THE CLASSIFICATION OF p-COMPACT GROUPS FOR p ODD

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Contents

1. Introduction

It has been a central goal in homotopy theory for about half a century to single out the homotopy theoretical properties characterizing compact Lie groups, and obtain a corresponding classification, starting with the work of Hopf [70] and Serre [116, §4] on H-spaces

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and loop spaces. Materializing old dreams of Sullivan [127, p. 5.96] and Rector [114], Dwyer and Wilkerson, in their seminal paper [51], introduced the notion of a p-compact group, as a p-complete loop space with finite mod p cohomology, and proved that p-compact groups have many Lie-like properties. Even before their introduction it has been the hope [113], and later the conjecture [54, 82, 43], that these objects should admit a classification much like the classification of compact connected Lie groups, and the work toward this has been carried out by many authors. The goal of this paper is to complete the proof of the classification theorem for p an odd prime, showing that there is a one-to-one correspondence between connected p-compact groups and finite reflection groups over the p-adic integers $\mathbf{Z}_p$. We do this by providing the last—and rather intricate—piece, namely that the p-completions of the exceptional compact connected Lie groups are uniquely determined as p-compact groups by their Weyl groups, seen as $\mathbf{Z}_p$-reflection groups. In fact our method of proof gives an essentially self-contained proof of the entire classification theorem.

We start by very briefly introducing p-compact groups and some objects associated to them necessary to state the classification theorem—we will later in the introduction return to the history behind the various steps of the proof. We refer the reader to [51] for more details on p-compact groups and also recommend the overview articles [43, 82, 88]; we likewise point out that it is the technical advances on homotopy fixed points by Miller [87], Lannes [81], and others which makes this theory possible.

A space X with a loop space structure, for short a *loop space*, is a triple (X, BX, e) where BX is a pointed connected space, called the *classifying space* of X, and $e : X \to \Omega BX$ is a homotopy equivalence. A *p-compact group* is a loop space with the two additional properties that $H^*(X; \mathbf{F}_p)$ is finite dimensional over $\mathbf{F}_p$ (to be thought of as 'compactness') and that BX is $\mathbf{F}_p$-local [18][51, §11] (or, in this connection, equivalently $\mathbf{F}_p$-complete [19]). Often we refer to a loop space simply as X. When working with a loop space we shall only be concerned with its classifying space BX, since this determines the rest of the structure—indeed, we could instead have defined a p-compact group to be a space BX with the above properties. The loop space $(G\hat{_p}, BG\hat{_p}, e)$, corresponding to a pair (G, p) (where p is a prime, G a compact Lie group with component group a finite p-group, and $(\cdot)\hat{_p}$ denotes $\mathbf{F}_p$-completion [19][51, §11]) is a p-compact group. (Note however that a compact Lie group G is not uniquely determined by $BG\hat{_p}$, since we are only focusing on the structure 'visible at the prime p'; e.g., $B\,\mathrm{SO}(2n+1)\hat{_p} \simeq B\,\mathrm{Sp}(n)\hat{_p}$ if $p \neq 2$, as originally proved by Friedlander [61]; see Theorem 11.4 for a complete analysis.)

A *morphism* $X \to Y$ between loop spaces is a pointed map of spaces $BX \to BY$. We say that two morphisms are *conjugate* if the corresponding maps of classifying spaces are freely homotopic. A morphism is called an *isomorphism* (or *equivalence*) if it has an inverse up to conjugation, or in other words if $BX \to BY$ is a homotopy equivalence. If X and Y are p-compact groups, we call a morphism a *monomorphism* if the homotopy fiber Y/X of the map $BX \to BY$ is $\mathbf{F}_p$-finite.

The loop space corresponding to the p-completed classifying space $BT = (B\mathrm{U}(1)^r)\hat{_p}$ is called a *p-compact torus* of rank r. A *maximal torus* in X is a monomorphism $i : T \to X$ such that the homotopy fiber of $BT \to BX$ has non-zero Euler characteristic. (We define the Euler characteristic as the alternating sum of the $\mathbf{F}_p$-dimensions of the $\mathbf{F}_p$-homology groups.) Fundamental to the theory of p-compact groups is the theorem of Dwyer-Wilkerson [51] that, analogously to the classical situation, any p-compact group admits a maximal torus. It is unique in the sense that for any other maximal torus $i' : T' \to X$, there exists an isomorphism $\varphi : T \to T'$ such that $i'\varphi$ and i are conjugate. (Note that there is a subtle difference between this statement and the classical statement of being 'unique up to conjugation' due to the fact that a maximal torus is defined to be a *map* and not a subgroup.)

Fix a p-compact group X with maximal torus $i : T \to X$ of rank r. Replace the map $Bi : BT \to BX$ by an equivalent fibration, and define the *Weyl space* $\mathcal{W}_X(T)$ as the topological monoid of self-maps $BT \to BT$ over BX. The *Weyl group* is defined as $W_X(T) = \pi_0(\mathcal{W}_X(T))$ [51, Def. 9.2]. By [51, Prop. 9.5] $W_X(T)$ is a finite group of order $\chi(X/T)$. Furthermore, by [51, Thm. 9.7], if X is connected then $W_X(T)$ identifies with the set of conjugacy classes of self-equivalences φ of T such that i and $i\varphi$ are conjugate. In other words, the canonical homomorphism $W_X(T) \to \mathrm{Aut}(\pi_1(T))$ is injective, so we can view $W_X(T)$ as a subgroup of $\mathrm{GL}_r(\mathbf{Z}_p)$, and this subgroup is independent of T up to conjugation in $\mathrm{GL}_r(\mathbf{Z}_p)$. We will therefore suppress T from the notation.

Now, by [51, Thm. 9.7] this exhibits $(W_X, \pi_1(T))$ as a finite reflection group over $\mathbf{Z}_p$. Finite reflection groups over $\mathbf{Z}_p$ have been classified for p odd by Notbohm [100] extending the classification over $\mathbf{Q}_p$ by Clark-Ewing [31] (which again builds on the classification over $\mathbf{C}$ by Shephard-Todd [119]); we recall this classification in Section 11 and extend Notbohm's result to all primes. (Recall that a finite $\mathbf{Z}_p$-reflection group is a pair (W, L) where L is a finitely generated free $\mathbf{Z}_p$-module, and W is a finite subgroup of $\mathrm{Aut}(L)$ generated by elements α such that $1 - \alpha$ has rank one viewed as a matrix over $\mathbf{Q}_p$; we say that two finite $\mathbf{Z}_p$-reflection groups (W, L) and (W', L') are *isomorphic*, if we can find a $\mathbf{Z}_p$-linear isomorphism $\varphi : L \to L'$ such that the group $\varphi W \varphi^{-1}$ equals W'.)

Given any self-homotopy equivalence $Bf : BX \to BX$, there exists, by the uniqueness of maximal tori, a map $B\tilde{f} : BT \to BT$ such that $Bf \circ Bi$ is homotopy equivalent to $Bi \circ B\tilde{f}$. Furthermore, $B\tilde{f}$ is unique up to the action of the Weyl group, as is easily seen from the definitions (cf. Lemma 2.1). This sets up a homomorphism $\Phi : \pi_0(\mathrm{Aut}(BX)) \to N_{\mathrm{GL}(L_X)}(W_X)/W_X$, where $\mathrm{Aut}(BX)$ is the space of self-homotopy equivalences of BX (this map has precursors going back to Adams-Mahmud [2]; see Lemma 2.1 and Theorem 1.4 for a more elaborate version). The group $N_{\mathrm{GL}(L_X)}(W_X)/W_X$ can be completely calculated; see Section 13.

The main classification theorem which we complete in this paper, is the following.

Theorem 1.1. *Let p be an odd prime. The assignment which to the isomorphism class of the connected p-compact group X assigns the isomorphism class of the pair (W_X, L_X) via the canonical action of W_X on $L_X = \pi_1(T)$ defines a bijection between the set of isomorphism classes of connected p-compact groups and the set of isomorphism classes of finite $\mathbf{Z}_p$-reflection groups.*

Furthermore, for each connected p-compact group X the map $\Phi : \pi_0(\mathrm{Aut}(BX)) \xrightarrow{\cong} N_{\mathrm{GL}(L_X)}(W_X)/W_X$ is an isomorphism, i.e., the group of outer automorphisms of X is canonically isomorphic to the group of outer automorphisms of (W_X, L_X).

In particular this proves, for p odd, Conjecture 5.3 in [43] (see Theorem 1.4). The self-map part of the statement can be viewed as an extension to p-compact groups, p odd, of the main result of Jackowski-McClure-Oliver [78, 76]. Our method of proof via centralizers is 'dual', but logically independent, of the one in [78, 76] (see e.g. [42, 67]).

By [52] the identity component of $\mathrm{Aut}(BX)$ is the classifying space of a p-compact group ZX, which is defined to be the *center* of X—we call X *center-free* if ZX is trivial. Furthermore recall that a connected p-compact group X is called *simple* if $L_X \otimes \mathbf{Q}$ is an irreducible W-representation and X is called *exotic* if it is simple and (W_X, L_X) does not come from a

Z-reflection group (see Section 11). By inspection of the classification of finite $\mathbf{Z}_p$-reflection groups, Theorem 1.1 has as a' corollary that the theory of p-compact groups on the level of objects splits in two parts, as has been conjectured (Conjectures 5.1 and 5.2 in [43]).

Theorem 1.2. *Let X be a connected p-compact group, p odd. Then X can be written as a product of p-compact groups*

$$X \cong G_{\hat{p}} \times X'$$

where G is a compact connected Lie group, and X' is a direct product of exotic p-compact groups which are all simply connected, center-free, and have torsion free $\mathbf{Z}_p$-cohomology.

Theorem 1.1 has both an existence and a uniqueness part to it, the existence part being that all finite $\mathbf{Z}_p$-reflection groups are realized as Weyl groups of a connected p-compact group. The finite $\mathbf{Z}_p$-reflection groups which come from compact connected Lie groups are of course realizable, and the finite $\mathbf{Z}_p$-reflection groups where p does not divide the order of the group can also relatively easily be dealt with, as done by Sullivan [127, p. 5.96] and Clark-Ewing [31] long before p-compact groups were officially defined. The remaining cases were realized by Quillen [111, §10], Zabrodsky [138], Aguadé [4] and Notbohm-Oliver [101] [103, Thm. 1.4]. The classification of finite $\mathbf{Z}_p$-reflection groups, Theorem 11.1, guarantees that the construction of these examples actually enables one to construct all finite $\mathbf{Z}_p$-reflection groups as Weyl groups of connected p-compact groups.

