_-) & \xrightarrow{f^{-1}_{\nu,L_2}\circ f_{\nu,L_1}} & C_{\mathrm{PGL}(2n+1,\mathbf{H})}(H_-) \end{array}$$

also the identity on $\pi_0C_{\mathrm{PGL}(2n+1,\mathbf{H})}(H_-)$. Since the identity component $\mathrm{SL}(2n+1,\mathbf{R})$ of the centralizer $C_{\mathrm{PGL}(2n+1,\mathbf{H})}(H_-)$ has no center, this shows that $f^{-1}_{\nu,L_2}\circ f_{\nu,L_1}$ is the identity automorphism [25, 5.2].)

4.11. *Rank two nontoral objects of* $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))$. The category $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))$ contains up to isomorphism two nontoral rank two objects, H_+ and H_-, whose inverse images in $\mathrm{GL}(2n,\mathbf{H})$ are

$$2^{1+2}_+ = \langle \mathrm{diag}(R,\dots,R),\mathrm{diag}(T,\dots,T)\rangle,\qquad R=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, T=\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

$$2^{1+2}_- = \langle \mathrm{diag}(i,\dots,i),\mathrm{diag}(j,\dots,j)\rangle,$$

where the representation of the dihedral group 2^{1+2}_+ is of real type and the representation of the quaternion group 2^{1+2}_- of quaternion type. This follows from 2.8 because 2_+^{1+2} has one faithful irreducible $\mathbf{H}$-representation of degree two and 2_-^{1+2} has one faithful irreducible $\mathbf{H}$-representation of degree one. The centralizers are [32, Proposition 4]

$$C_{\mathrm{GL}(2n,\mathbf{H})}(2_+^{1+2}) = \mathrm{GL}(n,\mathbf{H}), \qquad C_{\mathrm{PGL}(2n,\mathbf{H})}(H_+) = H_+ \times \mathrm{PGL}(n,\mathbf{H}),$$
$$C_{\mathrm{GL}(2n,\mathbf{H})}(2_-^{1+2}) = \mathrm{GL}(2n,\mathbf{R}), \qquad C_{\mathrm{PGL}(2n,\mathbf{H})}(H_-) = H_- \times \mathrm{PGL}(2n,\mathbf{R}),$$

as we see by an argument similar to that of 2.51. This implies [24, Lemma 5.18] that $ZC_{\mathrm{PGL}(2n,\mathbf{H})}(H) = H$ for all nontoral rank two objects H of $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))$.

We need to know that these nontoral objects satisfy condition (3) of [24, Theorem 2.51]. To see this we use [24, Lemma 2.63].

$\underline{H_+}$: Condition (1) of [24, Lemma 2.63] is clearly satisfied since the identity component of $C_{\mathrm{PGL}(2n,\mathbf{H})}(H_+)$ is nontrivial when $n \geq 3$. The group $H_+^* = 2_+^{1+2}$ is contained in $N(\mathrm{GL}(2n,\mathbf{H})) = N(\mathrm{GL}(1,\mathbf{H})) \wr \Sigma_{2n}$ and its centralizer there is

$$\begin{aligned} C_{N(\mathrm{GL}(2n,\mathbf{H}))}(2_+^{1+2}) &= C_{N(\mathrm{GL}(1,\mathbf{H}))\wr\Sigma_{2n}}(2_+^{1+2}) = N(\mathrm{GL}(1,\mathbf{H})) \wr \Sigma_n \\ &= N(\mathrm{GL}(n,\mathbf{H})), \end{aligned}$$

and therefore H_- is contained in $N(\mathrm{GL}(2n,\mathbf{H}))/\langle -E\rangle = N(\mathrm{PGL}(2n,\mathbf{H}))$, where its centralizer is

$$C_{N(\mathrm{PGL}(2n,\mathbf{H}))}(H_+) = H_+ \times N(\mathrm{PGL}(n,\mathbf{H})) = N(C_{\mathrm{GL}(2n,\mathbf{H})}(H_+))$$

as in 2.51. This means that $H_+ \subset N(\mathrm{PGL}(2n,\mathbf{H}))$ is a preferred lift [27] of $H_+ \subset \mathrm{GL}(2n,\mathbf{H})$. Precomposing the inclusion $H_+ \subset N(\mathrm{PGL}(2n,\mathbf{H}))$ with the nontrivial element of $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))(H_+) = O^+(2,\mathbf{F}_2) \cong C_2$ (4.18) leads to another preferred lift. The third preferred lift is the quotient of

$$(2_+^{1+2})^{\mathrm{diag}(B,\dots,B)} = \langle \mathrm{diag}(R^B,\dots,R^B), \mathrm{diag}((RT)^B,\dots,(RT)^B)\rangle,$$
$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1\end{pmatrix}, \qquad R^B = T, \qquad (RT)^B = \begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}.$$

Note that these three preferred lifts all have the same image in the Weyl group $\pi_0 N(\mathrm{GL}(2n,\mathbf{H})) = \pi_0(N(\mathrm{GL}(1,\mathbf{H}))) \wr \Sigma_{2n}$, namely the subgroup generated by the permutation $(1,2)(3,4)\cdots(2n-1,2n) \in \Sigma_{2n}$.

Under the inductive assumption that $\mathrm{PGL}(n,\mathbf{H})$ has $\pi_*(N)$-determined automorphisms (or using [19]) we conclude from [24, Lemma 2.63, (2.64)] and (part of) [25, 5.2] that condition (3) of [24, Theorem 2.51] is satisfied for the nontoral rank two object H_+ of $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))$. (Namely, [24, Lemma 2.63(1)] says that ν'_L does not depend on the choice of $L < V$. The difference $f_{\nu,L_2}^{-1} \circ f_{\nu,L_1}$ between any two of the maps $f_{\nu,L}$ from [24, Theorem 2.51(3)] is an automorphism of $C_{\mathrm{PGL}(2n,\mathbf{H})}(H_+)$ that, by [24, Lemma 2.63(2)], is the identity on the identity component and by the commutative diagram [24, (2.64)]

$$\begin{array}{ccc} & H_+ & \\ \swarrow & & \searrow \\ C_{\mathrm{PGL}(2n,\mathbf{H})}(H_+) & \xrightarrow{f_{\nu,L_2}^{-1}\circ f_{\nu,L_1}} & C_{\mathrm{PGL}(2n,\mathbf{H})}(H_+) \end{array} \tag{4.12}$$

also the identity on $\pi_0 C_{\mathrm{PGL}(2n,\mathbf{H})}(H_+)$. Since the identity component of $C_{\mathrm{PGL}(2n,\mathbf{H})}(H_+)$ has no center, this shows that $f_{\nu,L_2}^{-1}\circ f_{\nu,L_1}$ is the identity automorphism [25, 5.2].)

$\underline{H_-}$: Condition (1) of [24, Lemma 2.63] is clearly satisfied since the identity component of $C_{\mathrm{PGL}(2n,\mathbf{H})}(H_-)$ is nontrivial when $n \geq 3$. The group $H_-^* = 2_-^{1+2}$ is contained in $N(\mathrm{GL}(2n,\mathbf{H})) = N(\mathrm{GL}(1,\mathbf{H})) \wr \Sigma_{2n}$ and its centralizer there is

$$\begin{aligned} C_{N(\mathrm{GL}(1,\mathbf{H}))\wr\Sigma_{2n}}(2_-^{1+2}) &\overset{[24,\,5.10]}{=} C_{N(\mathrm{GL}(1,\mathbf{H}))}(i,j) \wr \Sigma_n \\ &= \mathrm{GL}(1,\mathbf{R}) \wr \Sigma_{2n} = N(\mathrm{GL}(2n,\mathbf{R})), \end{aligned}$$

and therefore H_- is contained in $N(\mathrm{GL}(2n,\mathbf{H}))/\langle -E\rangle = N(\mathrm{PGL}(2n,\mathbf{H}))$, where its centralizer is

$$\begin{aligned} C_{N(\mathrm{PGL}(2n,\mathbf{H}))}(H_-) &= H_- \times N(\mathrm{GL}(2n,\mathbf{R}))/\langle -E\rangle \\ &= H_- \times N(\mathrm{PGL}(2n,\mathbf{R})) = N(C_{\mathrm{PGL}(2n,\mathbf{H})}(H_-)) \end{aligned}$$

as in 2.51. This means that $H_- \subset N(\mathrm{PGL}(2n,\mathbf{H}))$ is a preferred lift [27] of $H_- \subset \mathrm{GL}(2n,\mathbf{H})$. Precomposing the inclusion $H_- \subset N(\mathrm{PGL}(2n,\mathbf{H}))$ with elements of $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))(H_-) = O^-(2,\mathbf{F}_2) = \mathrm{GL}(2,\mathbf{F}_2)$ (4.18) leads to the other two preferred lifts of H_-.

Under the inductive assumption that the identity component $\mathrm{PSL}(2n,\mathbf{R})$ of $\mathrm{PGL}(2n,\mathbf{R})$ has $\pi_*(N)$-determined automorphisms (or using [19]) we conclude from [24, Lemma 2.63] and diagram [24, (2.64)] and (part of) [25, 5.2] that condition (3) of [24, Theorem 2.51] is satisfied for the nontoral rank two object H_- of $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))$. (The argument for this is the same as in the case of H_+ with the little extra complication that $\pi_0 C_{\mathrm{PGL}(2n,\mathbf{H})}(H_-)$ has an extra generator, so that we replace diagram (4.12) by

$$\begin{array}{ccc} & \langle H_-, \mathrm{diag}(-1,1,\ldots,1)\rangle & \\ \swarrow & & \searrow \\ C_{\mathrm{PGL}(2n,\mathbf{H})}(H_+) & \xrightarrow{f_{\nu,L_2}^{-1}\circ f_{\nu,L_1}} & C_{\mathrm{PGL}(2n,\mathbf{H})}(H_+) \end{array} \tag{4.13}$$

from [24, (2.64)] where the slanted arrows induce isomorphisms on the component groups.)

4.14. *Rank three nontoral objects of* $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))$. The nontoral rank three objects of the category $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))$ are the quotients of $H_+\# L(i,n-i)$, $1\le i\le [n/2]$, $H_-\# L(i,2n-i)$, $1\le i\le n$, and V_0. These subgroups of $\mathrm{GL}(2n,\mathbf{H})$ are defined to be

$$\langle \mathrm{diag}(\overbrace{R,\dots,R}^{n}), \mathrm{diag}(\overbrace{T,\dots,T}^{n}), \mathrm{diag}(\overbrace{E,\dots,E}^{i},\overbrace{-E,\dots,-E}^{n-i})\rangle,$$

$$\langle \mathrm{diag}(\overbrace{i,\dots,i}^{2n}), \mathrm{diag}(\overbrace{j,\dots,j}^{2n}), \mathrm{diag}(\overbrace{1,\dots,1}^{i},\overbrace{-1,\dots,-1}^{2n-i})\rangle,$$

$$\langle \mathrm{diag}(\overbrace{i,\dots,i}^{2n}), \mathrm{diag}(\overbrace{R,\dots,R}^{n}), \mathrm{diag}(\overbrace{T,\dots,T}^{n})\rangle$$

and their centralizers are

$$C_{\mathrm{PGL}(2n,\mathbf{H})}(H_+\# L(i,n-i)) = H_+\times C_{\mathrm{PGL}(n,\mathbf{H})}(L(i,n-i)),$$
$$C_{\mathrm{PGL}(2n,\mathbf{H})}(H_-\# L(i,n-i)) = H_-\times C_{\mathrm{PGL}(2n,\mathbf{R})}(L(i,2n-i)),$$
$$C_{\mathrm{PGL}(2n,\mathbf{H})}(V_0) = H_+\times C_{\mathrm{PGL}(n,\mathbf{H})}(I) \overset{(4.2)}{=} H_+\times \frac{\mathrm{GL}(n,\mathbf{C})}{\langle -E\rangle}\rtimes\langle j\langle -E\rangle\rangle,$$

so that (4.1, 2.55, 4.2) $ZC_{\mathrm{PGL}(2n,\mathbf{H})}(V)=V$ for all nontoral rank three objects V of $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))$. The elements of $H_+\# L(i,n-i)$, $H_-\# L(i,2n-i)$, and V_0 have traces (computed in $\mathrm{GL}(4n,\mathbf{C})$) in the sets $\pm\{0,4n-8i,4n\}$, $\pm\{0,4n-4i,4n\}$, and $\pm\{0,4n\}$.

4.15. *Rank four nontoral objects of* $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))$. The elementary abelian 2-group $H_-\# P(1,i-1,2n-i,0)\subset \mathrm{GL}(2n,\mathbf{H})$, $1<i\le n$, is

$$\langle \mathrm{diag}(\overbrace{i,\dots,i}^{2n}), \mathrm{diag}(\overbrace{j,\dots,j}^{2n}), \mathrm{diag}(1,\overbrace{-1,\dots,-1}^{i-1},\overbrace{1,\dots,1}^{2n-i}),$$
$$\mathrm{diag}(1,\overbrace{1,\dots,1}^{i-1},\overbrace{-1,\dots,-1}^{2n-i})\rangle.$$

The elements of P have traces in $\{2n+2-2i,-2n+2i,2n+1\}$ and these three integers are all distinct, so that the Quillen automorphism group (4.18) has order $3\cdot 2^5$. This nontoral rank four object contains the two nontoral rank three objects $H_-\# L(1,2n-1)$, $H_-\# L(2,2n-2)$ when $i=2$ and the three nontoral rank three objects $H_-\# L(1,2n-1)$, $H_-\# L(i-1,2n-i+1)$, $H_-\# L(i,2n-i)$ when $i>2$.

The elementary abelian 2-group $V_0\# L(i,n-i)\subset \mathrm{GL}(2n,\mathbf{C})\subset \mathrm{GL}(2n,\mathbf{H})$, $1\le i\le [n/2]$, is the subgroup

$$\langle \mathrm{diag}(\overbrace{i,\dots,i}^{2n}), \mathrm{diag}(\overbrace{R,\dots,R}^{n}), \mathrm{diag}(\overbrace{T,\dots,T}^{n}),$$
$$\mathrm{diag}(\overbrace{E,\dots,E}^{i},\overbrace{-E,\dots,-E}^{n-i})\rangle$$

containing the three rank three objects $H_+\#L(i,n-i)$, $H_-\#L(2i,2n-2i)$, and V_0.

For these nontoral rank four objects $E \subset \mathrm{GL}(2n,\mathbf{H})$, the center of the centralizer is finite (2.55) and as, of course, $E \subset ZC_{\mathrm{PGL}(2n,\mathbf{H})}(E)$ we see that $\mathrm{Hom}_{\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))}(\mathrm{St}(E),E)$ is a subspace of the $\mathbf{F}_2$-vector space $\mathrm{Hom}_{\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))}(\mathrm{St}(E),\pi_1 BZC_{\mathrm{PGL}(2n,\mathbf{H})}(E))$.

4. Higher limits of the functor $\pi_i(BZC)$ on $\mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))^{[\,,\,]\neq 0}$. In this section we compute the first higher limits of the center functors $\pi_i BZC_{\mathrm{PGL}(n,\mathbf{H})}$, $i=1,2$ [24, (2.47)].

4.16. Lemma. *The first higher limits of the center functors are*

$$\lim{}^1 \pi_1 BZC_{\mathrm{PGL}(n,\mathbf{H})} = 0 = \lim{}^2 \pi_1 BZC_{\mathrm{PGL}(n,\mathbf{H})},$$
$$\lim{}^2 \pi_2 BZC_{\mathrm{PGL}(n,\mathbf{H})} = 0 = \lim{}^3 \pi_2 BZC_{\mathrm{PGL}(n,\mathbf{H})}.$$

The case $i=2$ is easy. Since $\pi_2 BZC_{\mathrm{PGL}(n,\mathbf{H})}$ has value 0 on all objects of $\mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))^{[\,,\,]\neq 0}$ of rank ≤ 4, it is immediate from Oliver's cochain complex [31] that $\lim^2$ and $\lim^3$ of this functor are trivial. We shall therefore now concentrate on the case $i=1$.

For any elementary abelian 2-group E in $\mathrm{PGL}(n,\mathbf{H})$ we shall write

$$[E] = \mathrm{Hom}_{\mathbf{A}(\mathrm{PGL}(n,\mathbf{H})(E))}(\mathrm{St}(E),E) \tag{4.17}$$

for the $\mathbf{F}_2$-vector space of $\mathbf{F}_2\mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))(E)$-equivariant maps from the Steinberg representation $\mathrm{St}(E)$ over $\mathbf{F}_2$ of $\mathrm{GL}(E)$ to E. Oliver's cochain complex has the form (2.33).

4.18. Proposition. *Regardless of the parity of n, the Quillen automorphism groups are*

$$\mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))(H_-) = O^-(2,\mathbf{F}_2),$$
$$\mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))(H_-\#V) = \begin{pmatrix} O^-(2,\mathbf{F}_2) & * \\ 0 & \mathbf{A}(\mathrm{GL}(n,\mathbf{R}))(V) \end{pmatrix},$$

and $\dim_{\mathbf{F}_2}[H_-] = 1 = \dim_{\mathbf{F}_2}[H_-\#L(i,2n+1-i)]$ *as described in* 2.35 *and* 2.42.

Proof. $\mathbf{A}(\mathrm{GL}(n,\mathbf{H}))(2_-^{1+2}) = \mathrm{Out}(2_-^{1+2})$ since all automorphisms of 2_-^{1+2} preserve the trace. This group maps (isomorphically) to the subgroup $O^-(2,\mathbf{F}_2) \subset \mathrm{GL}(H_-)$ of automorphisms that preserve the quadratic function q on H_-. The Quillen automorphism group of $H_-\#V$ consists of the automorphisms that lift to trace preserving automorphisms of $2_-^{1+2}\#V$. The dimension of the vector spaces of equivariant maps was computed by *magma*. ∎

In the odd case of $\mathrm{GL}(2n+1,\mathbf{H})$ the cochain complex (2.33) takes the form

$$(4.19)\qquad 0 \to [H_-] \xrightarrow{d^1} \prod_{1\le i\le n} [H_-\#L(i,2n+1-i)] \xrightarrow{d^2} \prod_{|E|=2^4} [E] \xrightarrow{d^3} \cdots$$

and we need to show that d^1 is injective and that $\dim(\operatorname{im} d^2) \geq n-1$.

If $E = H_-\#P(i)$, where $P(i)$ is as in (4.3), then

$$\mathbf{A}(\mathrm{PGL}(2n+1,\mathbf{H}))(H_-\#P(i)) = \begin{pmatrix} O^-(2,\mathbf{F}_2) & * \\ 0 & \mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))(P(i)) \end{pmatrix}$$

where $\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))(P(i))$ is the group of trace preserving automorphisms of $P(i)$. It turns out that

$$\dim_{\mathbf{F}_2}[H_-\#P(i_0,i_1,i_2,i_3)] = \begin{cases} 2, & \mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))(P(i)) = \{E\}, \\ 1, & \mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))(P(i)) = C_2, \\ 0, & \mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))(P(i)) = \mathrm{GL}(2,\mathbf{F}_2). \end{cases}$$

When $n=1$ or $n=2$, the cochain complex (4.19) has the form

$$0 \to [H_-] \xrightarrow{d^1} [H_-\#L(1,2)] \xrightarrow{d^2} [H_-\#P(1,1,1,0)] \to \cdots$$

or respectively

$$0 \to [H_-] \xrightarrow{d^1} [H_-\#L(1,4)] \times [H_-\#L(2,3)] \\ \xrightarrow{d^2} [H_-\#P(1,1,3,0)] \times [H_-\#P(1,2,2,0)] \to \cdots,$$

where all vector spaces are one-dimensional. In the case of $n=1$, d^1 is an isomorphism, and in the case $n=2$, d^1 has matrix $(1 \quad 1)$ and d^2 has matrix $\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$. In case $n\geq 3$, it is enough to show that d^1 is injective and d^2 has rank $n-1$ in the cochain complex

$$0 \to [H_-] \xrightarrow{d^1} \prod_{1\le i\le n} [H_-\#L(i,2n+1-i)] \xrightarrow{d^2} \prod_{2<i\le n} [H_-\#P(1,i-1,2n-i+1,0)]$$

that agrees with (4.19) in degrees one, a product of one-dimensional vector spaces, and two, a product of two-dimensional vector spaces. The elementary abelian 2-group $H_-\#P(1,i-1,2n-i+1,0) \subset \mathrm{GL}(2n+1,\mathbf{H})$ contains the nontoral subspaces $H_-\#L(1,2n)$, $H_-\#L(i-1,2n-i+2)$, and $H_-\#L(i,2n-i+1)$. The map f_-, defined exactly as in (2.37), is the nonzero element of $[H_-]$, and the maps df_-, defined exactly as in (2.43), are nonzero in $H_-\#L(i,2n+1-i)$. It follows that d^1 is injective. A *magma* computation reveals that $\{ddf_{-L(i-1,2n-i+2)}, ddf_{-L(i,2n-i+1)}\}$, where these $\mathbf{F}_2\mathbf{A}(\mathrm{GL}(2n+1,\mathbf{H}))(H_-\#P(1,i-1,2n-i+1,0))$-maps are defined as in (2.45), is a basis for the two-dimensional space $H_-\#P(1,i-1,2n-i+1,0)$ and that $ddf_{-L(1,2n)} = ddf_{-L(i-1,2n-i+2)} + ddf_{-L(i,2n-i+1)}$. This shows that d^2 has rank $n-1$.

In the even case of $\mathrm{GL}(2n,\mathbf{H})$ the cochain complex (2.33) takes the form

$$0 \to [H_-]\times[H_+] \xrightarrow{d^1} \prod_{1\le i\le n}[H_-\#L(i,2n-i)]\times \prod_{1\le i\le [n/2]}[H_+\#L(i,n-i)]\times[V_0] \xrightarrow{d^2} \prod_{|E|=2^4}[E].$$

4.20. Proposition. *The automorphism groups of the low-degree nontoral objects of the Quillen category* $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))$ *are*

$$\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))(H_+) = O^+(2,\mathbf{F}_2),$$

$$\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))(H_+\#V) = \begin{pmatrix} O^+(2,\mathbf{F}_2) & * \\ 0 & \mathbf{A}(\mathrm{GL}(n,\mathbf{H}))(V)\end{pmatrix},$$

$$\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))(V_0) \cong \mathrm{Sp}(2,\mathbf{F}_2),$$

$$\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))(V_0\#L(i,n-i)) \cong \begin{pmatrix} \mathrm{Sp}(2,\mathbf{F}_2) & * \\ 0 & 1\end{pmatrix}.$$

Furthermore, $\dim_{\mathbf{F}_2}[H_+]=2$, $\dim_{\mathbf{F}_2}[H_+\#L(i,n-i)]=3$, $\dim_{\mathbf{F}_2}[V_0]=4$, *and* $\dim_{\mathbf{F}_2}[V_0\#L(i,n-i)]=5$ *as described in* 2.35, 2.40, 2.38, *and* (2.45).

Proof. The Quillen automorphism groups of the dihedral group 2^{1+2}_+ and the generalized extraspecial group $4\circ 2^{1+2}_\pm$ are the full outer automorphism groups because the traces are nonzero only on the derived groups which are characteristic. The images in $\mathrm{GL}(H_+)$, respectively, $\mathrm{GL}(V_0)$, isomorphic to $O^+(2,\mathbf{F}_2)\cong C_2$ and to $\mathrm{Sp}(2,\mathbf{F}_2)=\mathrm{GL}(2,\mathbf{F}_2)$, are the Quillen automorphism groups for H_+ and V_0. For the middle formula, recall that the trace of $H_\pm\#V$ is the product of the traces. ∎

As in the real case (Chap. 2) we see that d^1 embeds $[H_-]\times[H_+]$ into $[V_0]$. The only problem is to show that the rank of d^2 is $\ge n+3[n/2]+4-3 = n+3[n/2]+1$. We have to show that

$$\dim(\operatorname{im} d^2)\ge n+3[n/2]+1.$$

We do this by mapping the $n+[n/2]+1$ nontoral rank three objects (4.14)

- $[H_-\#L(i,2n-i)]$, $1\le i\le n$, with basis $\{df_-\}$ as in (2.43),
- $[H_+\#L(i,n-i)]$, $1\le i\le [n/2]$, with basis $\{df_+, df_0, f_0\}$ as in (2.41),
- $[V_0]$ with basis $\{df_+, df_0, df_-, f_0\}$ as in (2.39)

into the $(n-2)+[n/2]$ nontoral rank four objects (4.15)

- $H_-\#P(1,i-1,2n+1-i)$, $2<i\le n$, with basis

$$\{ddf_{-L(i-1,2n+1-i)}, ddf_{-L(i,2n-i)}\}$$

  where these maps are defined as the similar maps in (2.45),

- $V_0 \# L(i, n-i)$, $1 \le i \le [n/2]$, with basis
$$\{ddf_{+L(i,n-i)}, ddf_{0L(i,n-i)}, df_{0L(i,n-i)}, ddf_{-L(2i,2n-2i)}, df_{0V_0}\}$$
as in (2.45).

Computations with *magma* show that the resulting $(n+3[n/2]+4) \times (2n+5[n/2])$ matrix has rank $n+3[n/2]+1$. The matrix has the form (shown here for $n=5$)

	$[H_-\#P(1,2,7)]$	$[H_-\#P(1,3,6)]$	$[H_-\#P(1,4,5)]$	$V_0\#L(1,4)$	$V_0\#L(2,3)$
$H_-\#L(1,9)$	$(1\ \ 1)$	$(1\ \ 1)$	$(1\ \ 1)$		
$H_-\#L(2,8)$	$(1\ \ 0)$			$(0\,0\,0\,1\,0)$	
$H_-\#L(3,7)$	$(0\ \ 1)$	$(1\ \ 0)$			
$H_-\#L(4,6)$		$(0\ \ 1)$	$(1\ \ 0)$		$(0\,0\,0\,1\,0)$
$H_-\#L(5,5)$			$(0\ \ 1)$		
$H_+\#L(1,4)$				A	
$H_+\#L(2,3)$					A
V_0				B	B

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

5. PROOFS OF THE MAIN THEOREMS

This chapter contains the proofs of the main results stated in the introduction of this Part II and also of three theorems from Part I.

1. Proof of Theorem 1.1. The proof of Theorem 1.1 uses induction over n simultaneously applied to the three infinite families $\mathrm{PSL}(2n, \mathbf{R})$, $\mathrm{SL}(2n+1, \mathbf{R})$, and $\mathrm{PGL}(n, \mathbf{H})$.

Note first that the proof of [24, Theorem 2.51] goes through with only insignificant changes if we replace hypotheses (1) and (2) by

(1 & 2) The centralizer of any toral $(V, \nu) \in \mathrm{Ob}(\mathbf{A}(X)^{\le t}_{\le 2})$ is uniquely N-determined

and leave the other conditions unchanged.

Proof of Theorem 1.1. The statement of the theorem means [24, Definition 2.10] that the 2-compact groups

- $\mathrm{PSL}(2n, \mathbf{R})$, $\mathrm{SL}(2n+1, \mathbf{R})$, and $\mathrm{PGL}(n, \mathbf{H})$ have $\pi_*(N)$-determined automorphisms,
- $\mathrm{PSL}(2n, \mathbf{R})$, $\mathrm{SL}(2n+1, \mathbf{R})$, and $\mathrm{PGL}(n, \mathbf{H})$ are N-determined.

We may inductively assume the connected 2-compact groups $\mathrm{PSL}(2i, \mathbf{R})$, $1 \le i \le n-1$, $\mathrm{SL}(2i+1, \mathbf{R})$, $1 \le i < n-1$, and $\mathrm{PGL}(i, \mathbf{H})$, $1 \le i < n$, to be uniquely N-determined. From [24, Theorem 1.4] we know that $\mathrm{PGL}(i, \mathbf{C})$ is uniquely N-determined for all $i \ge 1$. The plan is now to use [24, 2.48, 2.51] inductively.

Consider first the connected, centerless 2-compact group $\mathrm{PSL}(2n, \mathbf{R})$.

$\mathrm{PSL}(2n, \mathbf{R})$ has N-determined automorphisms: According to [24, 2.48] it suffices to show that:

(1) $C_{\mathrm{PSL}(2n,\mathbf{R})}(L)$ has N-determined automorphisms for any rank one elementary abelian 2-group $L \subset \mathrm{PSL}(2n, \mathbf{R})$.
(2) The limit $\lim^1(\mathbf{A}(\mathrm{PSL}(2n, \mathbf{R})); \pi_1 BZC_{\mathrm{PSL}(2n,\mathbf{R})})$ is 0 and the limit $\lim^2(\mathbf{A}(\mathrm{PSL}(2n, \mathbf{R})); \pi_2 BZC_{\mathrm{PSL}(2n,\mathbf{R})})$ is 0.

Item (2) is proved in 2.32. The centralizers that occur in item (1) are listed in (2.14) and (2.15). That the centralizers from (2.14) have N-determined automorphisms follows, under the induction hypothesis that the 2-compact groups $\mathrm{PSL}(2i, \mathbf{R})$, $1 \le i \le n-1$, have N-determined automorphisms, from general hereditary properties of N-determined 2-compact groups [24, Chapter 2]. Note here that $\check{Z}(C_0) = \check{T}(C_0)^{W(C_0)}$ for $C = C_{\mathrm{PSL}(2n,\mathbf{R})}(L)$ by [22, 1.6]. Similarly, the centralizers from (2.15) have N-determined automorphisms because the 2-compact groups $\mathrm{PGL}(n, \mathbf{C})$, $1 \le n < \infty$, have N-determined automorphisms [24, Theorem 1.4].