The work toward the uniqueness part, to show that a connected p-compact group is uniquely determined by its Weyl group, also predates the introduction of p-compact groups. The quest was initiated by Dwyer-Miller-Wilkerson [46, 47] (building on [3]) who proved the statement, using slightly different language, in the case where p is prime to the order of W_X as well as for $\mathrm{SU}(2)_{\hat{2}}$ and $\mathrm{SO}(3)_{\hat{2}}$. Notbohm [98] and Møller-Notbohm [94, Thm. 1.9] extended this to a uniqueness statement for all p-compact groups X where $\mathbf{Z}_p[L_X]^{W_X}$ (the ring of W_X-invariant polynomial functions on L_X) is a polynomial algebra and (W_X, L_X) comes from a finite **Z**-reflection group. Notbohm [101, 103] subsequently also handled the cases where (W, L) does not come from a finite **Z**-reflection group. (Beware that in [101, 103] Notbohm proves an apparently weaker uniqueness statement, from which the above statement however can be deduced; see Remark 7.11.)

To get general statements beyond the case where $\mathbf{Z}_p[L_X]^{W_X}$ is a polynomial algebra, i.e., to attack the cases where there exists p-torsion in the cohomology ring, the first step is to reduce the classification to the case of simple, center-free p-compact groups. The results necessary to obtain this reduction were achieved by the splitting theorem of Dwyer-Wilkerson [53] and Notbohm [104] along with properties of the center of a p-compact group established by Dwyer-Wilkerson [52] and Møller-Notbohm [93]. We carry out this reduction in Section 4; see also [91].

An analysis of the classification of finite $\mathbf{Z}_p$-reflection groups together with explicit calculations (see [102] and Theorem 12.2) shows that, for p odd, $\mathbf{Z}_p[L_X]^{W_X}$ is a polynomial algebra for all irreducible center-free finite $\mathbf{Z}_p$-reflection groups except the reflection groups coming from the p-compact groups $\mathrm{PU}(n)_{\hat{p}}$, $(E_8)_{\hat{5}}$, $(F_4)_{\hat{3}}$, $(E_6)_{\hat{3}}$, $(E_7)_{\hat{3}}$, and $(E_8)_{\hat{3}}$. For exceptional compact connected Lie groups the notation E_6 etc. denotes their *adjoint* form.

The case $\mathrm{PU}(n)_{\hat{p}}$ was handled by Broto-Viruel [22], using a Bockstein spectral sequence argument to deduce it from the result for $\mathrm{SU}(n)$, generalizing earlier partial results of Broto-Viruel [21] and Møller [90]. The remaining step in the classification is hence to handle the exceptional compact connected Lie groups, in particular the problematic E-family at the prime 3, and this is what is carried out in this paper. (The fourth named author has also given alternative proofs for $(F_4)\hat{_3}$ and $(E_8)\hat{_5}$ in [130] and [129].)

Theorem 1.3. *Let X be a connected p-compact group, for p odd, with Weyl group equal to $(W_G, L_G \otimes \mathbf{Z}_p)$ for one of the pairs $(F_4, p = 3)$, $(E_8, p = 5)$, $(E_6, p = 3)$, $(E_7, p = 3)$, or $(E_8, p = 3)$. Then X is isomorphic, as a p-compact group, to the p-completion of the corresponding exceptional group G.*

We will in fact give an essentially self-contained proof of the entire classification Theorem 1.1, since this comes rather naturally out of our inductive approach to the exceptional cases. We however still rely on the classification of finite $\mathbf{Z}_p$-reflection groups (see [100, 102] and Section 11 and 12) as well as the above mentioned structural results from [51, 52, 93, 53, 104, 92].

The main ingredient in handling the exceptional groups is to get sufficiently detailed information about their many conjugacy classes of elementary abelian p-subgroups (carried out in Section 8, expanding on the work of Griess [65]), and then to use this information to show the relevant obstruction groups are trivial (carried out in Section 10), using formulas of Oliver [106] (see also [67]).

It is possible to formulate a more topological version of the uniqueness part of Theorem 1.1 which holds for all p-compact groups (p odd), not necessarily connected, which is however easily seen to be equivalent to the first one using [6, Thm. 1.2]. It should be viewed as a topological analog of Chevalley's isomorphism theorem for linear algebraic groups (see [71, §32] [126, Thm. 1.5] and [37, 109, 99]). To state it, we define the *maximal torus normalizer* $\mathcal{N}_X(T)$ to be the loop space such that $B\mathcal{N}_X(T)$ is the Borel construction of the canonical action of $\mathcal{W}_X(T)$ on BT. Note that by construction we have a morphism $\mathcal{N}_X(T) \to X$, and that this map is independent of the choice of T, up to conjugacy. By [51, Prop. 9.5], $\mathcal{W}_X(T)$ is a discrete space so $B\mathcal{N}_X(X)$ has only two non-trivial homotopy groups and fits into a fibration sequence $BT \to B\mathcal{N}_X(T) \to BW_X$. (Beware that in general $\mathcal{N}_X(T)$ will not be a p-compact group since its group of components W_X need not be a p-group.)

Theorem 1.4 (Topological isomorphism theorem for p-compact groups, p odd)**.** *Let p be an odd prime and let X and X' be p-compact groups with maximal torus normalizers $\mathcal{N}_X$ and $\mathcal{N}_{X'}$. Then $X \cong X'$ if and only if $B\mathcal{N}_X \simeq B\mathcal{N}_{X'}$.*

Furthermore the spaces of self-homotopy equivalences $\mathrm{Aut}(BX)$ *and* $\mathrm{Aut}(B\mathcal{N}_X)$ *are equivalent as grouplike topological monoids. Explicitly, turn $i : B\mathcal{N}_X \to BX$ into a fibration which we will again denote by i, and let* $\mathrm{Aut}(i)$ *denote the grouplike topological monoid of self-homotopy equivalences of the map i. Then the following canonical zig-zag, given by restrictions, is a zig-zag of homotopy equivalences:*

$$B\,\mathrm{Aut}(BX) \xleftarrow{\simeq} B\,\mathrm{Aut}(i) \xrightarrow{\simeq} B\,\mathrm{Aut}(B\mathcal{N}_X).$$

In the above theorem, the fact that the evaluation map $\mathrm{Aut}(i) \to \mathrm{Aut}(BX)$ is an equivalence follows by a short general argument (Lemma 2.1), whereas the equivalence $\mathrm{Aut}(i) \to \mathrm{Aut}(B\mathcal{N}_X)$ requires a detailed case-by-case analysis.

We point out that the classification of course gives easy, although somewhat unsatisfactory, proofs that many theorems from Lie theory extend to p-compact groups, by using that the theorem is known to be true in the Lie case, and then checking the exotic cases. Since the classifying spaces of the exotic p-compact groups have cohomology ring a polynomial algebra, this can turn out to be rather straightforward. In this way one for instance sees that Bott's celebrated result about the structure of G/T [14] still holds true for p-compact groups, at least on cohomology.

Theorem 1.5 (Bott's theorem for p-compact groups)**.** *Let X be a connected p-compact group, p odd, with maximal torus T and Weyl group W_X. Then $H^*(X/T;\mathbf{Z}_p)$ is a free $\mathbf{Z}_p$-module of dimension $|W_X|$, concentrated in even degrees.*

Likewise combining the classification with a case-by-case verification for the exotic p-compact groups by Castellana [26, 27], we get that the Peter-Weyl theorem holds for connected p-compact groups, p odd:

Theorem 1.6 (Peter-Weyl theorem for connected p-compact groups)**.** *Let X be a connected p-compact group, p odd. Then there exists a monomorphism $X \to \mathrm{U}(n)\hat{_p}$ for some n.*

We also still have the 'standard' formula for the fundamental group (the subscript denotes coinvariants).

Theorem 1.7. *Let X be a connected p-compact group, p odd. Then*

$$\pi_1(X) = (L_X)_{W_X}$$

The classification also gives a verification that results of Borel, Steinberg, Demazure, and Notbohm [103, Prop. 1.11] extend to p-compact groups, p odd.

Theorem 1.8. *Let X be a connected p-compact group, p odd. The following conditions are equivalent:*

(1) *X has torsion free $\mathbf{Z}_p$-cohomology.*
(2) *BX has torsion free $\mathbf{Z}_p$-cohomology.*
(3) *$\mathbf{Z}_p[L_X]^{W_X}$ is a polynomial algebra over $\mathbf{Z}_p$.*
(4) *All elementary abelian p-subgroups of X factor through a maximal torus.*

Even in the Lie case, the proof of the above theorem is still not entirely satisfactory despite much effort—see the comments surrounding our proof as well as Borel's comments [10, p. 775] and the references [8, 38, 125]. Recall that the *centralizer* $\mathcal{C}_X(\nu)$ of an elementary abelian p-subgroup $\nu : E \to X$ is defined as $\mathcal{C}_X(\nu) = \Omega\,\mathrm{map}(BE, BX)_{B\nu}$; cf. Section 6. The following related result from Lie theory also holds true.

Theorem 1.9. *Let X be a connected p-compact group, p odd. Then the following conditions are equivalent:*

(1) *$\pi_1(X)$ is torsion free.*
(2) *Every rank one elementary abelian p-subgroup $\nu : \mathbf{Z}/p \to X$ has connected centralizer $\mathcal{C}_X(\nu)$.*
(3) *Every rank two elementary abelian p-subgroup factor through a maximal torus.*

Results about p-compact groups can in general, via Sullivan's arithmetic square, be translated into results about finite loop spaces, and the last theorem in this introduction is an example of such a translation. Recall that a finite loop space is a loop space (X, BX, e), where X is a finite CW-complex. A maximal torus of a finite loop space is simply a map $B\mathrm{U}(1)^r \to BX$ for some r, such that the homotopy fiber is homotopy equivalent to a finite CW-complex of non-zero Euler characteristic. The classical maximal torus conjecture (stated in 1973 by Wilkerson [133, Conj. 1] as "a popular conjecture toward which the author is biased"), asserts that compact connected Lie groups are the *only* connected finite loop spaces which admit maximal tori. A slightly more elaborate version states that the classifying space functor should set up a bijection between isomorphism classes of compact connected Lie groups and isomorphism classes of connected finite loop spaces admitting a maximal torus, under which the outer automorphism group of the Lie group G equals the outer automorphism group of the corresponding loop space (G, BG, e). (The last part is true by [76].) It is well known that a proof of the conjectured classification of p-compact groups for all primes p would imply the maximal torus conjecture. Our results at least imply that the conjecture is true after inverting the single prime 2.