$\mathrm{PSL}(2n, \mathbf{R})$ is N-determined: We verify the four conditions of [24, Theorem 2.51]. Let $V \subset \mathrm{PSL}(2n, \mathbf{R})$ be a toral elementary abelian 2-group of rank at most 2. The centralizer $C = C_{\mathrm{PSL}(2n,\mathbf{R})}(V)$ is one of the 2-compact groups listed in (2.14), (2.16), (2.15), or (2.17), so it is LHS (2.19). The identity component C_0 of C satisfies the equation $\check{Z}(C_0) = \check{T}(C_0)^{W(C_0)}$ [22, 1.6] and the adjoint form is

$$PC_0 = \begin{cases} \mathrm{PSL}(2i_0, \mathbf{R}) \times \mathrm{PSL}(2i_1, \mathbf{R}), & i_0 + i_1 = n, \\ \prod_{j=0}^{3} \mathrm{PSL}(2i_j, \mathbf{R}), & i_0 + i_1 + i_2 + i_3 = n, \\ \mathrm{PGL}(n, \mathbf{C}), & \\ \mathrm{PGL}(i_0, \mathbf{C}) \times \mathrm{PGL}(i_1, \mathbf{C}), & i_0 + i_1 = n, \end{cases}$$

in these four cases. The induction hypothesis and the general results of [24, Chapter 2, §2] imply that C_0 is uniquely N-determined and that C is totally N-determined. Since also the homomorphism

$$H^1(W; \check{T}) \twoheadrightarrow \lim{}^1(\mathbf{A}(\mathrm{PSL}(2n, \mathbf{R}))^{\le t}_{\le 2}, H^1(W_0; \check{T})^{W/W_0})$$

is surjective (2.20), we deduce from [24, Lemma 2.54] that the first two conditions of [24, Theorem 2.51] are satisfied. The third condition has been verified in Chapter 2, §4, and the fourth, and final, condition in 2.32.

$\mathrm{PSL}(2n,\mathbf{R})$ has $\pi_*(N)$-determined automorphisms: This means that the only automorphism of $\mathrm{PSL}(2n,\mathbf{R})$ that restricts to the identity on the maximal torus is the identity, i.e. that

$$H^1(W;\check{T})(\mathrm{PSL}(2n,\mathbf{R}))\cap \mathrm{AM}(\mathrm{Aut}(\mathrm{PSL}(2n,\mathbf{R})))=\{0\}$$

where AM is the Adams–Mahmud homomorphism [24, (2.4)]. For $n>4$, $H^1(W;\check{T})(\mathrm{PSL}(2n,\mathbf{R}))=0$, and there is nothing to prove. Consider the case $n=4$. Let f be an automorphism of $\mathrm{PSL}(8,\mathbf{R})$ such that $\mathrm{AM}(f)\in H^1(W;\check{T})$. Let $L\subset \mathrm{PSL}(8,\mathbf{R})$ be any rank one elementary abelian 2-group. Since f is the identity on the maximal torus, $f(L)$ is conjugate to L, so that f restricts to an automorphism of $C_{\mathrm{PSL}(8,\mathbf{R})}(L)$ and to an automorphism of the identity component of $C_{\mathrm{PSL}(8,\mathbf{R})}(L)$. Since $C_{\mathrm{PSL}(8,\mathbf{R})}(L)_0$ has $\pi_*(N)$-determined automorphisms by [24, Lemmas 2.38–2.39], it follows that $f\in H^1(W;\check{T})(\mathrm{PSL}(2n,\mathbf{R}))$ restricts to 0 in $H^1(W;\check{T})(C_{\mathrm{PSL}(8,\mathbf{R})}(L)_0)$. However, the restriction map is injective (see the proof of 2.20) so that $f=0$. This shows that $\mathrm{PSL}(8,\mathbf{R})$ has $\pi_*(N)$-determined automorphisms.

Consider next the 2-compact group $\mathrm{SL}(2m+1,\mathbf{R})$ where $m=n-1$.

$\mathrm{SL}(2m+1,\mathbf{R})$ has N-determined automorphisms: We verify the conditions of [24, Lemma 2.48]. Let $L\subset \mathrm{SL}(2m+1,\mathbf{R})$ be an elementary abelian 2-group of rank 1. The centralizer $C=C_{\mathrm{SL}(2m+1,\mathbf{R})}(L)$ is given in (3.3). According to [24, Chapter 2, §2], C has N-determined automorphisms. (Use the natural splitting of (3.10) in connection with [24, Lemma 2.35].) See 3.8 for the vanishing of the higher limits.

$\mathrm{SL}(2m+1,\mathbf{R})$ is N-determined: Conditions (1) and (2) of [24, Theorem 2.51] are verified in Chapter 3, §2, condition (3) in Chapter 3, §3, and condition (4) in 3.8.

$\mathrm{SL}(2m+1,\mathbf{R})$ has $\pi_*(N)$-determined automorphisms: To prove this, it suffices to find a rank one elementary abelian 2-group $L\subset \mathrm{SL}(2m+1,\mathbf{R})$ such that $C_{\mathrm{SL}(2m+1,\mathbf{R})}(L)_0$ has $\pi_*(N)$-determined automorphisms and such that $C_{\mathrm{SL}(2m+1,\mathbf{R})}(L)_0\to \mathrm{SL}(2m+1,\mathbf{R})$ induces a monomorphism on $H^1(W;\check{T})$. Such a line is provided by $L=L(2m-1,2)$ with centralizer identity component $C_{\mathrm{SL}(2m+1,\mathbf{R})}(L)_0=\mathrm{SL}(2m-1,\mathbf{R})\times \mathrm{SL}(2,\mathbf{R})$; see the proof of 3.14.

Consider finally the 2-compact group $\mathrm{PGL}(n,\mathbf{H})$ for $n\geq 3$.

$\mathrm{PGL}(n,\mathbf{H})$ has N-determined automorphisms: We verify the conditions of [24, Lemma 2.48]. Let $L\subset \mathrm{PGL}(n,\mathbf{H})$ be an elementary abelian 2-group of rank one. The centralizer $C=C_{\mathrm{PGL}(n,\mathbf{H})}(L)$ is given in (4.1) and (4.2).

According to the general results of [24, Chapter 2, §2], C has N-determined automorphisms, and according to 4.16, the higher limits vanish.

PGL(n, **H**) is N-determined: Note that $\mathrm{PGL}(3,\mathbf{H})$ satisfies condition (1 & 2) so that we may apply the variant of [24, Theorem 2.51] mentioned above. When $n > 3$, conditions (1) and (2) of that theorem follow if we can verify that the conditions of [24, Lemma 2.54] are satisfied. That the centralizer $C_{\mathrm{PGL}(n,\mathbf{H})}(V)$ (4.1, 4.2, 4.3, 4.4), where V is an elementary abelian 2-group of rank at most two, satisfies the conditions of [24, Lemma 2.54] is a consequence of the general results of [24, Chapter 2, §2] and 4.7, 4.8. See 4.9 and 4.11 for condition (3) and 4.16 for condition (4) of [24, Theorem 2.51].

PGL(n, **H**) has $\pi_*(N)$-determined automorphisms: We only need to consider the cases $n = 3$ and $n = 4$ as $H^1(W;\check{T})(\mathrm{PGL}(n,\mathbf{H})) = 0$ for $n > 4$ [16, Main Theorem]. In those two cases, it suffices, as above, to find a rank one elementary abelian 2-group $L \subset \mathrm{PGL}(n,\mathbf{H})$ such that $C_{\mathrm{PGL}(n,\mathbf{H})}(L)_0$ has $\pi_*(N)$-determined automorphisms and such that $C_{\mathrm{PGL}(n,\mathbf{H}))}(L)_0 \to \mathrm{PGL}(n,\mathbf{H})$ induces a monomorphism on $H^1(W;\check{T})$. Such a line is provided by $L = I$, for which $C_{\mathrm{PGL}(n,\mathbf{H})}(I)_0 = \mathrm{GL}(n,\mathbf{C})/\langle -E\rangle$ (4.2).

Since $\mathrm{PSL}(2n,\mathbf{R})$, $n \geq 4$, is uniquely N-determined and has a split maximal torus normalizer, we see that its automorphism group is isomorphic to $W\backslash N_{\mathrm{GL}(L)}(W)$ by [24, Lemma 2.16]. When $n = 4$, the group, $\mathrm{Out}_{\mathrm{tr}}(W)$, on the right in the exact sequence [24, (2.8)] is the permutation group Σ_3. There are Lie group outer automorphisms inducing Σ_3. When $n > 4$,

$$\begin{aligned}
&\mathrm{Aut}(\mathrm{PSL}(2n,\mathbf{R}))\\
&\qquad \cong W\backslash N_{\mathrm{GL}(L)}(W) = W\backslash\langle \mathbf{Z}_2^*, W(\mathrm{PGL}(2n,\mathbf{R}))\rangle = W\backslash\langle \mathbf{Z}_2^\times, W, c_1\rangle\\
&\qquad = (W\cap\langle \mathbf{Z}_2^\times, c_1\rangle)\backslash\langle \mathbf{Z}_2^\times, c_1\rangle\\
&\qquad = \begin{cases} \langle -c_1\rangle\backslash\langle \mathbf{Z}_2^\times, c_1\rangle = \mathbf{Z}_2^\times, & n \text{ odd},\\ \langle -1\rangle\backslash\langle \mathbf{Z}_2^\times, c_1\rangle = \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times \times \langle c_1\rangle, & n \text{ even}.\end{cases}
\end{aligned}$$

Similarly,

$$\begin{aligned}\mathrm{Aut}(\mathrm{SL}(2n+1,\mathbf{R})) &\cong W\backslash N_{\mathrm{GL}(L)}(W) = W\backslash\langle \mathbf{Z}_2^\times, W\rangle\\ &= (W\cap \mathbf{Z}_2^\times)\backslash\mathbf{Z}_2^\times = \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times\end{aligned}$$

for $n \geq 2$ by [24, Lemma 2.16].

The automorphism group $\mathrm{Aut}(\mathrm{PGL}(n,\mathbf{H}))$, $n \geq 3$, is contained in the quotient group $W\backslash N_{\mathrm{GL}(L)}(W) \cong \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$ [24, Lemma 2.16]. Since $H^2(W;\check{T})$ is an elementary abelian 2-group [21], it is isomorphic to the second cohomology group $H^2(W;t(\mathrm{PGL}(n,\mathbf{H})))$ with coefficient module $t(\mathrm{PGL}(n,\mathbf{H}))$, the maximal elementary abelian 2-group in the maximal torus. The unstable

Adams operations with index in $\mathbf{Z}_2^\times$ act trivially here since they act as coefficient group automorphisms. Thus all elements of $W\backslash N_{\mathrm{GL}(L)}(W)$ preserve the extension class $e \in H^2(W;\check{T})$ and we conclude that $\mathrm{Aut}(\mathrm{PGL}(n,\mathbf{H})) \cong \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$. ■

Proof of Corollary 1.6. Note first that $\mathrm{GL}(n,\mathbf{R})$ is LHS for all $n \geq 1$. If n is odd, $\mathrm{GL}(n,\mathbf{R}) = \mathrm{SL}(n,\mathbf{R}) \times \langle -E\rangle$ is LHS because its Weyl group is the direct product of the Weyl group of the identity component with the component group. If n is even, see [24, Example 2.29(5)]. According to [24, 2.35, 2.40], $\mathrm{GL}(n,\mathbf{R})$ is totally N-determined.

If n is odd, the identity component has trivial center, so that the automorphism group is $\mathrm{Aut}(\mathrm{GL}(n,\mathbf{R})) = \mathrm{Aut}(\mathrm{SL}(n,\mathbf{R})) = \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$ by the short exact sequence [25, 5.2].

Suppose next that $n = 2m$ is even. When $m = 1$, $\mathrm{Aut}(\mathrm{GL}(2,\mathbf{R})) = \mathrm{Aut}(\mathbf{Z}/2,\mathbf{Z}/2^\infty,0) = \mathrm{Aut}(\mathbf{Z}/2^\infty) = \mathbf{Z}_2^\times$ according to [24, (2.6)]. When $m > 1$, $H^1(\pi;\check{Z}(\mathrm{SL}(2m,\mathbf{R}))) = H^1(\pi;\langle -E\rangle)$ is the order two subgroup $\langle\delta\rangle$ of $\mathrm{Aut}(\mathrm{GL}(2m,\mathbf{R}))$ generated by the group isomorphism $\delta(A) = (\det A)A$, $A \in \mathrm{GL}(2m,\mathbf{R})$, and $H^1(W;\check{T}) = \mathrm{Hom}(W_{\mathrm{ab}},\langle -E\rangle) = \mathbf{Z}/2 \times \mathbf{Z}/2$ (for $m > 2$) [16, 21] is the middle group of an exact sequence

$$0 \to H^1(\pi;\langle -E\rangle) \to H^1(W;\check{T}) \to H^1(W_0;\check{T}) \to 0$$

because $\mathrm{GL}(2m,\mathbf{R})$ is LHS. (For $m = 2$, the cohomology group $H^1(W_0;\check{T})$ is trivial and $H^1(\pi;\check{Z}(\mathrm{SL}(2m,\mathbf{R}))) = \mathbf{Z}/2$, though.) In the exact sequence [24, (2.5)] for the automorphism group of $N = N(\mathrm{GL}(2m,\mathbf{R})) = N(\mathrm{SL}(2m+1,\mathbf{R}))$, the group on the right hand side is $\mathrm{Aut}(W,\check{T},0) = \langle W,\mathbf{Z}_2^\times\rangle$ as for $\mathrm{SL}(2m+1,\mathbf{R})$. Thus $\mathrm{Aut}(N)$ is generated by $H^1(W;\check{T})$, W, and $\mathbf{Z}_2^\times$, so that $\mathrm{Aut}(N,N_0) = \mathrm{Aut}(N)$ as W_0 is normal in W. Note that these three subgroups of $\mathrm{Aut}(N,N_0)$ commute because of the special form of the elements of $H^1(W;\check{T}) = \mathrm{Hom}(W_{\mathrm{ab}},\langle -E\rangle)$. Hence

$$\begin{aligned}
\frac{\mathrm{Aut}(N,N_0)}{W_0}
&= \frac{\langle H^1(W;\check{T}),W,\mathbf{Z}_2^\times\rangle}{W_0} = \frac{\langle H^1(W;\check{T}),W_0,c_1,\mathbf{Z}_2^\times\rangle}{W_0}\\
&= \frac{\langle H^1(W;\check{T}),c_1,\mathbf{Z}_2^\times\rangle}{W_0 \cap \langle H^1(W;\check{T}),c_1,\mathbf{Z}_2^\times\rangle}\\
&= \begin{cases} \dfrac{\langle H^1(W;\check{T}),c_1,\mathbf{Z}_2^\times\rangle}{\langle -c_1\rangle} = H^1(W;\check{T}) \times \mathbf{Z}_2^\times, & m \text{ odd},\\[2ex] \dfrac{\langle H^1(W;\check{T}),c_1,\mathbf{Z}_2^\times\rangle}{\langle -1\rangle} = H^1(W;\check{T}) \times \langle c_1\rangle \times \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times, & m \text{ even}.\end{cases}
\end{aligned}$$

According to [24, Lemma 2.17], the automorphism group $\mathrm{Aut}(\mathrm{GL}(2m,\mathbf{R}))$

is a subgroup of the above group and

$$\operatorname{Aut}(\mathrm{GL}(2m,\mathbf{R})) = \begin{cases} \langle\delta\rangle \times \mathbf{Z}_2^\times, & m \text{ odd},\\ \langle\delta\rangle \times \langle c_1\rangle \times \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times, & m \text{ even},\end{cases}$$

for $m > 1$. ■

2. The 2-compact group G_2**.** The group $B\mathrm{G}_2$ is a rank two 2-compact group containing a rank three elementary abelian 2-group $E_3 \subset \mathrm{G}_2$ such that $\mathbf{A}(\mathrm{G}_2)(E_3) = \mathrm{GL}(3,\mathbf{F}_2)$ [14, 6.1] [11, 5.3] and

$$H^*(B\mathrm{G}_2;\mathbf{F}_2) \cong H^*(BE_3;\mathbf{F}_2)^{\mathrm{GL}(3,\mathbf{F}_2)} \cong \mathbf{F}_2[c_4,c_6,c_7]$$

realizes the mod 2 rank 3 Dickson algebra [23]. The Quillen category $\mathbf{A}(\mathrm{G}_2)$ contains exactly one isomorphism class of objects E_1, E_2, E_3 of ranks $1,2,3$, as Lannes theory [20] implies that the inclusion functor

$$\mathbf{A}(\mathrm{GL}(3,\mathbf{F}_2),E_3) \to \mathbf{A}(\mathrm{G}_2)$$

is an equivalence of categories. The centralizers of $E_1 \subset E_2 \subset E_3$ are

$$\mathrm{SO}(4) \supset T \rtimes \langle -E\rangle \supset E_3.$$

In all three cases, $ZC_{\mathrm{G}_2}(E_i) = E_i$ so that $\pi_2 BZC_{\mathrm{G}_2} = 0$ and $\pi_1 BZC_{\mathrm{G}_2} = H^0(\mathrm{GL}(3,\mathbf{F}_2)(-);E_3)$. Thus $\pi_1 BZC_{\mathrm{G}_2}$ is an exact functor [24, Lemma 2.69] with $\lim^0 \pi_1 BZC_{\mathrm{G}_2} = H^0(\mathrm{GL}(3,\mathbf{F}_2);E_3) = 0$.

The Weyl group $W(\mathrm{G}_2) \subset \mathrm{GL}(2,\mathbf{Z}) \subset \mathrm{GL}(2,\mathbf{Z}_2)$, of order 12, is generated by the two matrices [5, VI.4.13]

$$\begin{pmatrix} -1 & 0\\ 3 & 1\end{pmatrix}, \quad \begin{pmatrix} 1 & 1\\ 0 & -1\end{pmatrix}$$

and the maximal torus normalizer $N(\mathrm{G}_2)$ is the semidirect product of the maximal torus and the Weyl group [8].

It is known that $H^0(W;\check{T})(\mathrm{G}_2) = 0$, that $H^1(W;\check{T})(\mathrm{G}_2) = 0$, and that $H^2(W;\check{T})(\mathrm{G}_2) = 0$ [16, 15].

Proof of Theorem 1.2. The rank one centralizer

$$\mathrm{SL}(4,\mathbf{R}) = \mathrm{SL}(2,\mathbf{C}) \circ \mathrm{SL}(2,\mathbf{C})$$

is uniquely N-determined by [24, Theorem 1.4] and [24, Chapter 2.§2]. Condition (2) of [24, Theorem 2.51] is satisfied because $H^1(W(X);\check{T}(X)) = 0$ for $X = \mathrm{G}_2, \mathrm{SL}(4,\mathbf{R})$ [16, Main Theorem]. Conditions (1) and (3) are satisfied because the only rank two object in G_2 is toral and its centralizer is a 2-compact toral group. We noted above that the higher limits vanish. Now [24, 2.48, 2.51] show that G_2 is uniquely N-determined.

We have $\operatorname{Aut}(\mathrm{G}_2) = W(\mathrm{G}_2)\backslash N_{\mathrm{GL}(2,\mathbf{Z}_2)}(W(\mathrm{G}_2))$ [24, Lemma 2.16] as the extension class is $e(\mathrm{G}_2) = 0$ [8]. The exact sequence [24, (2.8)] can be used to calculate the automorphism group. Using the description of the root system from [5, VI.4.13] with short root $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and long root $\alpha_2 = 2\varepsilon - \varepsilon_2 - \varepsilon_3$ generating the integral lattice in $\mathbf{Z}_2^3$ one finds that

$$N_{\mathrm{GL}(2,\mathbf{Z}_2)}(W(\mathrm{G}_2)) = \langle \mathbf{Z}_2^\times, A, W(\mathrm{G}_2)\rangle, \qquad A = \sqrt{-3}\begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix},$$

and therefore $\mathrm{Aut}(\mathrm{G}_2) = \mathbf{Z}_2^\times/\mathbf{Z}^\times \times C_2$ where the cyclic group of order two is generated by the exotic automorphism A interchanging the two roots. ∎

3. The 2-compact group $\mathrm{DI}(4)$**.** $B\mathrm{DI}(4)$ is a rank three 2-compact group containing a rank four elementary abelian 2-group $E_4 \subset \mathrm{DI}(4)$ such that $\mathbf{A}(\mathrm{DI}(4))(E_4) = \mathrm{GL}(4, \mathbf{F}_2)$ and [9]

$$H^*(B\mathrm{DI}(4); \mathbf{F}_2) \cong H^*(BE_4; \mathbf{F}_2)^{\mathrm{GL}(4,\mathbf{F}_2)} \cong \mathbf{F}_2[c_8, c_{12}, c_{14}, c_{15}]$$

realizes the mod 2 rank 4 Dickson algebra. Lannes theory [20] implies that the Quillen category $\mathbf{A}(\mathrm{DI}(4))$ is equivalent to $\mathbf{A}(\mathrm{GL}(4, \mathbf{F}_2), E_4)$ with exactly one elementary abelian 2-group (isomorphism class), $E_1, \ldots, E_4$, of each rank $1, \ldots, 4$. The centralizers of the toral subgroups E_1, E_2, E_3 and the nontoral subgroup E_4 are, respectively,

$$\mathrm{Spin}(7) \supset \mathrm{SU}(2)^3/\langle(-E, -E, -E)\rangle \supset T \rtimes \langle -E\rangle \supset E_4$$

and $ZC_{\mathrm{DI}(4)}(E_i) = E_i$ in all four cases, so that the functor

$$\pi_j BZC_{\mathrm{DI}(4)} \colon \mathbf{A}(\mathrm{GL}(4, \mathbf{F}_2), E_4) \to \mathbf{Ab}$$

is the 0-functor for $j = 2$ and equivalent to the functor $H^0(\mathrm{GL}(4, \mathbf{F}_2)(-); E_4)$ for $j = 1$. This is an exact functor [24, Lemma 2.69] and $\lim^0 \pi_1 BZC_{\mathrm{DI}(4)} = H^0(\mathrm{GL}(4, \mathbf{F}_2); E_4) = 0$.

As may be seen from [34], the Weyl group $W(\mathrm{DI}(4)) \subset \mathrm{GL}(3, \mathbf{Z}_2)$ of order $2|\mathrm{GL}(3, \mathbf{F}_2)| = 336$ is generated by the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -v & 0 & v^2+v \\ -1 & 1 & v \\ -2v & 0 & v \end{pmatrix}$$

where $v \in \mathbf{Z}_2$ is the unique 2-adic integer with $2v^2 - v + 1 = 0$. The first three matrices generate $W(\mathrm{Spin}(7))$ [7, 3.9, 3.11]. Since $W(\mathrm{DI}(4))$ is isomorphic to $\mathrm{GL}(3, \mathbf{F}_2) \times \langle -E\rangle$,

$$\begin{aligned} H^n(W; \check{T})(\mathrm{DI}(4)) &= \bigoplus_{2i \le n} H^{n-2i}(\mathrm{GL}(3, \mathbf{F}_2); H^{2i}(\langle -E\rangle; \check{T})) \\ &= \bigoplus_{2i \le n} H^{n-2i}(\mathrm{GL}(3, \mathbf{F}_2); (\mathbf{Z}/2)^3) \end{aligned}$$

and, in particular,

$$H^0(W;\check{T})(\mathrm{DI}(4)) = 0, \quad H^1(W;\check{T})(\mathrm{DI}(4)) = \mathbf{Z}/2,$$
$$H^2(W;\check{T})(\mathrm{DI}(4)) = \mathbf{Z}/2.$$

We may characterize the maximal torus normalizer short exact sequence for DI(4) as the unique nonsplit extension of $\check{T}$ by $W(\mathrm{DI}(4))$; it is nonsplit because the restriction to $W(\mathrm{Spin}(7)) \subset W(\mathrm{DI}(4))$ is nonsplit [8].

We cannot use [24, Theorem 2.51] as it stands because condition (2) fails: the restriction map

$$\mathbf{Z}/2 = H^1(W;\check{T})(\mathrm{DI}(4)) \to H^1(W;\check{T})(\mathrm{Spin}(7)) \overset{[16]}{=} (\mathbf{Z}/2)^2$$

is not surjective. Instead, we use the version of [24, Theorem 2.51] where the first two conditions have been replaced by (1 & 2).

Proof of Theorem 1.5. Condition (1 & 2) is satisfied for DI(4) since the connected 2-compact groups Spin(7) and $\mathrm{SU}(2)^2/\Delta$ are uniquely N-determined by [24, Theorem 1.4], Theorem 1.1, and the general results of [24, Chapter 2, §2]. Since also the relevant higher limits vanish [9, 2.4], DI(4) is uniquely N-determined by [24, 2.48, 2.51]. Since $\mathrm{Out_{tr}}(W(\mathrm{DI}(4)))$ is trivial and $Z(W(\mathrm{DI}(4))) = \langle -E\rangle$ has order two, Aut(DI(4)) can be read off from [24, (2.8), Lemma 2.16]. ∎

4. The 2-compact group F_4**.** $B\mathrm{F}_4$ is a rank four 2-compact group containing a rank five elementary abelian 2-group $E_5 \subset \mathrm{F}_4$ such that [35, 2.1]

$$H^*(B\mathrm{F}_4;\mathbf{F}_2) \cong H^*(BE_5;\mathbf{F}_2)^{\mathbf{A}(\mathrm{F}_4)(E_5)} \cong \mathbf{F}_2[y_4, y_6, y_7, y_{16}, y_{24}],$$

where the Quillen automorphism group is the parabolic subgroup

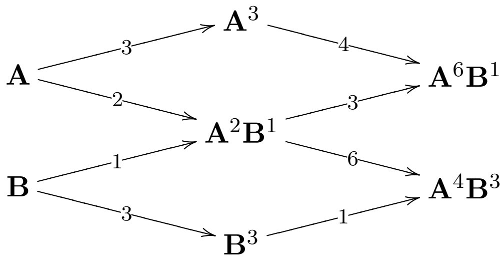
$$\mathbf{A}(\mathrm{F}_4)(E_5) = \begin{pmatrix} \mathrm{GL}(2,\mathbf{F}_2) & * \\ 0 & \mathrm{GL}(3,\mathbf{F}_2) \end{pmatrix} \subset \mathrm{GL}(5,\mathbf{F}_2)$$

of order $2^6|\mathrm{GL}(2,\mathbf{F}_2)|\,|\mathrm{GL}(3,\mathbf{F}_2)|$. The inclusion functor

$$\mathbf{A}(\mathbf{A}(\mathrm{F}_4)(E_5), E_5) \to \mathbf{A}(\mathrm{F}_4)$$

is a category equivalence by Lannes theory [20]. Inspection of the list of centralizers of elementary abelian 2-groups in F_4 [35, 3.2] shows that $ZC_{\mathrm{F}_4}(V) = V$ for each nontrivial $V \subset \mathrm{E}_5$ so that the functor $\pi_2 BZC_{\mathrm{F}_4}$ is 0 and $\pi_1 BZC_{\mathrm{F}_4} = H^0(\mathbf{A}(\mathrm{F}_4)(E_5)(-); E_5)$. Thus $\pi_1 BZC_{\mathrm{F}_4}$ is an exact functor [24, Lemma 2.69] and $\lim^0 \pi_1 BZC_{\mathrm{F}_4} = H^0(\mathbf{A}(\mathrm{F}_4)(E_5); E_5) = 0$.

It is known that $H^0(W;\check{T})(\mathrm{F}_4) = 0$, that $H^1(W;\check{T})(\mathrm{F}_4) = 0$, and that $H^2(W;\check{T})(\mathrm{F}_4) = \mathbf{Z}/2$ [16, 15]. The poset of the toral part of the Quillen category is



where the labels on the arrows indicate multiplicities.

Proof of Theorem 1.3. Table 1, based on [18, 6.11], [14, 2.14, 7.3, 7.4], [35, 3.2] and explicit computation, collects information about the toral objects of rank ≤ 2 of the Quillen category of F_4. (The third column shows the cohomological dimension of the centralizer and the fourth column the number of reflections in W_0, the Weyl group of the identity component of the centralizer.)

Table 1. Toral subgroups of F_4 of rank ≤ 2

Class	Centralizer	dim	refl.	$\lvert W_0\rvert$	$H^1(W_0;\check{T})$
$2\mathbf{A}$	$\mathrm{SU}(2)\times_{C_2}\mathrm{Sp}(3)$	24	10	$2^5 3^1$	$(\mathbf{Z}/2)^2$
$2\mathbf{B}$	$\mathrm{Spin}(9)$	36	16	$2^7 3^1$	$\mathbf{Z}/2$
$4\mathbf{A}^3$	$(\mathrm{U}(1)\times_{C_2}\mathrm{U}(3))\rtimes C_2$	10	3	$2^1 3^1$	$(\mathbf{Z}/2)^2$
$4\mathbf{A}^2\mathbf{B}^1$	$\mathrm{Spin}(4)\times_{C_2}\mathrm{Spin}(5)$	16	6	2^5	$(\mathbf{Z}/2)^5$
$4\mathbf{B}^3$	$\mathrm{Spin}(8)$	28	12	$2^6 3^1$	$(\mathbf{Z}/2)^2$

Condition (1 & 2) is satisfied for F_4 because centralizers of rank one objects and centralizers of rank two objects have uniquely N-determined centralizers. This follows from [24, Chapter 2, §2] as their simple factors are uniquely N-determined by [24, Theorem 1.4] and Theorem 1.1; note in particular that the unique nonconnected centralizer is uniquely N-determined according to [24, Lemma 2.37]. We already noted that the relevant higher limits vanish, and since there are no nontoral elementary abelian 2-groups of rank two [35, 3.2], F_4 is uniquely N-determined by [24, 2.48, 2.51].