Theorem 1.10. *Let X be a connected finite loop space with a maximal torus. Then there exists a compact connected Lie group G such that $BX[\frac{1}{2}]$ and $BG[\frac{1}{2}]$ are homotopy equivalent spaces, where $[\frac{1}{2}]$ indicates $\mathbf{Z}[\frac{1}{2}]$-localization.*

Relationship to the Lie case and the conjectural picture for $p = 2$. We now state a common formulation of both the classification of compact connected Lie groups and the classification of connected p-compact groups for p odd, which conjecturally should also hold for $p = 2$. We have not encountered this—in our opinion quite natural—description before in the literature (compare [43] and [82]).

Let R be an integral domain and W a finite R-reflection group. For an *RW-lattice L* (i.e., an RW-module which is finitely generated and free as an R-module) define SL to be the sublattice of L generated by $(1 - w)x$ where $w \in W$ and $x \in L$. Define an *R-reflection datum* to be a triple (W, L, L_0) where (W, L) is a finite R-reflection group and L_0 is an RW-lattice such that $SL \subseteq L_0 \subseteq L$ and L_0 is of the form SL' for some other RW-lattice L'. (If $R = \mathbf{Z}_p$, p odd, then "S" is idempotent (since $H_1(W; L_W) = 0$ for all (W, L) by a case-by-case computation given in [6, Thm. 3.2]) so $L_0 = SL$ in this case.) Two reflection data (W, L, L_0) and (W', L', L'_0) are said to be isomorphic if there exists an R-linear isomorphism $\varphi : L \to L'$ such that $\varphi W \varphi^{-1} = W'$ and $\varphi(L_0) = L'_0$.

If $\mathcal{D}$ is either the category of connected compact Lie groups or connected p-compact groups, then we can consider the assignment which to each object X in $\mathcal{D}$ assigns the triple (W, L, L_0), where W is the Weyl group, $L = \pi_1(T)$ is the *dual weight lattice*, and $L_0 = \ker(\pi_1(T) \to \pi_1(X))$ is the *coroot lattice*.

Theorem 1.1 and 1.7 as well as the classification of compact connected Lie groups [17, §4, no. 9] can now be reformulated as follows:

Theorem 1.11. *If $\mathcal{D}$ is the category of compact connected Lie groups or connected p-compact groups for p odd, then (W, L, L_0) is an R-reflection datum ($R = \mathbf{Z}$ for compact connected Lie groups and $\mathbf{Z}_p$ for connected p-compact groups) and this assignment sets up a bijection between the objects of $\mathcal{D}$ up to isomorphism and R-reflection data, up to isomorphism. Furthermore the group of outer automorphisms of X equals the group of outer automorphisms of the corresponding R-reflection datum.*

Conjecture 1.12. *Theorem 1.11 is also true if $\mathcal{D}$ is the category of connected 2-compact groups.*

One can check that the conjecture on objects is equivalent to the conjecture given in [43] and [82], and the self-map statement would then follow from [76] and [105]. The role of the coroot lattice L_0 in the above theorem and conjecture is in fact only to be able to distinguish direct summands isomorphic to $\mathrm{SO}(2n+1)$ from direct summands isomorphic to $\mathrm{Sp}(n)$, cf. Theorem 11.4(1). Alternatively one can use the extension class $\gamma \in H^3(W; L_X)$ of the maximal torus normalizer (see Section 3) rather than L_0 but in that picture it is not a priori clear which triples (W, L, γ) are realizable. It would be desirable to have a 'topological' version of Theorem 1.11 and Conjecture 1.12, i.e., statements on the level of automorphism spaces like Theorem 1.4 but we do not know a general formulation which incorporates this feature.

Organization of the paper and notation. The sections of this paper can be read in an almost arbitrary order. The short Section 2 sets up the map from the space of automorphisms of X to the space of automorphisms of $\mathcal{N}_X$, and in Section 3 we give an algebraic description of the automorphism group of $\mathcal{N}_X$ (which we expand on in Section 13). In Section 4 we reduce the classification Theorem 1.1 to the case of simple, center-free, connected p-compact groups. In Section 5 we prove a theorem about invariant rings and show how this leads to an easy construction of the exotic p-compact groups. In Section 6 we give proofs of the main theorem, modulo obstruction group calculations which are carried out in Section 10. The applications listed in the introduction are proved in Section 7. The purely algebraic Section 8 contains complete information about all non-toral elementary abelian p-subgroups of the exceptional compact connected Lie groups, along with their Weyl groups and centralizers, and Section 9 gives the analogous (but much easier) results for the projective unitary groups. (This information is used in a crucial way for the calculations in Section 10 as well as, in a milder way, directly in Section 6 for information about rank two non-toral subgroups.) In the appendix Section 11 we give a classification of finite $\mathbf{Z}_p$-reflection groups generalizing Notbohm's classification to all primes and in the appendix Section 12 we recall Notbohm's results on invariant rings of finite $\mathbf{Z}_p$-reflection groups. Finally in the appendix Section 13 we briefly calculate the outer automorphism groups of the exotic $\mathbf{Z}_p$-reflection groups to make the result of Theorem 1.1 more explicit.

We have tried to introduce the definitions relating to p-compact group as they are used, but it is nevertheless probably helpful for the reader unfamiliar with p-compact groups to keep copies of the excellent papers [51] and [52] of Dwyer-Wilkerson (whose terminology we follow) within reach. As a technical term we say that a p-compact group X is *determined by* $\mathcal{N}_X$ if it is true that any other p-compact group X' with the same maximal torus normalizer is isomorphic to X (which will be true for all p-compact groups, p odd, by Theorem 1.4).

We tacitly assume that any space in this paper has the homotopy type of a CW-complex, if necessary replacing a given space by the realization of its singular complex [86].

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2. The map $\mathrm{Aut}(BX) \to \mathrm{Aut}(B\mathcal{N}_X)$

The purpose of this very short section is to construct the map $\mathrm{Aut}(BX) \to \mathrm{Aut}(B\mathcal{N}_X)$ which we will later prove is an equivalence. We have been unable to find this description in the literature.

For a fibration $f : \mathcal{E} \to \mathcal{B}$ we let $\mathrm{Aut}(f)$ denote the space of commutative diagrams

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{E} \\ \downarrow f & & \downarrow f \\ \mathcal{B} & \longrightarrow & \mathcal{B} \end{array}$$

such that the horizontal maps are homotopy equivalences. (This is a subspace of $\mathrm{Aut}(\mathcal{E}) \times \mathrm{Aut}(\mathcal{B})$.)

Lemma 2.1 (Adams-Mahmud lifting)**.** *Let X be a p-compact group with maximal torus normalizer $\mathcal{N}_X$. Turn the inclusion of the maximal torus normalizer into a fibration $i : B\mathcal{N}_X \to BX$. Then the restriction map $\mathrm{Aut}(i) \to \mathrm{Aut}(BX)$ is an equivalence of grouplike topological monoids.*

In particular any self-homotopy equivalence of BX lifts to a self-homotopy equivalence of $B\mathcal{N}_X$, which is unique in the strong sense that the space of lifts is contractible. Choosing a homotopy inverse to the homotopy equivalence $B\,\mathrm{Aut}(i) \to B\,\mathrm{Aut}(BX)$, we get a canonical homomorphism of grouplike topological monoids

$$\Phi : \mathrm{Aut}(BX) \xrightarrow{\cong} \mathrm{Aut}(i) \to \mathrm{Aut}(B\mathcal{N}_X).$$

Proof. For any $\varphi \in \mathrm{Aut}(BX)$, there exists, e.g. by [92, Thm. 1.2(3)], a map $\psi \in \mathrm{Aut}(B\mathcal{N}_X)$ such that φi is homotopic to $i\psi$. Since i is assumed to be a fibration, ψ can furthermore be modified such that the equality is strict. This shows that the evaluation map $\mathrm{Aut}(i) \to \mathrm{Aut}(BX)$ is surjective on components. This map of grouplike topological monoids is furthermore easily seen to have the homotopy lifting property. To see that it is a homotopy equivalence we hence just have to verify that the fiber $\mathrm{Aut}_{BX}(B\mathcal{N}_X)$ over the identity map is contractible. We have the following diagram with rows and columns fibrations

$$\begin{array}{ccccc}
\mathcal{W}_X & \longrightarrow & X/T & \longrightarrow & X/\mathcal{N}_X \\
\| & & \downarrow & & \downarrow \\
\mathcal{W}_X & \longrightarrow & BT & \longrightarrow & B\mathcal{N}_X \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & BX & = & BX.
\end{array}$$

Taking homotopy T-fixed points of the top row produces a fibration sequence $\mathcal{W}_X \to (X/T)^{hT} \to (X/\mathcal{N}_X)^{hT}$, where, by the definition of $\mathcal{W}_X$, the inclusion of the fiber in the total space is a homotopy equivalence. (We refer to [51, 3.3, §10] for basic facts and definitions about homotopy actions.) Hence $(X/\mathcal{N}_X)^{hT}$ is contractible. Let $\breve{\mathcal{N}}_X$ and $\breve{T}$ denote discrete approximations to $\mathcal{N}_X$ and T respectively, i.e., $\breve{T} \cong (\mathbf{Z}/p^\infty)^r \subseteq T$ and $\breve{\mathcal{N}}_X$ is an extension of W_X by $\breve{T}$ such that $B\breve{\mathcal{N}}_X \to B\mathcal{N}_X$ is an $\mathbf{F}_p$-equivalence (cf. [52, §3] for facts about discrete approximations). The space $(X/\mathcal{N}_X)^{h\mathcal{N}_X}$ is contractible as well, since $(X/\mathcal{N}_X)^{h\mathcal{N}_X} \simeq (X/\mathcal{N}_X)^{h\breve{\mathcal{N}}_X} \simeq ((X/\mathcal{N}_X)^{h\breve{T}})^{hW_X}$ and $(X/\mathcal{N}_X)^{h\breve{T}}$ is contractible. Since the space of maps $B\mathcal{N}_X \to B\mathcal{N}_X$ over BX identifies with $(X/\mathcal{N}_X)^{h\mathcal{N}_X}$, we see that any self-map of $B\mathcal{N}_X$ over BX is an equivalence, and that $\mathrm{Aut}_{BX}(B\mathcal{N}_X)$ is contractible as wanted. □

3. Automorphisms of maximal torus normalizers

The aim of this short section is to establish some easy facts about automorphisms of maximal torus normalizers which are needed to carry out the reduction to connected, center-free simple p-compact groups in Section 4. At the same time the section serves to make the automorphism statement of Theorem 1.1 more explicit.

Recall that an *extended p-compact torus* is a loop space $\mathcal{N}$ such that $W = \pi_0(\mathcal{N})$ is a finite group and the identity component $\mathcal{N}_1$ of $\mathcal{N}$ is a p-compact torus T. Let $\breve{\mathcal{N}}$ be the discrete approximation to $\mathcal{N}$ (see [52, 3.12]), and recall that $\breve{\mathcal{N}}$ will have a unique largest p-divisible subgroup $\breve{T}$, which will be a discrete approximation to T.