The automorphism group of the 2-compact group F_4 is the middle term of the exact sequence [24, (2.8), Lemma 2.16]. (All automorphisms of F_4 automatically preserve the extension class $e(\mathrm{F}_4)$, which is the nontrivial element of $H^2(W;\check{T}) = \mathbf{Z}/2$ [8, 21].) The group $\mathrm{Out}_{\mathrm{tr}}(W(\mathrm{F}_4))$ of trace preserving outer automorphisms is cyclic of order two but the nontrivial outer automorphism of $W(\mathrm{F}_4)$ cannot be realized as conjugation with an element of $N_{\mathrm{GL}(L)}(W)$. The center of $W(\mathrm{F}_4)$ is $C_2 = \langle -E\rangle$. We conclude that $\mathrm{Aut}(\mathrm{F}_4) = \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$ consists entirely of unstable Adams operations. ∎

5. The E-family. We consider the centerfree simple 2-compact groups E_6, PE_7, and E_8.

Proof of Theorem 1.4. Using the information from [14, 18] one finds that the Quillen category for E_6 contains six isomorphism classes of toral objects of rank at most two. Their class distribution and centralizers are listed in Table 2, in which $S(\mathrm{U}(m)\times\mathrm{U}(n)) = \mathrm{U}(1)\times_{C_{\mathrm{lcm}(m,n)}}(\mathrm{SU}(m)\times\mathrm{SU}(n))$

Table 2. Elementary abelian toral subgroups of E_6 of rank ≤ 2

Class	Centralizer	dim	refl.	$\lvert W_0\rvert$	$H^1(W_0;\check{T})$
$2\mathbf{A}$	$\mathrm{SU}(2)\times_{C_2}\mathrm{SU}(6)$	38	16	$2^5 3^2 5^1$	$\mathbf{Z}/2$
$2\mathbf{B}$	$\mathrm{U}(1)\times_{C_4}\mathrm{Spin}(10)$	46	20	$2^7 3^1 5^1$	$\mathbf{Z}/2$
$4\mathbf{A}^3$	$(\mathrm{U}(1)\times_{C_2} S(\mathrm{U}(3)\times\mathrm{U}(3)))\rtimes C_2$	18	6	$2^2 3^2$	$(\mathbf{Z}/2)^4$
$4\mathbf{A}^2\mathbf{B}$	$\mathrm{SU}(2)\times_{C_2} S(\mathrm{U}(2)\times\mathrm{U}(4))$	32	8	$2^5 3^1$	$(\mathbf{Z}/2)^4$
$4\mathbf{A}\mathbf{B}^2$	$\mathrm{U}(1)\times_{C_2} S(\mathrm{U}(1)\times\mathrm{U}(5))$	26	10	$2^3 3^1 5^1$	$(\mathbf{Z}/2)^2$
$4\mathbf{B}^3$	$\mathrm{U}(1)\times_{C_4}(\mathrm{U}(1)\times_{\langle(-1,z)\rangle}\mathrm{Spin}(8))$	30	12	$2^6 3^1$	$(\mathbf{Z}/2)^2$

stands for $(\mathrm{U}(m)\times\mathrm{U}(n))\cap\mathrm{SU}(m+n)$. Note that E_6 satisfies condition (1 & 2), because the only nonconnected centralizer in the table happens to be uniquely N-determined by [24, Lemma 2.37], and that all rank two elementary abelian 2-groups are [4] toral in the simply connected compact Lie group E_6. As the higher limits vanish (5.1 below), the 2-compact group E_6 is uniquely N-determined by [24, 2.48, 2.51].

According to [14, 9.4] and [18, 6.11, 6.12], the Lie group PE_7 contains three conjugacy classes of elements of order two with centralizers

$$C_{\mathrm{PE}_7}(2\mathbf{B}) = \mathrm{SU}(2)\circ\mathrm{SSpin}(12) = \frac{\mathrm{SU}(2)\times\mathrm{Spin}(12)}{\langle(E,x),(-E,xz)\rangle},$$
$$C_{\mathrm{PE}_7}(2\mathbf{H}) = (\mathrm{U}(1)\times\mathrm{E}_6)\rtimes C_2,$$
$$C_{\mathrm{PE}_7}(2\mathbf{A}) = \mathrm{SU}(8)/\langle i\rangle\rtimes C_2.$$

(The classes $2\mathbf{B}$, $2\mathbf{H}$, $2\mathbf{A}$ correspond to $2\mathbf{B}$, $4\mathbf{H}$, $4\mathbf{A}$ from [14, Table IV]; $z\in\mathrm{Spin}(2n)$ is the nontrivial element in the kernel of $\mathrm{Spin}(2n)\to\mathrm{SO}(2n)$ and $x\in\mathrm{Spin}(2n)$ is an element in the fiber over $-E\in\mathrm{SO}(2n)$.) We verify the first two conditions of [24, Theorem 2.51] by using [24, Lemma 2.54]. For all centralizers C of Table 3, $\check{Z}(C_0)=\check{T}^{W_0}$ [24, 2.32] as the identity component C_0 does not contain $\mathrm{SO}(2n+1)$ as a direct factor, and direct computation shows that they are all LHS [24, 2.26]. At this stage we are assuming that C_0 is uniquely N-determined. Since $H^1(W;\check{T})(\mathrm{PE}_7)=0$ [16, Main Theorem] it remains to show that the limit from [24, Lemma 2.54] is trivial. This is a machine computation in the toral subcategory of PE_7: there is a unique rank two toral object in $\mathbf{A}(\mathrm{PE}_7)$ with class distribution $\mathbf{BHA}$. It turns out that $H^1(W_0;\check{T})=\mathbf{Z}/2$, because the three rank one objects embed into $H^1(W_0;\check{T}) = (\mathbf{Z}/2)^3$ for this single rank two object as pairwise complementary subspaces, and therefore the limit is indeed trivial.

Table 3. Elementary abelian toral subgroups of PE_7 of rank ≤ 2

Class	Centralizer	dim	refl.	$\lvert W_0\rvert$	$H^1(W_0)$
$2\mathbf{B}$	$\mathrm{SU}(2)\circ \mathrm{SSpin}(12)$	69	31	$2^{10}3^25^1$	$\mathbf{Z}/2$
$2\mathbf{H}$	$(\mathrm{U}(1)\times \mathrm{E}_6)\rtimes C_2$	79	36	$2^73^45^1$	$\mathbf{Z}/2$
$2\mathbf{A}$	$\mathrm{SU}(8)/\langle iE\rangle \rtimes C_2$	63	28	$2^73^25^17^1$	$\mathbf{Z}/2$
$4\mathbf{BHA}$	$\dfrac{\mathrm{U}(1)\times \mathrm{SU}(2)\times \mathrm{SU}(6)}{(-1,-E,-E),\langle(-i,E,E)\rangle}\rtimes C_2$	39	16	$2^53^25^1$	$(\mathbf{Z}/2)^3$
$4\mathbf{BA}^3$	$\dfrac{\mathrm{U}(1)\times \mathrm{SU}(4)\times \mathrm{SU}(4)}{\langle(i,-iE,iE),(1,iE,iE)\rangle}\rtimes (C_2\times C_2)$	31	12	2^63^2	$(\mathbf{Z}/2)^4$
$4\mathbf{BH}^3$	$\dfrac{\mathrm{U}(1)\times \mathrm{U}(1)\times \mathrm{Spin}(10)}{\langle(1,-1,x),(-1,-1,xz)\rangle}\rtimes C_2$	47	20	$2^73^15^1$	$(\mathbf{Z}/2)^2$
$4\mathbf{B}^3$	$\dfrac{\mathrm{U}(1)\times \mathrm{U}(1)\times \mathrm{SU}(6)}{\langle(1,1,-E),(-1,-1,E)\rangle}\rtimes C_2$	37	15	$2^43^25^1$	$(\mathbf{Z}/2)^2$
$4\mathbf{B}^3$	$\dfrac{\mathrm{SU}(2)\times \mathrm{SU}(2)\times \mathrm{SU}(2)\times \mathrm{Spin}(8)}{\langle(E,E,-E,x),(-E,-E,E,xz),(E,-E,-E,z)\rangle}$	37	15	2^93^1	$(\mathbf{Z}/2)^5$

Still according to [14], there are three nontoral rank two elementary abelian 2-groups in PE_7. One is $2\mathbf{A}$-pure, has centralizer $(\mathbf{Z}/2)^2\times \mathrm{PSO}(8)$ and automorphism group $\mathrm{GL}(2,\mathbf{F}_2)$, one has class distribution $2^2\mathbf{A}^2\mathbf{H}$, centralizer $(\mathbf{Z}/2)^2\times \mathrm{PSp}(4)$ and automorphism group C_2, and the third one is $2\mathbf{H}$-pure, has centralizer $(\mathbf{Z}/2)^2\times \mathrm{F}_4$ and automorphism group $\mathrm{GL}(2,\mathbf{F}_2)$. We use [24, Lemma 2.63] to verify the third condition of [24, Theorem 2.51]. It turns out that $W(\mathrm{PE}_7)$ contains two elements, v_1 and v_2, of order two with $+1$-eigenspace of dimension four. The image of $C_{W(\mathrm{PE}_7)}(v_1)$ in $\mathrm{GL}(\pi_1(T^{v_1})\otimes \mathbf{Q}) = \mathrm{GL}(4,\mathbf{Q})$ has order $2^5\cdot 3^1$ while in the case of v_2 we get an image of order $2^7\cdot 3^2$. We conclude that if $V_2\subset N(\mathrm{PE}_7)$ is the preferred lift of the nonpure nontoral rank two object, then the image in $W(\mathrm{PE}_7)$ is v_1 [27, 1.3, 4.2]. This observation can be used in connection with [24, Lemma 2.63] to verify the third condition of [24, Theorem 2.51]. Since the higher limits vanish (5.2), the 2-compact group PE_7 is uniquely N-determined by [24, 2.48, 2.51].

Table 4, based on [14, 18] and explicit computations, collects information about centralizers of elementary abelian 2-groups in E_8 of rank ≤ 2. We verify the first two conditions of [24, Theorem 2.51] by using [24, Lemma 2.54]. For all centralizers C of Table 4, $\check{Z}(C_0) = \check{T}^{W_0}$ [24, 2.32] as the identity component C_0 does not contain $\mathrm{SO}(2n+1)$ as a direct factor, and direct computation shows that they are all LHS. At this stage we are assuming that C_0 is uniquely N-determined. Since $H^1(W;\check{T})(\mathrm{E}_8) = 0$ [16, Main The-

Table 4. Elementary abelian toral subgroups of E_8 of rank ≤ 2

Class	Centralizer	dim	refl	$\lvert W_0 \rvert$	$H^1(W_0;\check{T})$
$2\mathbf{A}$	$SU(2) \circ E_7$	136	64	$2^{11}3^4 5^1 7^1$	$\mathbf{Z}/2$
$2\mathbf{B}$	$SSpin(16)$	120	56	$2^{14}3^2 5^1 7^1$	$\mathbf{Z}/2$
$4\mathbf{A}^1\mathbf{B}^2$	$\frac{U(1)\times SU(8)}{\langle(i,-iE),(1,-E)\rangle} \rtimes C_2$	64	28	$2^7 3^2 5^1 7^1$	$(\mathbf{Z}/2)^2$
$4\mathbf{A}^2\mathbf{B}^1$	$\frac{Spin(4)\times Spin(12)}{\langle(z_1,z_2),(x_1,x_2)\rangle}$	72	32	$2^{11}3^2 5^1$	$(\mathbf{Z}/2)^4$
$4\mathbf{A}^3$	$\frac{U(1)\times U(1)\times E_6}{\langle(-1,-1,e)\rangle} \rtimes C_2$	80	36	$2^7 3^4 5^1$	$(\mathbf{Z}/2)^2$
$4\mathbf{B}^3$	$\frac{Spin(8)\times Spin(8)}{\langle(z_1,z_2),(x_1,x_2)\rangle} \rtimes C_2$	56	24	$2^{12}3^2$	$(\mathbf{Z}/2)^6$

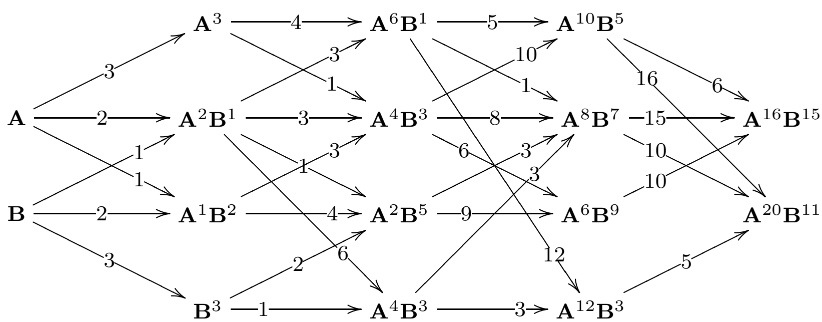
orem] it remains to show that the limit from [24, Lemma 2.54] is trivial. This is a machine computation in the toral subcategory of E_8: there is a unique rank two toral object in $\mathbf{A}(E_8)$ with class distribution $\mathbf{A}^1\mathbf{B}^2$. It turns out that $H^1(W_0;\check{T}) = \mathbf{Z}/2$, because the two rank one objects embed into $H^1(W_0;\check{T}) = (\mathbf{Z}/2)^2$ for this single rank two object as complementary subspaces, and therefore the limit is indeed trivial. There are no nontoral rank two elementary abelian 2-groups in the simply connected compact Lie group E_8 [4]. Since the higher limits vanish (5.3), the 2-compact group E_8 is uniquely N-determined by [24, 2.48, 2.51]. In all three cases, $H^1(W;\check{T}) = 0$ [16, Main Theorem], $H^2(W;\check{T}) = \mathbf{Z}/2$ (where the extension class $e(X)$ is nontrivial) [21], and the group $\mathrm{Out}_{\mathrm{tr}}(W)$ is trivial so that [24, Lemma 2.16, (2.8)] immediately gives the formulas for the automorphism groups. ∎

Alternatively, for $X = E_6, PE_7, E_8$, we may apply the method used by Vavpetič and Viruel in [35] and shift from the category $\mathbf{A}(X)$ of elementary abelian subgroups to the category $\mathcal{R}_2(X)$ of 2-stubborn subgroups of X. In the situation of [24, Theorem 2.51], their functor $\mathcal{R}_2(X)^{\mathrm{op}} \to \mathbf{A}(X)\colon P \to {}_2Z(P)$ gives induced maps

$$BP \to BC_X({}_2Z(P)) \to BX',$$

for each 2-stubborn $P \subset X$, that respect the morphisms of the stubborn category up to homotopy. Since the obstruction groups for the stubborn category are known to vanish, these maps rigidify to a map $BX \to BX'$ under the maximal torus. This way one circumvents the problem of computing the higher limits over the Quillen category and relies instead on the result from [18] that the higher limits over $\mathcal{R}_2(X)$ are known to vanish.