Proposition 3.1. *For an extended p-compact torus $\mathcal{N}$, the obvious map associating to a self-homotopy equivalence of $B\breve{\mathcal{N}}$ a self-homotopy equivalence of $B\mathcal{N}$ via fiberwise $\mathbf{F}_p$-completion* [19, Ch. I §8] *induces an equivalence of aspherical grouplike topological monoids*

$$\mathrm{Aut}(B\breve{\mathcal{N}})^{\wedge}_p \xrightarrow{\cong} \mathrm{Aut}(B\mathcal{N}).$$

If $\pi_0(\mathcal{N})$ acts faithfully on $\pi_1(\mathcal{N}_1)$ then $\mathrm{Aut}_1(B\breve{\mathcal{N}})$, the component of $\mathrm{Aut}(B\breve{\mathcal{N}})$ of the identity map, has the homotopy type of $B(\breve{T}^W)$ where $\breve{T}$ is a discrete approximation to T.

Sketch of proof. The statement on the level of component groups follows directly from [52, 3.12]. (The point is that the homotopy fiber of $B\breve{\mathcal{N}} \to B\mathcal{N}$ will have homotopy type $K(V,1)$ for a $\mathbf{Q}_p$-vector space V, and hence the existence and uniqueness obstructions to lifting a map $B\breve{\mathcal{N}} \to B\breve{\mathcal{N}}$ to $B\mathcal{N}$ lie in $H^n(\breve{\mathcal{N}};V)$ where $n = 2,1$ which are easily seen to be zero.) It is likewise easy to see that both spaces are aspherical and that we get a homotopy equivalence of the identity components. The last statement is also obvious. □

Let L be a finitely generated free $\mathbf{Z}_p$-module and suppose that $W \subseteq \mathrm{GL}(\breve{T})$, where we set $\breve{T} = L \otimes \mathbf{Z}/p^\infty$. Consider the second cohomology group $H^2(W;\breve{T})$ which classifies extensions of W by $\breve{T}$ with the fixed action of W on $\breve{T}$. Given an isomorphism $\alpha : L \to L'$ sending $W \subseteq \mathrm{GL}(L)$ to $W' \subseteq \mathrm{GL}(L')$ we get an isomorphism of cohomology groups $H^2(W;\breve{T}) \to H^2(W';\breve{T}')$ by sending an extension $\breve{T} \xrightarrow{i} \breve{\mathcal{N}} \xrightarrow{\pi} W$ to the extension $\breve{T}' \xrightarrow{i\circ\alpha^{-1}} \breve{\mathcal{N}} \xrightarrow{c_\alpha\circ\pi} W'$, where c_α denotes conjugation by α. An isomorphism between two triples (W,L,γ) and (W',L',γ'), where γ and γ' are extension classes, is an isomorphism $L \to L'$ sending W to W' and γ to γ'. The automorphism group of a triple (W,L,γ) thus identifies with

$${}^{\gamma}N_{\mathrm{GL}(\breve{T})}(W) = \{\alpha \in N_{\mathrm{GL}(\breve{T})}(W) \,|\, \alpha(\gamma) = \gamma \in H^2(W;\breve{T})\}.$$

It follows directly from the definition (and using that $\breve{T}$ is characteristic in $\breve{\mathcal{N}}$) that two triples as above are isomorphic if and only if the associated groups $\breve{\mathcal{N}}$ and $\breve{\mathcal{N}}'$ are isomorphic, where $\breve{\mathcal{N}}$ is obtained from the extension $1 \to \breve{T} \to \breve{\mathcal{N}} \to W \to 1$ given by γ, and analogously for γ'. However, $\breve{\mathcal{N}}$ and (W,L,γ) in general have slightly different automorphisms, as is described in the following lemma (see also [132]):

Proposition 3.2. *In the notation above, for any exact sequence $1 \to \breve{T} \to \breve{\mathcal{N}} \xrightarrow{\pi} W \to 1$ with extension class γ we have a canonical exact sequence*

$$1 \to \mathrm{Der}(W,\breve{T}) \to \mathrm{Aut}(\breve{\mathcal{N}}) \to {}^{\gamma}N_{\mathrm{GL}(\breve{T})}(W) \to 1 \tag{3.1}$$

where we embed the derivations $\mathrm{Der}(W,\breve{T})$ in $\mathrm{Aut}(\breve{\mathcal{N}})$ by sending a derivation s to the automorphism given by $x \mapsto s(\pi(x))x$, and the map $\mathrm{Aut}(\breve{\mathcal{N}}) \to {}^{\gamma}N_{\mathrm{GL}(\breve{T})}(W)$ is given by restricting an automorphism $\varphi \in \mathrm{Aut}(\breve{\mathcal{N}})$ to $\breve{T}$.

This exact sequence has an exact subsequence $1 \to \breve{T}/\breve{T}^W \to \breve{\mathcal{N}}/Z\breve{\mathcal{N}} \to W \to 1$ and the quotient exact sequence is

$$1 \to H^1(W;\breve{T}) \to \mathrm{Out}(\breve{\mathcal{N}}) \to {}^{\gamma}N_{\mathrm{GL}(\breve{T})}(W)/W \to 1.$$

In particular if (W,L) is a finite $\mathbf{Z}_p$-reflection group and p is odd then by [6],[78, Pf. of Prop. 3.5] *$H^1(W;\breve{T}) = 0$, so we get an isomorphism $\mathrm{Out}(\breve{\mathcal{N}}) \xrightarrow{\cong} {}^{\gamma}N_{\mathrm{GL}(\breve{T})}(W)/W$.*

Proof. Let $\varphi \in \mathrm{Aut}(\breve{\mathcal{N}})$, and consider the restriction map $\varphi \mapsto \varphi_{|\breve{T}} \in \mathrm{Aut}(\breve{T})$. Note that for all $x \in \breve{\mathcal{N}}$, $l \in \breve{T}$ we have

$$(\varphi \circ c_x)(l) = \varphi(xlx^{-1}) = \varphi(x)\varphi(l)\varphi(x)^{-1} = (c_{\varphi(x)} \circ \varphi)(l)$$

so $\varphi_{|\breve{T}} \in N_{\mathrm{GL}(\breve{T})}(W)$. That the image is the elements which fixes the extension class follows easily from the definitions: The diagram

$$\begin{array}{ccccc} \breve{T} & \xrightarrow{i\circ\varphi} & \breve{\mathcal{N}} & \xrightarrow{c_\varphi\circ\pi} & W \\ \| & & \downarrow{\scriptstyle\varphi} & & \| \\ \breve{T} & \xrightarrow{i} & \breve{\mathcal{N}} & \xrightarrow{\pi} & W \end{array}$$

shows that φ leaves γ invariant. Likewise, to see that the right map in (3.1) is surjective let $\psi \in {}^{\gamma}N_{\mathrm{GL}(\breve{T})}(W)$ and let $\breve{T} \to \tilde{N} \to W$ be the extension obtained by first pushing forward along $\psi : \breve{T} \to \breve{T}$ and then pulling back along $\psi^{-1}(-)\psi : W \to W$. Since ψ fixes γ there exists an isomorphism $\tilde{N} \to \breve{\mathcal{N}}$ making the following diagram commute

$$\begin{array}{ccccc} \breve{T} & \xrightarrow{\psi} & \breve{T} & = & \breve{T} \\ \downarrow & & \downarrow & & \downarrow \\ \breve{\mathcal{N}} & \longrightarrow & \tilde{N} & \longrightarrow & \breve{\mathcal{N}} \\ \downarrow & & \downarrow & & \downarrow \\ W & \xrightarrow{\psi(-)\psi^{-1}} & W & = & W \end{array}$$

which shows that $\mathrm{Aut}(\breve{\mathcal{N}}) \to {}^{\gamma}N_{\mathrm{GL}(\breve{T})}(W)$ is surjective.

Now suppose $\varphi \in \mathrm{Aut}(\breve{\mathcal{N}})$ restricts to the identity on $\breve{T}$. For a given $x \in \breve{\mathcal{N}}$ we have

$$\varphi(x)l\varphi(x^{-1}) = \varphi(x)\varphi(l)\varphi(x^{-1}) = \varphi(xlx^{-1}) = xlx^{-1}$$

so the induced map $\varphi : W \to W$ is the identity since W acts faithfully on $\breve{T}$. This means that we can define a map $s : W \to \breve{T}$ by $s(w) = \tilde{w}^{-1}\varphi(\tilde{w})$ where $\tilde{w}$ is a lift of w, and this is easily seen to be a derivation. Furthermore taking the automorphism of $\breve{\mathcal{N}}$ associated to s gives back φ, which establishes exactness in the middle, and we have proved the existence of the first exact sequence.

The existence of the short exact subsequence is clear, noting that $Z\breve{\mathcal{N}} = \breve{T}^W$ (since W acts faithfully on $\breve{T}$) and that $\breve{T}/\breve{T}^W$ embeds in $\mathrm{Der}(W, \breve{T})$ as the principal derivations by sending l to the derivation $w \mapsto c_w(l)l^{-1}$. The last exact sequence is now obvious. □

Remark 3.3. See [68] for a related exact sequence for compact connected Lie groups, fitting with the conjectured classification of connected p-compact groups for $p = 2$.

Proposition 3.4. *Suppose $\{(W_i, L_i, \gamma_i)\}_{i=1}^k$ is a collection of pair-wise non-isomorphic triples where L_i is a finitely generated free $\mathbf{Z}_p$-module, W_i is a finite subgroup of $\mathrm{GL}(L_i)$ such that $L_i \otimes \mathbf{Q}$ is irreducible, and $\gamma_i \in H^2(W_i; \breve{T}_i)$. Let $(W, L, \gamma) = \prod_{i=1}^k (W_i, L_i, \gamma_i)^{m_i}$ denote the product. Then*

$$\prod_{i=1}^{k} ({}^{\gamma_i}N_{\mathrm{GL}(L_i)}(W_i)/W_i) \wr \Sigma_{m_i} \xrightarrow{\cong} {}^{\gamma}N_{\mathrm{GL}(L)}(W)/W.$$

Proof. Assume for ease of notation that $L = L_1 \oplus L_2$; the general case follows from this by induction. Consider $\varphi \in N_{\mathrm{GL}(L_1 \oplus L_2)}(W_1 \times W_2)$. For every $w \in W_1 \times W_2$ there exists a unique $\tilde{w} \in W_1 \times W_2$ such that

$$\varphi(wx) = \tilde{w}\varphi(x) \text{ for all } x \in L.$$

Let α denote the element in $\mathrm{Aut}(W_1 \times W_2)$ given by $w \mapsto \tilde{w}$.

By the definition of α the canonical map

$$\varphi_{ji} : L_i \to L_1 \oplus L_2 \xrightarrow{\varphi} {}^{\alpha}(L_1 \oplus L_2) \to {}^{\alpha}L_j$$

is $W_1 \times W_2$-equivariant, where the superscript α means that we are acting through α. Hence this map, after tensoring with $\mathbf{Q}$ has to be either an isomorphism or zero, since $L_i \otimes \mathbf{Q}$ and ${}^{\alpha}L_j \otimes \mathbf{Q}$ are irreducible. But combined with the fact that φ is an isomorphism, this means that

$$\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$$

has to consist of either just 'diagonal' elements or just 'off diagonal' elements. If we furthermore require φ to respect the extension classes it is clear that we can only have the 'diagonal' case if (W_1, L_1, γ_1) is not isomorphic to (W_2, L_2, γ_2) whereas if they are isomorphic both cases can really occur. This proves the lemma. □

4. Reduction to connected, center-free simple p-compact groups

In this section we prove some lemmas, which, together with the splitting theorems of Dwyer-Wilkerson [53] and Notbohm [104], reduce the proof of Theorem 1.4 to the case of connected, center-free simple p-compact groups. This reduction is known and most of it appears in [91] (relying on earlier work of that author). We here provide a self-contained and a bit more direct proof using [52].

Lemma 4.1 (Behavior with respect to products)**.** *Let X and X' be p-compact groups with maximal torus normalizers $\mathcal{N}$ and $\mathcal{N}'$. Then $\mathcal{N} \times \mathcal{N}'$ is a maximal torus normalizer for $X \times X'$ and the following statements hold:*

1. $\mathrm{Aut}_1(BX) \times \mathrm{Aut}_1(BX') \xrightarrow{\cong} \mathrm{Aut}_1(BX \times BX')$ *and* $\mathrm{Aut}_1(B\mathcal{N}) \times \mathrm{Aut}_1(B\mathcal{N}') \xrightarrow{\cong} \mathrm{Aut}_1(B\mathcal{N} \times B\mathcal{N}')$*, where* Aut_1 *denotes the set of homotopy equivalences homotopic to the identity.*
2. *If* $\mathrm{Aut}(BX) \to \mathrm{Aut}(B\mathcal{N})$ *and* $\mathrm{Aut}(BX') \to \mathrm{Aut}(B\mathcal{N}')$ *are injective on* π_0*, then so is* $\mathrm{Aut}(B(X \times X')) \to \mathrm{Aut}(B(\mathcal{N} \times \mathcal{N}'))$.
3. *Suppose p is odd and that X and X' are connected and center-free. If* $\mathrm{Aut}(BX) \to \mathrm{Aut}(B\mathcal{N})$ *and* $\mathrm{Aut}(BX) \to \mathrm{Aut}(B\mathcal{N}')$ *are surjective on* π_0*, then so is* $\mathrm{Aut}(B(X \times X')) \to \mathrm{Aut}(B(\mathcal{N} \times \mathcal{N}'))$.

Proof. Recall that the map $\mathrm{Aut}(BX) \to \mathrm{Aut}(B\mathcal{N})$ was described in Section 2. To see (1) first note that

$$\mathrm{map}(BX \times BX', BX \times BX') \simeq \mathrm{map}(BX, \mathrm{map}(BX', BX)) \times \mathrm{map}(BX', \mathrm{map}(BX, BX')). \tag{4.1}$$

The evaluation map $\mathrm{map}(BX', BX)_{\mathrm{const}} \to BX$ is an equivalence by the Sullivan conjecture for p-compact groups [52, Thm. 9.3 and Prop. 10.1], (and likewise with X and X' switched). Since the component of the identity map on the left hand side of (4.1) lands in the component of the constant map in $\mathrm{map}(BX', BX)$ this shows that $\mathrm{map}(BX \times BX', BX \times BX')_1 \simeq \mathrm{map}(BX, BX)_1 \times \mathrm{map}(BX', BX')_1$ as wanted. (The statement just says that the center of a product of p-compact groups is the product of the centers, which of course also follows from the equivalence of the different definitions of the center from [52].)

To see (2) suppose that φ is a self-equivalence of $BX \times BX'$ such that its restriction to a self-equivalence of $B(\mathcal{N} \times \mathcal{N}')$ becomes homotopic to the identity. The restriction $\varphi_{|BX\times *}$ composed with the projection onto BX' becomes null homotopic upon restriction to $B\mathcal{N}$, which by e.g., [89, Thm. 6.1] implies that it is null homotopic. Likewise the projection of $\varphi_{|*\times BX'}$ onto BX' becomes homotopic to the identity map upon restriction to $B\mathcal{N}$, which by assumption means that the projection of $\varphi_{*\times BX'}$ onto BX' is the identity. But by adjointness, repeating the argument of the first claim, this implies that φ composed with the projection onto BX' is homotopic to the projection map onto BX' (this is [52, Lem. 5.3]). By symmetry this holds for the projection onto BX as well, and we conclude that φ is homotopic to the identity as wanted.

Finally, combining Propositions 3.2 and 3.4 gives (3), since $\pi_0(B\mathcal{N}) = \mathrm{Out}(\breve{\mathcal{N}})$ by Proposition 3.1. □

Remark 4.2. Part (3) of the above lemma is in general false for $p = 2$. For instance if $X = \mathrm{SO}(3)\hat{_2}$ then both for $Y = X$ and $Y = X \times X$ we have $\pi_0(\mathrm{Aut}(BY)) \cong N_{\mathrm{GL}(L_Y)}(W_Y)/W_Y$. But for $Y = X \times X$ we have $H^1(W_Y; \breve{T}_Y) \cong \mathbf{Z}/2 \times \mathbf{Z}/2$, so $BN_Y(T)$ has non-trivial automorphisms which restrict to the identity on BT (see Proposition 3.2).

Recall the observation that for p odd the component group of X is determined by W_X:

Lemma 4.3. *Let X be a p-compact group for p odd, with maximal torus normalizer $j : \mathcal{N} \to X$. By definition $W_X = \pi_0(\mathcal{N})$. The map $\pi_0(j) : W_X = \pi_0(\mathcal{N}) \to \pi_0(X)$ is surjective. The kernel is $O^p(W_X)$, the subgroup generated by elements of order prime to p. It can also be identified with the Weyl group of the connected component X_1 of X, and is the largest $\mathbf{Z}_p$-reflection subgroup of W_X.*

Proof. By [52, Rem. 2.11] $\pi_0(j)$ is surjective with kernel the Weyl group of the identity component of X. Since $\pi_0(X)$ is a p-group, $O^p(\pi_0(\mathcal{N}))$ is contained in the kernel. On the other hand, since p is odd, the Weyl group of X_1 is generated by elements of order prime to p, since it is a $\mathbf{Z}_p$-reflection group, so equality has to hold. □

Remark 4.4. For $p = 2$ the component group of X cannot be read off from $\mathcal{N}_X$, and one would have to remember $\pi_0(X)$ as part of the data. For instance the 2-compact groups $\mathrm{SO}(3)\hat{_2}$ and $\mathrm{O}(2)\hat{_2}$ have the same maximal torus normalizers, namely $\mathrm{O}(2)\hat{_2}$. Note however that if X is the centralizer of a toral abelian subgroup A of a connected p-compact group Y, then the component group of X *can* be read off from A and $\mathcal{N}_Y$ (see [52, Thm. 7.6]); a case of frequent interest.

Before proceeding recall that by [45] (see also [52, Prop. 11.9]) we have, for a fibration $\mathcal{F} \to \mathcal{E} \xrightarrow{f} \mathcal{B}$, a fibration sequence

$$\mathrm{map}(\mathcal{B}, B\,\mathrm{Aut}(\mathcal{F}))_{C(f)} \to B\,\mathrm{Aut}(f) \to B\,\mathrm{Aut}(\mathcal{B}).$$

Here $C(f)$ denotes the components corresponding to the orbit of the $\pi_0(\mathrm{Aut}(\mathcal{B}))$-action on the class in $[\mathcal{B}, B\,\mathrm{Aut}(\mathcal{F})]$ classifying the fibration.

We are interested in when the map of grouplike topological monoids $\mathrm{Aut}(f) \to \mathrm{Aut}(\mathcal{E})$ is a homotopy equivalence. This will follow if we can see that $\mathrm{Aut}_1(f) \to \mathrm{Aut}_1(\mathcal{E})$ and $\pi_0(\mathrm{Aut}(f)) \to \pi_0(\mathrm{Aut}(\mathcal{E}))$ are equivalences. By an easy general argument given as [52, Prop. 11.10] the statement about identity components follows if $\mathcal{B} \to \mathrm{map}(\mathcal{F}, \mathcal{B})_0$ is an equivalence, there the subscript 0 denotes the component of the trivial map.

Lemma 4.5 (Behavior with respect to components)**.** *Let X be a p-compact group with maximal torus normalizer $\mathcal{N}$, and assume that p is odd (so that $\pi_0(X)$ can be read off from $\mathcal{N}$). Let $\mathcal{N}_1$ denote the kernel of the map $\mathcal{N} \to \pi_0(X)$, which is a maximal torus normalizer for X_1.*

If $\operatorname{Aut}(BX_1) \xrightarrow{\cong} \operatorname{Aut}(B\mathcal{N}_1)$*, then* $\operatorname{Aut}(BX) \xrightarrow{\cong} \operatorname{Aut}(B\mathcal{N})$*. If furthermore BX_1 is determined by $B\mathcal{N}_1$ then BX is determined by $B\mathcal{N}$.*

Proof. First note that by an inspection of Euler characteristics and using [92, Thm. 1.2(3)], $\mathcal{N}_1$ is indeed a maximal torus normalizer in X_1. Set $\pi = \pi_0(X)$ for short. We want to apply the setup described before the lemma to the fibrations $BX_1 \to BX \to B\pi$ and $B\mathcal{N}_1 \to B\mathcal{N} \to B\pi$ and to see that in both cases the map of monoids $\operatorname{Aut}(f) \to \operatorname{Aut}(\mathcal{E})$ are homotopy equivalences. By the remarks above this follows if it is an isomorphism on π_0 and that $\mathcal{B} \to \operatorname{map}(\mathcal{F}, \mathcal{B})_0$ is an equivalence. The statement about π_0 is true in both cases since a self-map of $\mathcal{E}$ determines a unique self-map of $B\pi$. Likewise it is easy to see that $B\pi \xrightarrow{\simeq} \operatorname{map}(BX_1, B\pi)_0$ and that $B\pi \xrightarrow{\simeq} \operatorname{map}(B\mathcal{N}_1, B\pi)_0$. This means that our map $B\operatorname{Aut}(BX) \to B\operatorname{Aut}(B\mathcal{N})$ (from Lemma 2.1) fits in a map of fibration sequences

$$\begin{array}{ccccc} \operatorname{map}(B\pi, B\operatorname{Aut}(BX_1))_{C(f)} & \longrightarrow & B\operatorname{Aut}(BX) & \longrightarrow & B\operatorname{Aut}(B\pi) \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{map}(B\pi, B\operatorname{Aut}(B\mathcal{N}_1))_{C(f)} & \longrightarrow & B\operatorname{Aut}(B\mathcal{N}) & \longrightarrow & B\operatorname{Aut}(B\pi). \end{array}$$

Here the maps between the fibers and base spaces are homotopy equivalences by assumption, and we conclude that we get a homotopy equivalence between the total spaces as well.