5.1. *Limits over the Quillen category of* E_6. The poset for the toral part, $\mathbf{A}(E_6)^{\leq t}$, of the Quillen category for E_6:



contains three objects V_2, V_3, V_4, with class distributions $4\mathbf{A}^3$, $8\mathbf{A}^6\mathbf{B}^1$, and $16\mathbf{A}^{12}\mathbf{B}^3$, with nonconnected centralizers

$$S(\mathrm{U}(1)^2)\times_{C_2} S(\mathrm{U}(3)^2)\rtimes C_2 \supset S(\mathrm{U}(1)^2)\times_{C_2} S(\mathrm{U}(1)^2\times \mathrm{U}(2)^2)\rtimes C_2$$
$$\supset S(\mathrm{U}(1)^2)\times_{C_2} S(\mathrm{U}(1)^6)\rtimes C_2,$$

where the component group C_2 is generated by

$$c=\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\begin{pmatrix}0 & -E\\ E & 0\end{pmatrix}\right)\in \mathrm{SU}(2)\times_{C_2}\mathrm{SU}(6).$$

It follows from [24, 2.33, 5.14] that $\check{Z}C_{\mathrm{E}_6}(V)=\check{T}^{W(E_6)(V)}$ for every toral object V of the Quillen category. As in [24, 3.13–3.15], this implies that $\lim^*(\mathbf{A}(\mathrm{E}_6);\pi_i(BZC_{\mathrm{E}_6})_{\not\leq t})\cong\lim^*(\mathbf{A}(\mathrm{E}_6);\pi_i(BZC_{\mathrm{E}_6}))$ for $i=1,2$. The elementary abelian subgroups $\langle V_2,c\rangle$, $\langle V_3,c\rangle$, $\langle V_4,c\rangle$, with class distributions $2^3\mathbf{A}^7$, $2^4\mathbf{A}^{14}\mathbf{B}^1$, and $2^5\mathbf{A}^{28}\mathbf{B}^3$, are nontoral and, according to [14, 8.2], they are the only nontoral elementary abelian subgroups of E_6. Their centralizers are $\langle V_2,c\rangle\times\mathrm{SU}(3)$, $\langle V_2,c\rangle\times\mathrm{U}(2)$, and $\langle V_2,c\rangle\times\mathrm{U}(1)^2$. The automorphism groups in $\mathbf{A}(\mathrm{E}_6)$ for the first two subgroups contain the automorphism groups in $\mathbf{A}(\mathrm{F}_4)$. Thus

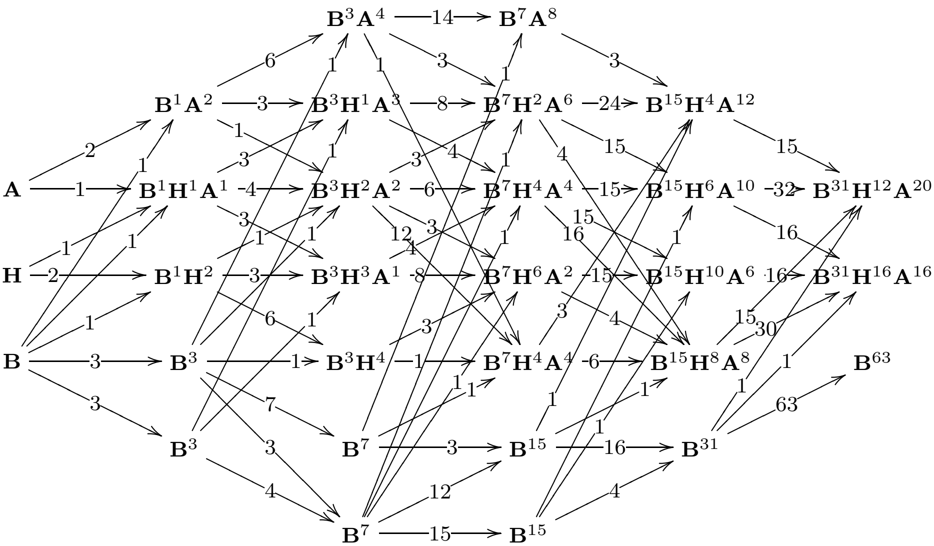
$$\mathbf{A}(\mathrm{E}_6)(2^3\mathbf{A}^7)=\mathrm{GL}(3,\mathbf{F}_2),\qquad \mathbf{A}(\mathrm{E}_6)(2^4\mathbf{A}^{14}\mathbf{B}^1)=\begin{pmatrix}\mathrm{GL}(3,\mathbf{F}_2) & 0\\ * & 1\end{pmatrix}$$

of order 168 and 1344. Their contributions to Oliver's cochain complex [31] for the higher limits of the functor $\pi_1(BZC_{\mathrm{E}_6})$ [24, (2.47)] are

$$\mathrm{Hom}_{\mathbf{A}(\mathrm{E}_6)(2^3\mathbf{A}^7)}(\mathrm{St}(2^3),2^3)=0,\qquad \mathrm{Hom}_{\mathbf{A}(\mathrm{E}_6)(2^4\mathbf{A}^{14}\mathbf{B}^1)}(\mathrm{St}(2^4),2^4)=0$$

and hence $\lim^j(\mathbf{A}(\mathrm{E}_6);\pi_1(BZC))=0$ for $j=1,2$. The functor $\pi_2(BZC_{\mathrm{E}_6})$ [24, (2.47)] is 0 on nontoral objects of rank ≤ 3 and has value $\mathbf{Z}_2$ on the nontoral object of rank four. It follows that $\lim^j(\mathbf{A}(\mathrm{E}_6);\pi_2(BZC_{\mathrm{E}_6}))$ for $j=2,3$.

5.2. *Limits over the Quillen category of* PE_7*.* As for E_6, inspection of $\mathbf{A}(\mathrm{PE}_7)^{\leq t}$:



shows that the problem of computing the higher limits of the functors $\pi_i(BZC_{\mathrm{PE}_7})$ is concentrated on the nontoral objects of the Quillen category.

Let V be a rank three elementary abelian 2-group in PE_7 containing a nontoral elementary abelian 2-group H of rank two. Then V is generated by H and an involution $L \subset C_{\mathrm{PE}_7}(H)$ not contained in H where $C_{\mathrm{PE}_7}(H) = \mathrm{F}_4$, $\mathrm{PSp}(4)$, or $\mathrm{PSO}(8)$. The orthogonality relations [14, 1.5] and the eigenvalue multiplicities from [14, Table VI] (with small corrections for the classes **A** and **B**) determine the class distribution for V as shown in the following table:

$C_{\mathrm{PE}_7}(H)$	L	$C = C_{PE_7}(H \times L)$	$\dim C$	$H \times L$
F_4	$2\mathbf{A}[\mathrm{F}_4]$	$\mathrm{SU}(2) \circ \mathrm{Sp}(3)$	24	$\mathbf{B}^1\mathbf{A}^3\mathbf{H}^3$
	$2\mathbf{B}[\mathrm{F}_4]$	$\mathrm{Spin}(9)$	36	$\mathbf{B}^1\mathbf{H}^6$
$\mathrm{PSp}(4)$	$L(1,3)$	$\mathrm{SU}(2) \circ \mathrm{Sp}(3)$	24	$\mathbf{B}^1\mathbf{A}^3\mathbf{H}^3$
	$L(2,2)$	$\mathrm{Spin}(5) \circ \mathrm{Spin}(5) \rtimes C_2$	20	$\mathbf{B}^1\mathbf{A}^4\mathbf{H}^2$
	I	$\mathrm{U}(4)/\langle -E\rangle \rtimes C_2$	16	$\mathbf{B}^1\mathbf{A}^5\mathbf{H}^1$
$\mathrm{PSO}(8)$	$L(2,6)$	$\mathrm{U}(4)/\langle -E\rangle \rtimes C_2$	16	$\mathbf{B}^1\mathbf{A}^5\mathbf{H}^1$
	$L(4,4)$	$\mathrm{SO}(4) \circ \mathrm{SO}(4) \rtimes (C_2 \times C_2)$	12	$\mathbf{B}^1\mathbf{A}^6$
	I, I^D	$\mathrm{U}(4)/\langle -E\rangle \rtimes C_2$	16	$\mathbf{B}^1\mathbf{A}^5\mathbf{H}^1$

Of course, some of the nontoral elementary abelian 2-groups in this table may be conjugate in PE_7.

Any elementary abelian 2-group of rank three in PE_7 is contained in and contains the center of the maximal rank subgroup $C_{\mathrm{PE}_7}(2\mathbf{B}) = \mathrm{SU}(2) \circ \mathrm{SSpin}(12)$ because the orthogonality relations combined with eigenvalue multiplicities [14, 1.5, Table VI] imply that any elementary abelian 2-group of rank three in PE_7 contains an element from the class $2\mathbf{B}$. Thus any elementary abelian 2-group of rank three in PE_7 is conjugate to the preimage $V^* \subset SU(2) \circ SSpin(12)$ of an elementary abelian 2-group of rank two $V \subset SO(3) \times PSO(12)$. The elementary abelian 2-group V^* is toral if and only if V is toral. Suppose from now on that V is nontoral. Then the image, V_2, of V in $PSO(12)$ must be a nontoral elementary abelian 2-group of rank two. Indeed, if V_2 is toral in $PSO(12)$, the image, V_1, of V in $SO(3)$ must be nontoral. Then $[V_2, V_2] = \{e\}$ in $SSpin(12)$ and $[V_1, V_1] = \{E, -E\}$ in $SU(2)$ so that $[V^*, V^*] \neq \{e\}$ in $SU(2) \circ SSpin(12)$, contradicting that V^* is abelian. Thus V_2 is one of the nontoral elementary abelian 2-groups described in Chapter 2, §4, with $q(V_2) = 0$ or $V_2 = H_\pm, H_\pm^D$ as in 2.51.

$\underline{q(V_2) = 0}$: The possibilities for $V_2 \subset PSO(12)$ are indexed by the five partitions $(i_0, i_1, i_2, i_3) \in \{(5,1,1,1), (4,2,1,1), (3,3,1,1), (3,2,2,1), (2,2,2,2)\}$ of 8 into four natural numbers. The nontoral elementary abelian 2-group corresponding to (i_0, i_1, i_2, i_3) is $V_2 = \langle v_1, v_2 \rangle$ generated by

$$v_1 = ((+1)^{2i_0-1}(-1)^{2i_1-1}(+1)^{2i_2-1}(-1)^{2i_3-1}),$$
$$v_2 = ((+1)^{2i_0-1}(+1)^{2i_1-1}(-1)^{2i_2-1}(-1)^{2i_3-1}).$$

The following table describes the preimage $V_2^* \subset SSpin(12)$ of $V_2 \subset PSO(12)$ using [24, Lemma 5.28].

	$(5,1,1,1)$	$(4,2,1,1)$	$(3,3,1,1)$	$(3,2,1,1)$	$(2,2,2,2)$
v_1^2	z	e	z	e	z
v_2^2	z	z	z	e	z
$[v_1, v_2]$	z	z	z	z	z
$V_2^* \subset SSpin(12)$	2_-^{1+2}	2_+^{1+2}	2_-^{1+2}	2_+^{1+2}	2_-^{1+2}

As the extraspecial 2-group 2_+^{1+2} does not imbed in $SU(2)$ [24, 5.5] no V with $V_2^* = 2_+^{1+2}$ has elementary abelian preimage in $SU(2) \circ SSpin(12)$. In the other three cases, we need $u_1, u_2 \in SU(2)$ such that $(u_1^2, u_2^2, [u_1, u_2]) = (-E, -E, -E)$. The only possibility is $2_-^{1+2} = \langle u_1, u_2 \rangle = \langle iR, iT \rangle$ [24, 5.5] and we see that

$$V^* = \langle (E, z), (iR, v_1), (iT, v_2) \rangle = \langle (-E, e), (iR, v_1), (iT, v_2) \rangle$$

is elementary abelian. The centralizer of V^* in PE_7 or $SU(2) \circ SSpin(12)$ is the preimage under the map $SU(2) \circ SSpin(12) \to SO(2) \times PSO(12)$ of a subgroup of $C_{SO(3) \times PSO(12)}(V)$ [28, 5.11]. In the case $(5,1,1,1)$, the centralizer of V^* has type B_4, dimension 36, and

$$V^* = 2^3\mathbf{B}^1\mathbf{H}^6, \qquad C_{PE_7}(V^*) = V^* \circ Spin(9) = V \times Spin(9),$$
$$\mathbf{A}(PE_7)(V^*) = \begin{pmatrix} GL(2, \mathbf{F}_2) & * \\ 0 & 1 \end{pmatrix}.$$

In the case $(3,3,1,1)$, the centralizer of V^* has type B_2^2, dimension 20, and

$$V^* = 2^3\mathbf{B}^1\mathbf{H}^2\mathbf{A}^4, \quad C_{\mathrm{PE}_7}(V^*) = V \times \mathrm{Spin}(5) \circ \mathrm{Spin}(5) \rtimes C_2,$$

$$\mathbf{A}(\mathrm{PE}_7)(V^*) = \begin{pmatrix} C_2 & * \\ 0 & 1 \end{pmatrix}.$$

In the case $(2,2,2,2)$, the centralizer of V^* has type A_1^4, dimension 12, and

$$V^* = 2^3\mathbf{B}^1\mathbf{A}^6, \quad C_{\mathrm{PE}_7}(V^*) = V \times \mathrm{SU}(2) \circ \mathrm{SU}(2) \circ \mathrm{SU}(2) \circ \mathrm{SU}(2) \rtimes (C_2 \times C_2),$$

$$\mathbf{A}(\mathrm{PE}_7)(V^*) = \begin{pmatrix} \mathrm{GL}(2,\mathbf{F}_2) & * \\ 0 & 1 \end{pmatrix}.$$

In all these cases, V^* contains a single element from the class $2\mathbf{B}$, which implies that $\mathbf{A}(\mathrm{PE}_7)(V^*) = \mathbf{A}(\mathrm{SU}(2) \circ \mathrm{SSpin}(12))(V^*)$. (If $g \in \mathrm{PE}_7$ normalizes V^*, conjugation by g, $c_g \colon V^* \to V^*$, must fix the unique element in V^* from the class $2\mathbf{B}$, and thus $g \in C_{\mathrm{PE}_7}(2\mathbf{B}) = \mathrm{SU}(2) \circ \mathrm{SSpin}(12)$.)