Now assume furthermore that X_1 is determined by $\mathcal{N}_1$, and let X' be another p-compact group with maximal torus normalizer $\mathcal{N}$. By Lemma 4.3 we get that $\pi = \pi_0(X) \cong \pi_0(X')$ and that $\mathcal{N}_1$ is also a maximal torus normalizer in X'_1.

We want to show that the two fibrations $BX \to B\pi$ and $BX' \to B\pi$ are equivalent as fibrations over $B\pi$, or equivalently that the π-spaces BX_1 and BX'_1 are $h\pi$-equivalent, i.e., that we can find a zig-zag of π-maps which are non-equivariant equivalences connecting the two (see e.g., [40] where this equivalence relation is called equivariant weak homotopy equivalence).

By the assumptions on X_1 we can choose a homotopy equivalence $Bf : BX_1 \to BX'_1$ such that

$$\begin{array}{ccc} & B\mathcal{N}_1 & \\ {}^{Bj}\swarrow & & \searrow^{Bj'} \\ BX_1 & \xrightarrow{\quad Bf \quad} & BX'_1 \end{array}$$

commutes up to homotopy, and Bf is unique up to homotopy.

We now want to see that we can change Bf so that it becomes a π-map. For this, consider the restriction π-map

$$\operatorname{map}(BX_1, BX'_1) \to \operatorname{map}(B\mathcal{N}_1, BX'_1).$$

By the assumptions on $\operatorname{Aut}(BX_1)$ this map sends distinct components of $\operatorname{map}(BX_1, BX'_1)$ corresponding to homotopy equivalences to distinct components of $\operatorname{map}(B\mathcal{N}_1, BX'_1)$. Moreover, by the proof of Lemma 2.1, we have a homotopy equivalence $\operatorname{map}(BX_1, BX'_1)_{Bf} \simeq \operatorname{map}(B\mathcal{N}_1, BX'_1)_{Bf \circ Bj}$. In particular the component $\operatorname{map}(BX_1, BX'_1)_{Bf}$ is preserved under the π-action, since this obviously is so for $\operatorname{map}(B\mathcal{N}_1, BX'_1)_{Bj'}$. Furthermore since

$\mathrm{map}(B\mathcal{N}_1, BX_1')^{\pi}_{Bj'}$ contains Bj' we see that $\mathrm{map}(BX_1, BX_1')^{h\pi}_{Bf} \simeq \mathrm{map}(B\mathcal{N}_1, BX_1')^{h\pi}_{Bj'}$ is non-empty, and so there exists a π-map $E\pi \times BX_1 \to BX_1'$ which is a homotopy equivalence. This shows that BX_1 and BX_1' are $h\pi$-homotopy equivalent as wanted. □

Remark 4.6. If X is a connected p-compact group, and p is odd, then it follows from [52, Thm. 7.5] that $Z(\breve{\mathcal{N}})$ is a discrete approximation to the center of X. The proof of the above lemma extends this to X non-connected *provided* we know that the self-maps of X_1 are detected by their restriction to $\mathcal{N}_1$, which will be a consequence of Theorem 1.4. Having to appeal to this is a bit unfortunate but seems unavoidable. The point is that if there for a connected p-compact group X, existed a self-equivalence σ of X of finite p-power order and not detected by $\mathcal{N}$, then we could form $X \rtimes \langle\sigma\rangle$, where σ would be central in the normalizer but not in the whole group. (See also Lemma 10.2.)

Lemma 4.7 (Behavior with respect to centers)**.** *Let X be a connected p-compact group with center $\mathcal{Z}$.*

1. *If $\pi_0(\mathrm{Aut}(BX/\mathcal{Z})) \to \pi_0(\mathrm{Aut}(B\mathcal{N}/\mathcal{Z}))$ is surjective and $X/\mathcal{Z}$ is determined by $\mathcal{N}/\mathcal{Z}$ then X is determined by $\mathcal{N}$.*
2. *If p is odd and* $\mathrm{Aut}(BX/\mathcal{Z}) \to \mathrm{Aut}(B\mathcal{N}/\mathcal{Z})$ *is a homotopy equivalence then* $\mathrm{Aut}(BX) \to \mathrm{Aut}(B\mathcal{N})$ *is as well.*

Proof. Suppose that X and X' have the same maximal torus normalizer $\mathcal{N}$ and choose fixed inclusions $j : \mathcal{N} \to X$ and $j' : \mathcal{N} \to X'$. By [52, Thm. 7.6] X' and X have the same center $\mathcal{Z}$. Suppose that $X/\mathcal{Z}$ is isomorphic to $X'/\mathcal{Z}$. If $\pi_0(\mathrm{Aut}(BX/\mathcal{Z})) \to \pi_0(\mathrm{Aut}(B\mathcal{N}/\mathcal{Z}))$ is surjective we furthermore have that we can choose the homotopy equivalence $BX/\mathcal{Z} \to BX'/\mathcal{Z}$ in such a way that

$$\begin{array}{ccc} & B\mathcal{N}/\mathcal{Z} & \\ {\scriptstyle j/\mathcal{Z}}\swarrow & & \searrow{\scriptstyle j'/\mathcal{Z}} \\ BX/\mathcal{Z} & \xrightarrow{\qquad\qquad} & BX'/\mathcal{Z} \end{array}$$

commutes up to homotopy.

We have canonical maps $BX \to B^2\mathcal{Z}$ and $BX' \to B^2\mathcal{Z}$ classifying the extensions, and we claim that in fact the bottom triangle in the diagram

$$\begin{array}{ccc} & B\mathcal{N}/\mathcal{Z} & \\ \swarrow & & \searrow \\ BX/\mathcal{Z} & \xrightarrow{\qquad\qquad} & BX'/\mathcal{Z} \\ \searrow & & \swarrow \\ & B^2\mathcal{Z} & \end{array}$$

commutes up to homotopy. By construction the outer square commutes up to homotopy (since both composites agree with the classifying map $B\mathcal{N}/\mathcal{Z} \to B^2\mathcal{Z}$ since j and j' are fixed). Since the top triangle also commutes up to homotopy, an application of the transfer [51, 9.12], using that $B^2\mathcal{Z}$ is a product of Eilenberg-Mac Lane spaces and that $\chi((X/\mathcal{Z})/(\mathcal{N}/\mathcal{Z})) = 1$, shows that the bottom triangle commutes up to homotopy as well. Since we have constructed a map $BX/\mathcal{Z} \to BX'/\mathcal{Z}$ over $B^2\mathcal{Z}$ we get an induced homotopy equivalence $BX \to BX'$. (Note that this construction does not a priori give this map as a map under $B\mathcal{N}$.)

We now want to get the second statement about automorphism groups. Consider the homotopy commutative diagram

$$\begin{array}{ccc} B\mathcal{N} & \xrightarrow{f'} & B\mathcal{N}/\mathcal{Z} \\ \downarrow & & \downarrow \\ BX & \xrightarrow{f} & BX/\mathcal{Z} \end{array}$$

where we can suppose that the two horizontal maps f' and f are fibrations.

We first claim that we can replace $B\operatorname{Aut}(f)$ with $B\operatorname{Aut}(BX)$ and $B\operatorname{Aut}(f')$ with $B\operatorname{Aut}(B\mathcal{N})$. As in the case of the component group (see the proof of Lemma 4.5) we just have to justify that in the appropriate fibration sequences we have equivalences $\mathcal{B} \to \operatorname{map}(\mathcal{F}, \mathcal{B})_0$ and $\pi_0(\operatorname{Aut}(f)) \to \pi_0(\operatorname{Aut}(\mathcal{E}))$. The map $BX/\mathcal{Z} \to \operatorname{map}(B\mathcal{Z}, BX/\mathcal{Z})_0$ is a homotopy equivalence since the trivial map is central [52, Prop. 10.1]. That $B\mathcal{N}/\mathcal{Z} \to \operatorname{map}(B\mathcal{Z}, B\mathcal{N}/\mathcal{Z})_0$ is an equivalence is a similar (but easier) argument.

By Lemma 2.1 a self-equivalence of BX induces a unique self-equivalence of $B\mathcal{N}$, and hence a canonical self-equivalence of $B\mathcal{Z}$. Now, by the description of $X/\mathcal{Z}$ as a Borel construction (given in [51, Pf. of Prop. 8.3]) we get a canonical self-equivalence of $BX/\mathcal{Z}$. This self-equivalence is furthermore unique, in the sense that given a diagram

$$\begin{array}{ccc} BX & \xrightarrow{g} & BX \\ \downarrow & & \downarrow \\ BX/\mathcal{Z} & \xrightarrow{g'} & BX/\mathcal{Z} \end{array}$$

the homotopy type of g' is uniquely given by that of g. To see this note that by Lemma 2.1 the diagram restricts to a unique diagram

$$\begin{array}{ccc} B\mathcal{N} & \xrightarrow{\tilde{g}} & B\mathcal{N} \\ \downarrow & & \downarrow \\ B\mathcal{N}/\mathcal{Z} & \xrightarrow{\tilde{g}'} & B\mathcal{N}/\mathcal{Z}. \end{array}$$

By looking at discrete approximations we see that the homotopy class of $\tilde{g}'$ is determined by $\tilde{g}$. Since by assumption the homotopy class of g' is determined by $\tilde{g}'$, we conclude that a self-equivalence of BX induces a unique self-equivalence of $BX/\mathcal{Z}$, and so $\pi_0(\operatorname{Aut}(f)) \cong \pi_0(\operatorname{Aut}(BX))$. The last part of the argument furthermore shows that also $\pi_0(\operatorname{Aut}(f')) \cong \pi_0(\operatorname{Aut}(B\mathcal{N}))$.

We hence have the following diagram where the rows are fibration sequences

$$\begin{array}{ccccc} \operatorname{map}(BX/\mathcal{Z}, B\operatorname{Aut}(B\mathcal{Z}))_{C(f)} & \longrightarrow & B\operatorname{Aut}(BX) & \longrightarrow & B\operatorname{Aut}(BX/\mathcal{Z}) \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{map}(B\mathcal{N}/\mathcal{Z}, B\operatorname{Aut}(B\mathcal{Z}))_{C(f')} & \longrightarrow & B\operatorname{Aut}(B\mathcal{N}) & \longrightarrow & B\operatorname{Aut}(B\mathcal{N}/\mathcal{Z}). \end{array}$$

Examining when the middle vertical arrow is a homotopy equivalence reduces to finding out when the restriction map $\mathrm{map}(BX/\mathcal{Z}, B\,\mathrm{Aut}(B\mathcal{Z}))_{C(f)} \to \mathrm{map}(B\mathcal{N}/\mathcal{Z}, B\,\mathrm{Aut}(B\mathcal{Z}))_{C(f')}$ is a homotopy equivalence, which we now analyze.

Note that since $B\mathcal{Z}$ is an product of Eilenberg-Mac Lane spaces we have a fibration sequence

$$B^2\mathcal{Z} \to B\,\mathrm{Aut}(B\mathcal{Z}) \to B\,\mathrm{Aut}(\breve{\mathcal{Z}})$$

where $\breve{\mathcal{Z}}$ is the discrete approximation to $\mathcal{Z}$ and $\mathrm{Aut}(\breve{\mathcal{Z}})$ is the discrete group of automorphisms. Since our extensions are central this gives a diagram of fibration sequences

$$\begin{array}{ccccc} \mathrm{map}(BX/\mathcal{Z}, B^2\mathcal{Z})_{C(f)} & \longrightarrow & \mathrm{map}(BX/\mathcal{Z}, B\,\mathrm{Aut}(B\mathcal{Z}))_{C(f)} & \longrightarrow & \mathrm{map}(BX/\mathcal{Z}, B\,\mathrm{Aut}(\breve{\mathcal{Z}}))_0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{map}(B\mathcal{N}/\mathcal{Z}, B^2\mathcal{Z})_{C(f')} & \longrightarrow & \mathrm{map}(B\mathcal{N}/\mathcal{Z}, B\,\mathrm{Aut}(B\mathcal{Z}))_{C(f')} & \longrightarrow & \mathrm{map}(B\mathcal{N}/\mathcal{Z}, B\,\mathrm{Aut}(\breve{\mathcal{Z}}))_0. \end{array}$$

Again, in this diagram the map between the base spaces is obviously an equivalence, so we are reduced to studying

$$\mathrm{map}(BX/\mathcal{Z}, B^2\mathcal{Z})_{C(f)} \to \mathrm{map}(B\mathcal{N}/\mathcal{Z}, B^2\mathcal{Z})_{C(f')}. \tag{4.2}$$

Since $B^2\mathcal{Z}$ is a product of Eilenberg-Mac Lane spaces a transfer argument (cf. [51, 9.12]) shows that this gives an embedding as a retract. Since we assume $\pi_0(\mathrm{Aut}(BX/\mathcal{Z})) \cong \pi_0(B\,\mathrm{Aut}(B\mathcal{N}/\mathcal{Z}))$ we furthermore get that this is an isomorphism on π_0 by the definition of $C(f)$ and $C(f')$. Now set $X' = X/\mathcal{Z}$ and $\mathcal{N}' = \mathcal{N}/\mathcal{Z}$ and let (W, L') denote the Weyl group of X'. Write $B\mathcal{Z} \simeq B^2A \times BA'$, where A is torsion free and A' is finite (see [52, Thm. 1.1]). On π_1 the map (4.2) identifies with

$$H^1(BX'; A') \oplus H^2(BX'; A) \to H^1(B\mathcal{N}'; A') \oplus H^2(B\mathcal{N}'; A).$$

The group $H^1(B\mathcal{N}'; A')$ is zero since $\pi_1(B\mathcal{N}') = W_X$ is generated by elements of order prime to p, since p is assumed to be odd.

Furthermore, $H^2(B\mathcal{N}'; A)$ is related via the Serre spectral sequence to the groups

$$H^2(BW; H^0(B^2L'; A)), H^1(BW; H^1(B^2L'; A)), \text{ and } H^0(BW; H^2(B^2L'; A)).$$

The first of these groups is zero since W is generated by elements prime to p by the assumption that p is odd. The second is obviously zero, and the last group is zero since $H^0(W; \mathrm{Hom}(L', \mathbf{Z}_p)) = \mathrm{Hom}((L')_W, \mathbf{Z}_p) = 0$ because $(L')_W$ is finite.

Hence we get an isomorphism on π_1, since we already know that the map is injective. On π_2 and π_3 the map identifies with $H^0(BX'; A') \oplus H^1(BX'; A) \to H^0(B\mathcal{N}'; A') \oplus H^1(B\mathcal{N}'; A)$ and $H^0(BX'; A) \to H^0(B\mathcal{N}'; A)$ respectively, and these maps are obviously isomorphisms. Hence $\mathrm{map}(BX/\mathcal{Z}, B^2\mathcal{Z})_{C(f)} \to \mathrm{map}(B\mathcal{N}/\mathcal{Z}, B^2\mathcal{Z})_{C(f')}$ is a homotopy equivalence, which via the fibration sequences above imply that $B\,\mathrm{Aut}(BX) \to B\,\mathrm{Aut}(B\mathcal{N})$ is a homotopy equivalence as wanted. □

Remark 4.8. Consider $BX = B(\mathrm{SO}(3) \times S^1)\hat{_2}$. This has center $\mathcal{Z} = (S^1)\hat{_2}$ and $X/\mathcal{Z} = \mathrm{SO}(3)\hat{_2}$. It is easy to calculate directly that $B\,\mathrm{Aut}(BX/\mathcal{Z}) \xrightarrow{\simeq} B\,\mathrm{Aut}(B\mathcal{N}/\mathcal{Z})$ (or appeal to [76]). However $\pi_0(\mathrm{Aut}(BX)) \to \pi_0(\mathrm{Aut}(B\mathcal{N}))$ is not onto by Proposition 3.2, since $\mathrm{Hom}(W_{\mathrm{SO}(3)}, \mathbf{Z}/2^\infty) = \mathbf{Z}/2$. This shows that the p odd assumption is necessary in the last part of the above lemma. (Compare also Remark 4.2.)

Remark 4.9. Suppose that X is a connected p-compact group. Fibration sequences with base space $B^2\pi_1(X)$ and fiber $B(X\langle 1\rangle)$ are in one-to-one correspondence with the set of maps $[B^2\pi_1(X), B\operatorname{Aut}(B(X\langle 1\rangle))]$. Likewise self-equivalences of BX can be expressed in terms of self-equivalences of $B(X\langle 1\rangle)$ and $\pi_1(X)$, analogously to the lemmas above. Hence if we a priori knew that Theorem 1.7 held true, i.e., if we could read off $\pi_1(X)$ from $\mathcal{N}_X$ then the above methods would reduce the proof of the main theorems to the simply connected case, which could be used advantageously in the proofs. (See also Remark 7.3.)

Remark 4.10. The assumption in Lemma 4.7(1) that $\pi_0(\operatorname{Aut}(BX/\mathcal{Z})) \to \pi_0(\operatorname{Aut}(B\mathcal{N}/\mathcal{Z}))$ is surjective has the following origin. We have a canonical restriction map $H^2(BX/\mathcal{Z}; \breve{\mathcal{Z}}) \to H^2(B\mathcal{N}/\mathcal{Z}; \breve{\mathcal{Z}})$, which is injective by a transfer argument. Two extension classes in $H^2(BX/\mathcal{Z}; \breve{\mathcal{Z}})$ give rise to isomorphic total spaces if the extension classes are conjugate via the actions of $\operatorname{Aut}(BX/\mathcal{Z})$ and $\operatorname{Aut}(\breve{\mathcal{Z}})$ on $H^2(BX/\mathcal{Z}; \breve{\mathcal{Z}})$. The total spaces have isomorphic maximal torus normalizers if the extension classes have images in $H^2(B\mathcal{N}/Z; \breve{\mathcal{Z}})$ which are conjugate under the actions of $\operatorname{Aut}(B\mathcal{N}/\mathcal{Z})$ and $\operatorname{Aut}(\breve{\mathcal{Z}})$, which could a priori be a weaker notion.

5. An integral version of a theorem of Nakajima and realization of p-compact groups

The goal of this section is to prove an integral version of an algebraic result of Nakajima (Theorem 5.1) and use this to prove a Theorem 5.3 which, as part of our induction proof of Theorem 1.1, will allow us to construct the center-free p-compact groups corresponding to $\mathbf{Z}_p$-reflection groups (W, L) such that $\mathbf{Z}_p[L]^W$ is a polynomial algebra. This will provide the existence part of Theorem 1.1. We feel that this way of showing existence, is perhaps more straightforward than previous approaches; compare for instance [103]. (We refer to the introduction for the history behind this result.)

Theorem 5.1. *Let p be an odd prime and let (W, L) be a finite $\mathbf{Z}_p$-reflection group. For a subspace V of $L \otimes \mathbf{F}_p$ we let W_V denote the pointwise stabilizer of V in W. Then the following conditions are equivalent:*

(1) $\mathbf{Z}_p[L]^W$ *is a polynomial algebra.*
(2) $\mathbf{Z}_p[L]^{W_V}$ *is a polynomial algebra for all non-trivial subspaces $V \subseteq L \otimes \mathbf{F}_p$.*
(3) (W_V, L) *is a $\mathbf{Z}_p$-reflection group for all non-trivial subspaces $V \subseteq L \otimes \mathbf{F}_p$.*

Remark 5.2. An analog of the implication (1) ⇒ (2) where the ring $\mathbf{Z}_p$ is replaced by a field was proven by Nakajima [95, Lem. 1.4] (in the case of finite fields see also [56] and [97]). For fields of positive characteristic the implication (3) ⇒ (1) does not hold; see [80] for more information about this case. Our proof unfortunately involves the classification of finite $\mathbf{Z}_p$-reflection groups and some case-by-case checking. (See the discussion following the proof of Theorem 1.8 for related information.)

Proof of Theorem 5.1. To start, note that the implication (2) ⇒ (3) follows from the fact that if $\mathbf{Z}_p[L]^{W_V}$ is a polynomial algebra then $\mathbf{Q}_p[L \otimes \mathbf{Q}]^{W_V}$ is as well, so (W_V, L) is a $\mathbf{Z}_p$-reflection group by the Shephard-Todd-Chevalley theorem ([7, Thm. 7.2.1] or [120, Thm. 7.4.1]).