$\underline{H_-}$: The elementary abelian 2-group $V_2 = H_- \subset \mathrm{PSO}(12)$ is the image of $2_-^{1+2} = \langle v_1, v_2 \rangle \subset \mathrm{SO}(12)$ generated by [24, 5.7]

$$v_1 = \mathrm{diag}\left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\right),$$

$$v_2 = \mathrm{diag}\left(\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right)$$

In $\mathrm{SSpin}(12)$ we have $(v_1^2, v_2^2, [v_1, v_2]) = (e,e,e)$ because, in $\mathrm{Spin}(12)$, we see that $(v_1^2, v_2^2, [v_1, v_2]) = (x,x,x)$ as $(v_1^2, v_2^2, [v_1, v_2]) = (-E,-E,-E)$ in $\mathrm{SU}(6) \subset \mathrm{Spin}(12)$. To match this we need $u_1, u_2 \in \mathrm{SU}(2)$ such that $(u_1^2, u_2^2, [u_1, u_2]) = (E,E,E)$. The only possibility is $u_1 = \pm E = u_2$ and then

$$V^* = \langle (E,z), (E,v_1), (E,v_2) \rangle.$$

The centralizer of V^* has type A_1D_3, dimension 24, and

$$V^* = 2^3\mathbf{B}^1\mathbf{H}^3\mathbf{A}^3, \quad C_{\mathrm{PE}_7}(V^*) = V \times \mathrm{SU}(2) \circ \mathrm{Sp}(3).$$

The unique occurrence of $2\mathbf{B}$ means that the automorphism group of V^* is $\mathbf{A}(\mathrm{PE}_7)(V^*) = \mathbf{A}(\mathrm{SU}(2) \circ \mathrm{SSpin}(12))(V^*)$ and this group is a subgroup of

$$\begin{pmatrix} \mathrm{GL}(2,\mathbf{F}_2) & * \\ 0 & 1 \end{pmatrix}$$

that maps onto $\mathbf{A}(\mathrm{SO}(3) \times \mathrm{PSO}(12))(V) = \mathrm{GL}(2,\mathbf{F}_2)$. The class distribution forces

$$\mathbf{A}(\mathrm{PE}_7)(V^*) = \begin{pmatrix} \mathrm{GL}(2,\mathbf{F}_2) & 0 \\ 0 & 1 \end{pmatrix}.$$

$\underline{H^D_-}$: The elementary abelian 2-group $V_2 = H^D_- \subset \mathrm{PSO}(12)$ is the image of $(2^{1+2}_-)^D = \langle v_1^D, v_2^D \rangle \subset \mathrm{SO}(12)$ where v_1, v_2 are as above. In Spin(12) we get $((v_1^2)^D, (v_2^2)^D, [v_1, v_2]^D) = (xz, xz, xz)$ so that $((v_1^2)^D, (v_2^2)^D, [v_1, v_2]^D) = (z, z, z)$ in SSpin(12). To match this we need the extraspecial 2-group $2^{1+2}_- = \langle iR, iT \rangle \subset \mathrm{SU}(2)$ and we get

$$V^* = \langle (E, z), (iR, v_1^D), (iT, v_2^D) \rangle.$$

The centralizer of V^* has type C_3, dimension 21, and

$$V^* = 2^3\mathbf{B}^7, \quad C_{\mathrm{PE}_7}(V^*) = V^* \times \mathrm{PSp}(3),$$

$$\mathbf{A}(\mathrm{PE}_7)(V^*) \supset \mathbf{A}(\mathrm{SU}(2) \circ \mathrm{SSpin}(12))(V^*) = \begin{pmatrix} \mathrm{GL}(2, \mathbf{F}_2) & * \\ 0 & 1 \end{pmatrix}.$$

There are two conjugacy classes of involutions, v_1 and v_2, in $W(\mathrm{PE}_7)$ with $+1$-eigenspace of dimension three corresponding, respectively, to this non-toral elementary abelian 2-group and to the one discussed later under item H^D_+. The index

$$\left| \mathbf{A}(\mathrm{PE}_7)(V^*) \cap \begin{pmatrix} \mathrm{GL}(2, \mathbf{F}_2) & 0 \\ * & 1 \end{pmatrix} : \mathbf{A}(N(\mathrm{PE}_7))(V^*) \cap \begin{pmatrix} \mathrm{GL}(2, \mathbf{F}_2) & 0 \\ * & 1 \end{pmatrix} \right|$$

equals the number of lifts

$$\begin{array}{ccc} & & N(\mathrm{PE}_7) \\ & \overset{\mu}{\nearrow} & \downarrow \\ V^* & \xrightarrow{v_1} & W(\mathrm{PE}_7) \end{array}$$

for which $C_N(\mathrm{PE}_7)(V^*) = N(V^* \times \mathrm{PSp}(3))$. According to *magma* computations using the short exact sequence [27, 4.2], this number is 12 and hence the Quillen automorphism group $\mathbf{A}(\mathrm{PE}_7)(V^*)$ equals $\mathrm{GL}(3, \mathbf{F}_2)$.

$\underline{H_+}$: The elementary abelian 2-group $V_2 = H_+ \subset \mathrm{PSO}(12)$ is the image of $2^{1+2}_+ = \langle v_1, v_2 \rangle \subset \mathrm{SO}(12)$ generated by [24, 5.7]

$$v_1 = \mathrm{diag}(R, R, R, R, R, R), \quad v_2 = \mathrm{diag}(T, T, T, T, T, T).$$

In Spin(12) we have $(v_1^2, v_2^2, [v_1, v_2]) = (z, z, x)$, and so in SSpin(12) we have $(v_1^2, v_2^2, [v_1, v_2]) = (z, z, e)$. To match this we need $(u_1^2, u_2^2, [u_1, u_2]) = (-E, -E, E)$. The only possibility is $u_1 = \pm\,\mathrm{diag}(i, -i) = u_2$ and we get

$$V^* = \langle (E, z), (\mathrm{diag}(i, -i), v_1), (\mathrm{diag}(i, -i), v_2) \rangle.$$

The centralizer of V^* has type $T_1 A_3$, dimension 16, and

$$V^* = 2^3\mathbf{B}^1\mathbf{H}^1\mathbf{A}^5, \quad C_{\mathrm{PE}_7}(V^*) = V \times \mathrm{U}(4)/\langle -E \rangle \rtimes C_2,$$

$$\mathbf{A}(\mathrm{PE}_7)(V^*) = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

$\underline{H^D_+}$: The elementary abelian 2-group $V_2 = H^D_+ \subset \mathrm{PSO}(12)$ is the image of $(2^{1+2}_+)^D = \langle v^D_1, v^D_2 \rangle \subset \mathrm{SO}(12)$ where v_1 and v_2 are as above. In $\mathrm{Spin}(12)$ we have $((v^2_1)^D, (v^2_2)^D, [v_1, v_2]^D) = (z, z, xz)$ and in $\mathrm{SSpin}(12)$ we have $((v^2_1)^D, (v^2_2)^D, [v_1, v_2]^D) = (z, z, z)$. To match this we need $2^{1+2}_- = \langle iR, iT \rangle \subset \mathrm{SU}(2)$, which gives

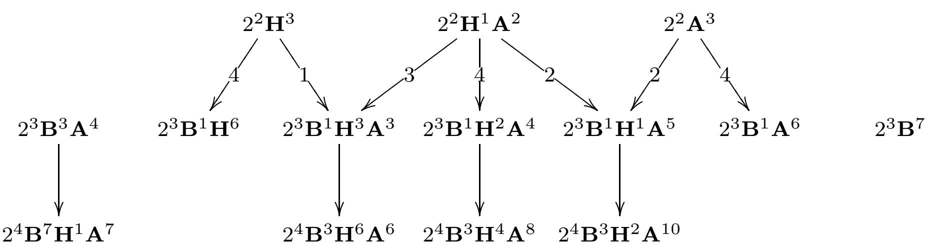
$$V^* = \{(E, z), (iR, v^D_1), (iT, v^D_2)\}.$$

The centralizer V^* has type A_3, dimension 15, and

$$V^* = 2^3\mathbf{B}^3\mathbf{A}^4, \qquad C_{\mathrm{PE}_7}(V^*) = V^* \circ \mathrm{SSpin}(6) = V \times \mathrm{SU}(4)/\langle -E \rangle,$$

$$\mathbf{A}(\mathrm{PE}_7)(V^*) = \begin{pmatrix} C_2 & * \\ 0 & 1 \end{pmatrix}.$$

From this list of rank three nontoral elementary abelian 2-groups one may find some rank four nontoral elementary abelian 2-groups. For instance, one may replace H^D_+ above by $H^D_+ \# L(2,4)$ to obtain a rank four nontoral elementary abelian 2-group with class distribution $2^4\mathbf{B}^7\mathbf{H}^1\mathbf{A}^7$ and use 2.40 to estimate the Quillen automorphism group. There are the following relations between the nontorals of rank two or three:



where the nontoral elementary abelian 2-groups of the third row contain no other of the nontoral elementary abelian 2-groups of the second row than the ones indicated. Using the bases $\{f_-\}$, $\{f_-\}$, and $\{f_+, f_0\}$ for $[\mathbf{H}^3]$, $[\mathbf{A}^3]$, and $[\mathbf{H}^2\mathbf{A}]$ (2.34) described in (2.37, 2.36), the $(1+2+1)\times(1+4+3+6+1)$ matrix for the first differential

$$0 \to [\mathbf{H}^3] \times [\mathbf{H}^1\mathbf{A}^2] \times [\mathbf{A}^3] \xrightarrow{d^1} [\mathbf{B}^1\mathbf{H}^6] \times [\mathbf{B}^1\mathbf{H}^3\mathbf{A}^3] \times [\mathbf{B}^1\mathbf{H}^2\mathbf{A}^4] \times [\mathbf{B}^1\mathbf{H}^1\mathbf{A}^5] \times [\mathbf{B}^1\mathbf{A}^6]$$

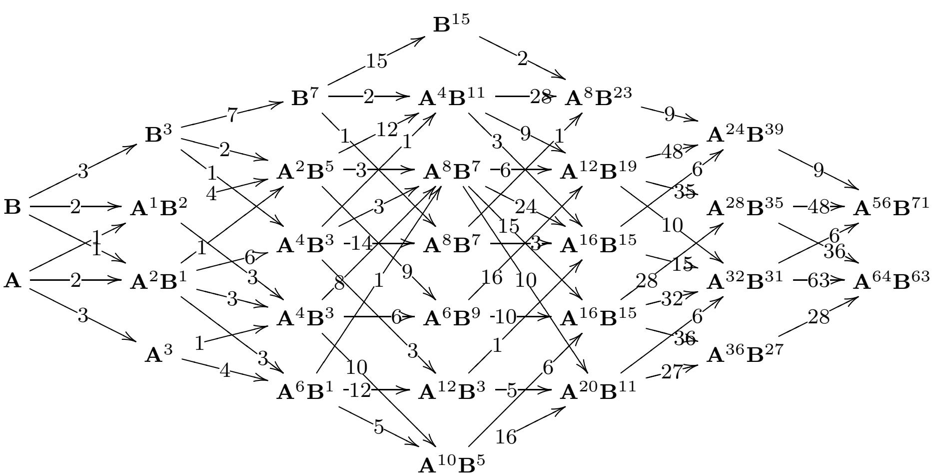
becomes

	$[\mathbf{B}^1\mathbf{H}^6]$	$[\mathbf{B}^1\mathbf{H}^3\mathbf{A}^3]$	$[\mathbf{B}^1\mathbf{H}^2\mathbf{A}^4]$	$[\mathbf{B}^1\mathbf{H}^1\mathbf{A}^5]$	$[\mathbf{B}^1\mathbf{A}^6]$
$[\mathbf{H}^3]$	(1)	$(1\ 0\ 0\ 0)$			
$[\mathbf{H}^1\mathbf{A}^2]$		$\begin{pmatrix}0&1&0&0\\0&0&1&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}1&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$	
$[\mathbf{A}^3]$				$(0\ 1\ 0\ 0\ 0\ 0)$	(1)

using the basis (2.39) for $[\mathbf{B}^1\mathbf{H}^3\mathbf{A}^3]$ and similar bases for the other summands. Computations with *magma* show that $[\mathbf{B}^1\mathbf{H}^3\mathbf{A}^3] \to [\mathbf{B}^3\mathbf{H}^6\mathbf{A}^6]$ is injective, $[\mathbf{B}^1\mathbf{H}^2\mathbf{A}^4] \to [\mathbf{B}^3\mathbf{H}^4\mathbf{A}^8]$ has rank at least 2, and $[\mathbf{B}^1\mathbf{H}^1\mathbf{A}^5] \to [\mathbf{B}^3\mathbf{H}^2\mathbf{A}^{10}]$ at least 5. This means that $\lim^j(\mathbf{A}(\mathrm{PE}_7); \pi_1(BZC_{\mathrm{PE}_7})) = 0$ for $j = 1, 2$.

The functor $\pi_2(BZC_{\mathrm{PE}_7})$ is trivial on all the rank three nontoral elementary abelian 2-groups in PE_7. Let now V be a nontoral elementary abelian 2-group of rank four. Again, the orthogonality relations imply that V contains an involution from class 2**B** so that V is contained in and contains the center of $\mathrm{SU}(2) \circ \mathrm{SSpin}(12)$. The image of V in $\mathrm{SO}(3) \times \mathrm{PSO}(12)$ must be nontoral of rank three and the image of V in $\mathrm{PSO}(12)$ nontoral of rank two or three. In any case, $V = V^* \times L$ where V^* is one of the rank three nontorals and L is generated by an involution in $C_{\mathrm{PE}_7}(V^*)$ not contained in V^*. Going through the list of nontoral rank three objects V^*, one sees that $\pi_2(BZC_{\mathrm{PE}_7})$ is trivial on all rank four objects $V^* \times L$ as well. In most cases, the identity component of $C_{\mathrm{PE}_7}(V^* \times L)$ is semisimple. In the case where $C_{\mathrm{PE}_7}(V^*) = V \times \mathrm{U}(4) \rtimes C_2$, note that the centralizers have the form $V \times \mathrm{U}(i) \circ \mathrm{U}(j) \rtimes C_2$, $i + j = 4$, and that these compact Lie groups have finite center. Consequently, $\lim^j(\mathbf{A}(\mathrm{PE}_7); \pi_2(BZC_{\mathrm{PE}_7})) = 0$ for $j = 2, 3$.

5.3. *Limits over the Quillen category of* E_8. As for E_6, inspection of $\mathbf{A}(\mathrm{E}_8)^{\le t}$:



shows that the problem of computing the higher limits of the functors $\pi_i(BZC_{\mathrm{PE}_7})$ is concentrated on the nontoral objects of the Quillen category.

Let V be a nontoral rank three elementary abelian 2-group in E_8. Unless V is $\mathbf{A}$-pure, V is contained in and contains the center of the maximal rank subgroup $\mathrm{SSpin}(16)$ of E_8. Let $\overline{V} \subset \mathrm{PSO}(16)$, $V^* \subset \mathrm{Spin}(16)$, and $\overline{V^*} \subset \mathrm{SO}(16)$ be the (pre)images of V under the isogenies

$$\begin{array}{ccc} \mathrm{SSpin}(16) & \xleftarrow{\langle x\rangle} & \mathrm{Spin}(16) \\ {\scriptstyle\langle z\rangle}\big\downarrow & & \big\downarrow{\scriptstyle\langle z\rangle} \\ \mathrm{PSO}(16) & \xleftarrow{\langle -E\rangle} & \mathrm{SO}(16) \end{array}$$

of compact Lie groups. Then $\overline{V}$ is a nontoral rank two elementary abelian 2-group in $\mathrm{PSO}(16)$. If V^* is abelian, also $\overline{V^*}$ is abelian and, in fact, elementary abelian, for if $\overline{V^*}$ contains an element of order 4 it is toral (2.10). Thus V is one of the nontoral elementary abelian 2-groups described in Chapter 2, §4, contradicting that $[V^*, V^*] = \langle z\rangle$ for all these groups [24, Lemma 5.28]. Therefore V^* is not abelian and then $[V^*, V^*] = \langle x\rangle$ so that $[\overline{V}, \overline{V}] \neq 0$ and $\overline{V} = H_\pm, H_\pm^D$ is one of the nontoral rank 2 elementary abelian 2-groups described in 2.51. However, V cannot be both H_+ and H_+^D as $[V^*, V^*] = \langle x\rangle$ for one of these alternatives and $[V^*, V^*] = \langle xz\rangle$ for the other choice [24, Lemma 5.28], [14, 2.9].

So there are, up to conjugation in $\mathrm{SSpin}(16)$, just two possibilities for V. The class distributions, which we get from the orthogonality relations [14, 1.5], are $2^3\mathbf{A}^1\mathbf{B}^6$ (projecting onto H_+) and $2^3\mathbf{A}^3\mathbf{B}^4$ (projecting onto H_-), and the centralizers are

$$\begin{aligned} C_{\mathrm{E}_8}(2^3\mathbf{A}^1\mathbf{B}^6) &= C_{\mathrm{SSpin}(16)}(V) = V \times \mathrm{PSO}(8), \\ C_{\mathrm{E}_8}(2^3\mathbf{A}^3\mathbf{B}^4) &= C_{\mathrm{SSpin}(16)}(V) = V \times \mathrm{PSp}(4). \end{aligned}$$

In particular, $\mathbf{A}(\mathrm{E}_8)(V)$ cannot be all of $\mathrm{GL}(3, \mathbf{F}_2)$. Computations with preferred lifts based on the exact sequence [27, 4.2] show that these automorphism groups have index at most 7 in $\mathrm{GL}(3, \mathbf{F}_2)$. Then there are only two possibilities and the class distributions force

$$\mathbf{A}(\mathrm{E}_8)(2^3\mathbf{A}^1\mathbf{B}^6) = \begin{pmatrix} 1 & * \\ 0 & \mathrm{GL}(2, \mathbf{F}_2) \end{pmatrix}, \quad \mathbf{A}(\mathrm{E}_8)(2^3\mathbf{A}^3\mathbf{B}^4) = \begin{pmatrix} \mathrm{GL}(2, \mathbf{F}_2) & * \\ 0 & 1 \end{pmatrix},$$

both of order $2^2 \cdot |\mathrm{GL}(2, \mathbf{F}_2)| = 24$. The remaining case is when V is $\mathbf{A}$-pure. Then V is contained in and contains the center of the maximal rank subgroup $\mathrm{SU}(2) \circ \mathrm{E}_7$ of E_8. Let $\overline{V} \subset \mathrm{PU}(2) \times \mathrm{PE}_7$ be the image of V. The centralizer of V in E_8 has rank four as seen by inspecting the nonconnected rank two centralizers (Table 4) in E_8. Therefore the embedding

$\overline{V} \to \mathrm{PU}(2) \times \mathrm{PE}_7$ must have the form $\overline{V} \ni v \mapsto (\varphi_1(v), \varphi_2(v))$ where φ_1 is the nontoral embedding of $\overline{V}$ in $\mathrm{PU}(2)$ [24, Corollary 3.19] and φ_2 one of the three nontoral embeddings in PE_7. The preimage V^* of V in $\mathrm{SU}(2) \times \mathrm{E}_7$ is then $V^* = \langle(-E, e)\rangle \times 2_-^{1+2} \to \mathrm{SU}(2) \times \mathrm{E}_7$, the monomorphism being $V^* \supset 2_-^{1+2} \ni g \mapsto (\varphi_1(g), \varphi_2(g))$ where φ_1 is the faithful representation of the extraspecial group 2_-^{1+2} in $\mathrm{SU}(2)$ and φ_2 is one of the three nontoral embeddings of 2_-^{1+2} in E_7 [14, 9.5], and $C_{\mathrm{SU}(2)\times \mathrm{E}_7}(V^*) = \langle -E\rangle \times \langle z\rangle \times X$ where X is $\mathrm{PSO}(8)$, $\mathrm{PSp}(4)$, or F_4 (and z generates the center of E_7). The commutative diagram, where the exact row is [28, 5.11],

$$\begin{array}{ccccccc} 1 & \longrightarrow & C_{\mathrm{SU}(2)\times \mathrm{E}_7}(V^*)/\langle(-E,z)\rangle & \longrightarrow & C_{\mathrm{SU}(2)\circ \mathrm{E}_7}(V) & \longrightarrow & \mathrm{Hom}(V, \langle(-E,z)\rangle) \\ & & & & \big\uparrow & \nearrow_{[\cdot,\cdot]} & \\ & & & & V & & \end{array}$$

can be used to show that $C_{\mathrm{SU}(2)\circ \mathrm{E}_7}(V) = V \times X$. Since we are assuming that V is $\mathbf{A}$-pure, the dimension of this centralizer is 52 so $C_{\mathrm{E}_8}(V) = C_{\mathrm{SU}(2)\circ \mathrm{E}_7}(V) = V \times \mathrm{F}_4$ and there is only one possibility in this case. The Quillen automorphism group $\mathbf{A}(\mathrm{E}_8)(2^3\mathbf{A}) = \mathrm{GL}(3, \mathbf{F}_2)$ can be determined by computations based on the short exact sequence [27, 4.2].

The contribution, $[V] = \mathrm{Hom}_{\mathbf{A}(\mathrm{E}_8)(V)}(\mathrm{St}(V), V)$, to Oliver's cochain complex [31] for computing $\lim^*(\mathbf{A}(\mathrm{E}_8); \pi_1(BZC_{\mathrm{E}_8}))$ is 0 when $V = 2^3\mathbf{A}^7$.

When $V = 2^3\mathbf{A}^1\mathbf{B}^6$, the stabilizers for the action of $\mathbf{A}(\mathrm{E}_8)(V)$ on planes $P < V$ have orders 6 or 8. The $\mathbf{F}_2$-vector space $[V]$ is 1-dimensional and the homomorphism

$$f[P > L] = \begin{cases} L, & |\mathbf{A}(\mathrm{E}_8)(V)_P| = 8, \\ 0, & |\mathbf{A}(\mathrm{E}_8)(V)_P| = 6, \end{cases}$$

where $\mathbf{A}(\mathrm{E}_8)(V)_P$ is the stabilizer at the plane $P < V$, is a nontrivial vector in $[V]$. There is a rank one elementary abelian 2-group $L(1,3) \subset \mathrm{PSO}(8)$ such that (2.14) the centralizer of $V \times L(1,3) \subset V \times \mathrm{PSO}(8) \subset \mathrm{E}_8$ is $C_{\mathrm{E}_8}(V \times L(1,3)) = V \times C_{\mathrm{PSO}(8)}(L(1,3)) = V \times \mathrm{SO}(2) \circ \mathrm{SO}(6) \rtimes C_2 = V \times \mathrm{U}(4)/\langle -E\rangle \rtimes C_2$. The class distribution of this rank four elementary abelian 2-group in E_8 is $2^4\mathbf{A}^9\mathbf{B}^6$ according to the orthogonality relations. The only rank three nontoral contained in this subgroup is $2^3\mathbf{A}^1\mathbf{B}^6$, for $\mathrm{SO}(2) \circ \mathrm{SO}(6)$ does not occur as the identity component of a rank one centralizer in F_4 (Table 1) or $\mathrm{PSp}(4)$ (Chap. 4, §1). The Quillen automorphism group of $V \times L(1,3)$ is contained in the subgroup

$$A = \begin{pmatrix} \mathbf{A}(\mathrm{E}_8)(V) & 0 \\ * & 1 \end{pmatrix}$$

of order $2^3 \cdot |\mathbf{A}(\mathrm{E}_8)(V)| = 192$ because conjugation in E_8 takes the subgroup

V into itself since this subgroup is determined up to conjugacy by its class distribution. The differential d in Oliver's cochain complex is an isomorphism between $[V]$ and the 1-dimensional subspace

$$\mathrm{Hom}_A(\mathrm{St}(V\times L), V\times L)\subset [V\times L]=\mathrm{Hom}_{\mathbf{A}(\mathrm{E}_8)(V\times L)}(\mathrm{St}(V\times L), V\times L)$$

as

$$df[E>P>L]=\begin{cases} L, & |A_E|=192,\ |A_P|=8,\\ 0, & \text{otherwise,}\end{cases}$$

is nontrivial in $\mathrm{Hom}_A(\mathrm{St}(V\times L), V\times L)\cong \mathbf{F}_2$.

When $V=2^3\mathbf{A}^3\mathbf{B}^4$, the stabilizers for the action of $\mathbf{A}(\mathrm{E}_8)(V)$ on planes $P<V$ have orders 24 or 4. The $\mathbf{F}_2$-vector space $[V]$ is 1-dimensional and the homomorphism

$$f[P>L]=\begin{cases} L, & |\mathbf{A}(\mathrm{E}_8)(V)_P|=24,\\ 0, & |\mathbf{A}(\mathrm{E}_8)(V)_P|=4,\end{cases}$$

where $\mathbf{A}(\mathrm{E}_8)(V)_P$ is the stabilizer at the plane $P<V$, is a nontrivial vector in $[V]$. There is a rank one elementary abelian 2-group $L(1,3)\subset \mathrm{PSp}(4)$ such that the centralizer of $V\times L(1,3)\subset V\times \mathrm{PSp}(4)\subset \mathrm{E}_8$ is $C_{\mathrm{E}_8}(V\times L(1,3))=V\times C_{\mathrm{PSp}(4)}(L(1,3))=V\times \mathrm{Sp}(1)\circ\mathrm{Sp}(3)$ (4.1). The class distribution is $2^4\mathbf{A}^5\mathbf{B}^{10}$. The only rank three nontoral contained in this subgroup is $2^3\mathbf{A}^3\mathbf{B}^4$, for $\mathrm{Sp}(1)\circ\mathrm{Sp}(3)$ does not occur as the identity component of a rank one centralizer in F_4 (Table 1) or $\mathrm{PSO}(8)$ (Chap. 2, §1). The Quillen automorphism group is contained in the group A defined as above but in fact the class distribution forces

$$\mathbf{A}(\mathrm{E}_8)(V\times L(1,3))\subset \begin{pmatrix}\mathbf{A}(\mathrm{E}_8)(V) & 0\\ 0 & 1\end{pmatrix}=A$$

of order 24. The corresponding space of equivariant linear maps $\mathrm{St}(V\times L)\to V\times L$ is 10-dimensional and the image of f under the differential d,

$$df[E>P>L]=\begin{cases} L, & |A_E|=24 \text{ and } E\cong_{\mathbf{F}_2\mathbf{A}(\mathrm{E}_8)(V)} V,\\ 0, & \text{otherwise,}\end{cases}$$

is nonzero.

We conclude that $\lim^1(\mathbf{A}(\mathrm{E}_8);\pi_1(BZC_{\mathrm{E}_8}))=0$ since there are no nontoral elementary abelian 2-groups of rank two, and that

$$\lim{}^2(\mathbf{A}(\mathrm{E}_8);\pi_1(BZC_{\mathrm{E}_8}))=0$$

since the differential is injective.

We have already seen that the functor $\pi_2(BZC_{\mathrm{E}_8})$ has value 0 on all nontoral elementary abelian 2-groups of rank at most three. Consider a nontoral elementary abelian 2-groups of rank four, $E\subset \mathrm{E}_8$ say. If E is $\mathbf{A}$-pure, then E contains the $\mathbf{A}$-pure nontoral elementary abelian 2-group V of rank three. Otherwise, E is contained in and contains the center of $\mathrm{SSpin}(16)$. As before, let $\overline{E} \subset \mathrm{PSO}(16)$, $\overline{E^*} \subset \mathrm{SO}(16)$, and $E^* \subset \mathrm{Spin}(16)$ be the groups corresponding to E. The image $\overline{E}$ in $\mathrm{PSO}(16)$ is a rank three nontoral elementary abelian 2-group. If $[\overline{E}, \overline{E}] \neq 0$, then $\overline{E}$ contains a nontoral rank two group (2.52), which means that E contains a nontoral rank three group. In these cases, $C_{\mathrm{E}_8}(E) = V \times C_X(L)$ where V has rank three and L rank one and X is $\mathrm{PSO}(8)$, $\mathrm{PSp}(4)$, or F_4. In all cases, the center $ZC_{\mathrm{E}_8}(E)$ of the centralizer equals E (2.14, 2.15, 4.1, 4.2, Table 1). In the remaining cases, $[\overline{E}, \overline{E}] = 0$ and $q(\overline{E}) = 0$ as $\overline{E}$ is nontoral (2.10) so that $\overline{E^*} \subset \mathrm{SO}(16)$ is elementary abelian. This faithful representation of E has the form $\sum_{\varrho \in E^\vee} i_\varrho \varrho$ for certain integers $i_\varrho \geq 0$. As $[E^*, E^*] \subset \langle x \rangle$ it follows from [24, Lemma 5.28] that all the numbers i_ϱ have the same parity, which must be odd since $\overline{E^*}$ is nontoral. Since $C_{\mathrm{SO}(16)} = \mathrm{SO}(16) \cap \prod_\varrho \mathrm{O}(i_\varrho)$, where the i_ϱ are odd, the centers of this centralizer and the closely related centralizer $C_{\mathrm{SSpin}(16)}(E)$ are finite. We conclude that the functor $\pi_2(BZC_{\mathrm{E}_8})$ has value zero on all nontoral elementary abelian 2-groups of rank at most four. Oliver's cochain complex now immediately shows that $\lim^j(\mathbf{A}(\mathrm{E}_8); \pi_2(BZC_{\mathrm{E}_8})) = 0$ for $j = 2, 3$.

6. Proofs of 1.1, 1.2, and 1.3 from Part I. At this stage we know that DI(4) and any compact, connected simple, centerless Lie group G are uniquely N-determined when considered as 2-compact groups.

Proof of Theorem 1.1 and Corollary 1.2 from Part I. Let X be a connected 2-compact group. From [12, 1.12] we know that $N(X) = N(G) \times N(\mathrm{DI}(4))^t$ for some integer $t \geq 0$. From [24, 2.38, 2.39, 2.42, 2.43] we know that $G \times \mathrm{DI}(4)^t$ is uniquely N-determined. In particular, X and $G \times \mathrm{DI}(4)^t$ are isomorphic.

Let next X be any 2-compact group, not necessarily connected. By the remarks at the beginning of [24, Chapter 3, §2], the H^i-injectivity, $i = 1, 2$, condition holds for X. Thus X has N-determined automorphisms by [24, Lemma 2.35] and X is N-determined by [24, Lemma 2.40] if it is LHS [24, Definition 2.27]. ∎

Proof of Corollary 1.3 from Part I. Let BL be a connected finite loop space with maximal torus $BT \to BL$ and let $BN \to BL$ be the normalizer of the maximal torus [29, 1.1, 1.3, 1.4]. For each prime p, the p-completion, BL_p, of BL is a connected p-compact group with maximal torus normalizer $BN_p \to BL_p$, the fiberwise p-completion of $BN \to BL$. In particular, at $p = 2$, BN_2 is $BN(G)_2$ for some connected compact Lie group G [12, 1.12] and there is [24, Theorem 1.1] a commutative diagram

$$\begin{array}{ccc} BN_2 & \xrightarrow{\cong} & BN(G)_2 \\ \downarrow & & \downarrow \\ BL_2 & \xrightarrow{\cong} & BG_2 \end{array}$$

where the horizontal maps are homotopy equivalences. There is no BDI(4)-factor in the 2-compact group BL_2 because the Weyl group of the finite loop space L is a reflection group at all primes [29, 1.2]. We also see that $BN_p \cong BN(G)_p$ at all odd primes because the extension class is trivial at odd primes [2]. The classification theorem for p-compact groups at $p > 2$ [3] provides homotopy equivalences $BL_p \to BG_p$ for primes $p > 2$. Using Sullivan's Arithmetic Square, these maps combine, since they are all defined as maps under BN, to a homotopy equivalence $BL \to BG$. ∎

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