To go further we want to see that the theorem is well behaved under products, i.e., that if $(W, L) = (W', L') \times (W'', L'')$, then the theorem holds for (W, L) if it holds for (W', L') and (W'', L''). This follow from the fact that the stabilizer in $W' \times W''$ of an arbitrary subgroup in $(L' \otimes \mathbf{F}_p) \oplus (L'' \otimes \mathbf{F}_p)$ equals the stabilizer of the smallest product subgroup containing it, combined with the fact that the tensor product of two algebras is a polynomial algebra if and only if each of the factors are. Hence to prove the remaining implications it follows from Theorem 11.1 that it suffices to consider separately the cases where (W, L) comes from a compact connected Lie group and the cases where (W, L) is one of the exotic $\mathbf{Z}_p$-reflection groups.

Assume first that $(W, L) = (W_G, L_G \otimes \mathbf{Z}_p)$ for a compact connected Lie group G. If $\mathbf{Z}_p[L]^W$ is a polynomial algebra then by Theorem 12.2 (which involves case-by-case considerations and p odd) $BX = BG\hat{_p}$ satisfies $H^*(BX; \mathbf{Z}_p) \cong H^*(B^2L; \mathbf{Z}_p)^W$. We can identify $V \subseteq L \otimes \mathbf{F}_p$ with a toral elementary abelian p-subgroup in X and by [56, Rem. 1.3] $H^*(BC_X(V); \mathbf{Z}_p)$ is again a polynomial algebra concentrated in even degrees. In particular $C_X(V)$ is connected and by [52, Thm. 7.6] $W_{C_X(V)} = W_V$. Hence, by Theorem 12.1, $H^*(BC_X(V); \mathbf{Z}_p) \cong H^*(B^2L; \mathbf{Z}_p)^{W_V}$, so $\mathbf{Z}_p[L]^{W_V}$ is a polynomial algebra. This shows that (1) $\Rightarrow$ (2) when (W, L) comes from a compact connected Lie group. To prove (3) $\Rightarrow$ (1) for Lie groups suppose that (W, L) is a finite $\mathbf{Z}_p$-reflection group corresponding to a p-compact group $X = G\hat{_p}$ such that (W_V, L) is a $\mathbf{Z}_p$-reflection group for all non-trivial $V \subseteq L \otimes \mathbf{F}_p$. Since p is odd it follows by [52, Thm. 7.6] that $C_X(V)$ is connected for all non-trivial $V \subseteq L \otimes \mathbf{F}_p$. Hence, since X is assumed to come from a compact connected Lie group [8, Thm. B] (or [125, Thm. 2.28]) implies that $H^*(BX; \mathbf{Z}_p)$ does not have p-torsion and hence $H^*(BX; \mathbf{Z}_p) \cong H^*(B^2L; \mathbf{Z}_p)^W$ (cf. Theorem 12.1). So $\mathbf{Z}_p[L]^W$ is a polynomial algebra as wanted.

Next we assume that (W, L) is one of the exotic $\mathbf{Z}_p$-reflection groups. By Theorem 12.2, $\mathbf{Z}_p[L]^W$ is a polynomial algebra, so we only need to prove that $\mathbf{Z}_p[L]^{W_V}$ is a polynomial algebra for any non-trivial $V \subseteq L \otimes \mathbf{F}_p$. Furthermore, by Theorem 12.2(2), $\mathbf{F}_p[L \otimes \mathbf{F}_p]^W$ is a polynomial algebra. Nakajima's result [95, Lem. 1.4] shows that $\mathbf{F}_p[L \otimes \mathbf{F}_p]^{W_V}$ is a polynomial algebra as well. Thus we are done if $p \nmid |W_V|$ by Lemma 12.6, which in particular covers the cases where $p \nmid |W|$.

If (W, L) belongs to family number 2 on the Clark-Ewing list, then since p is odd, it is easily seen from the form of the representing matrices (see Section 11 for a concrete description) that reduction mod p gives a bijection between reflections in (W, L) and $(W, L \otimes \mathbf{F}_p)$. As $\mathbf{F}_p[L \otimes \mathbf{F}_p]^{W_V}$ is a polynomial algebra it follows by the Shephard-Todd-Chevalley theorem [7, Thm. 7.2.1] that $W_V \subseteq \mathrm{GL}(L \otimes \mathbf{F}_p)$ is a reflection group. Thus (W_V, L) is a $\mathbf{Z}_p$-reflection group. Since the representing matrices are monomial, it follows by [95, Thm. 2.4] that $\mathbf{Z}_p[L]^{W_V}$ is a polynomial algebra.

By Theorem 11.1 only four cases remain, namely the Zabrodsky-Aguadé cases $(W_{12}, p = 3)$, $(W_{29}, p = 5)$, $(W_{31}, p = 5)$ and $(W_{34}, p = 7)$. For each of these a direct computation (for instance easily done with the aid of a computer) shows that if S is a Sylow p-subgroup of W, then $U = (L \otimes \mathbf{F}_p)^S$ is 1-dimensional and (W_U, L) is isomorphic to $(\Sigma_p, L_{\mathrm{SU}(p)} \otimes \mathbf{Z}_p)$ (the construction of such a subgroup U can also be found in Aguadé [4]). Hence we see that if $V \subseteq L \otimes \mathbf{F}_p$ is non-trivial then either $p \nmid |W_V|$ or V is W-conjugate to U. But in these cases we already know that $\mathbf{Z}_p[L]^{W_V}$ is a polynomial algebra. $\square$

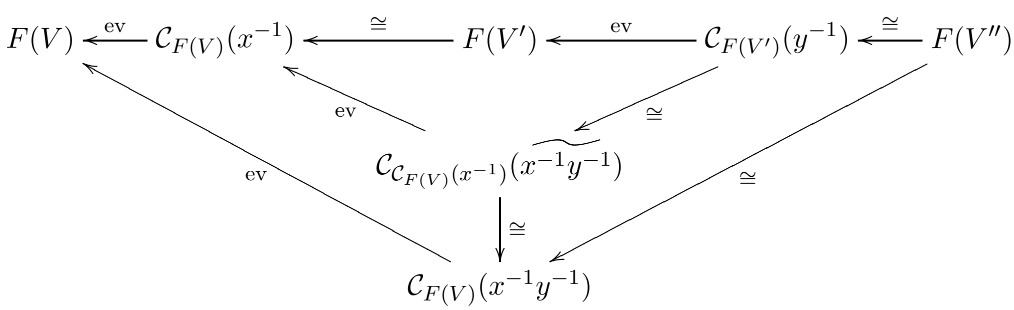
Theorem 5.3. *Let p be an odd prime and let (W, L) be a finite $\mathbf{Z}_p$-reflection group with the property that $\mathbf{Z}_p[L]^W$ is a polynomial algebra over $\mathbf{Z}_p$.*

Assume that for all non-trivial elementary abelian p-subgroups $V \subseteq \breve{T} = L \otimes \mathbf{Z}/p^\infty$ there exists a p-compact group $F(V)$ with discrete approximation to its maximal torus normalizer given by $\breve{T} \rtimes W_V$ such that $F(V)$ is determined by $\mathcal{N}_{F(V)}$, $\Phi : \mathrm{Aut}(BF(V)) \xrightarrow{\cong} \mathrm{Aut}(B\mathcal{N}_{F(V)})$, *and* $H^*(BF(V);\mathbf{Z}_p) \xrightarrow{\cong} H^*(B^2L;\mathbf{Z}_p)^{W_V}$. *Then there exists a* p*-compact group* X *with discrete approximation to its maximal torus normalizer given by* $\breve{T}\rtimes W$ *satisfying the same properties as listed for* $F(V)$.

Proof. First recall that by Theorem 5.1 (W_V, L) is again a $\mathbf{Z}_p$-reflection group and $\mathbf{Z}_p[L]^{W_V}$ is a polynomial algebra. Set $\breve{\mathcal{N}} = \breve{T}\rtimes W$. We want to construct a candidate 'centralizer decomposition' diagram. Let $\mathbf{A}$ be the category with objects the non-trivial elementary abelian p-subgroups V of $\breve{T}$ and morphisms the homomorphisms between them induced by inclusions of subgroups and conjugation by elements in W. We now define a functor F from $\mathbf{A}^{\mathrm{op}}$ to p-compact groups and conjugation classes of morphisms. On objects we send V to $F(V)$. By assumption $j_V : \mathcal{C}_{\breve{\mathcal{N}}}(V) \to F(V)$ is a discrete approximation to the maximal torus normalizer in $F(V)$. Now let $\varphi : V \to V'$ be a morphism in $\mathbf{A}$, induced by conjugation by an element $x \in W$ and consider the diagram

$$\begin{array}{ccccc} V' & \hookrightarrow & \mathcal{C}_{\breve{\mathcal{N}}}(V') & \xrightarrow{c_{x^{-1}}} & \mathcal{C}_{\breve{\mathcal{N}}}(V) \\ & & \downarrow{\scriptstyle j_{V'}} & & \downarrow{\scriptstyle j_V} \\ & & F(V') & & F(V) \end{array}$$

Taking the centralizer of the composite map $x^{-1} : V' \to F(V)$ we get a space $\mathcal{C}_{F(V)}(x^{-1}) = \Omega\,\mathrm{map}(BV', BF(V))_{Bx^{-1}}$, which has discrete approximation to its maximal torus normalizer equal to $C_{\breve{\mathcal{N}}}(V')$. By assumption we get a unique (up to conjugacy) isomorphism $F(V') \to \mathcal{C}_{F(V)}(x^{-1})$ under $C_{\breve{\mathcal{N}}}(V')$. By composing with the evaluation $\mathcal{C}_{F(V)}(x^{-1}) \to F(V)$, we get a morphism $F(\varphi) : F(V') \to F(V)$. We need to check that this gives us a well defined functor from $\mathbf{A}^{\mathrm{op}}$ to the homotopy category of spaces, i.e., that for $V \xrightarrow{\varphi} V' \xrightarrow{\psi} V''$, $F(\psi\varphi)$ is conjugate to $F(\varphi)F(\psi)$. To see this suppose that ψ is induced by conjugation by $y \in W$ and consider the following diagram with obvious maps



(Here $\widetilde{(\cdot)}$ denotes the adjoint map which is explained in Remark 6.3.) Note that the bottom composite from $F(V'')$ to $F(V)$ is $F(\psi\varphi)$ and the top composite is $F(\varphi)F(\psi)$. The top triangle is commutative, since the lower isomorphism in that triangle is just the map obtained by taking centralizers of the upper one. The rightmost square is homotopy commutative, since the corresponding square of isomorphisms between centralizers in $\breve{\mathcal{N}}$ is commutative, using our assumptions that maps are detected here. Finally, the leftmost square is homotopy commutative, by definition of the adjoint construction.

We hence get a well defined functor $BF : \mathbf{A}^{\mathrm{op}} \to Ho(\mathrm{Spaces})$, where $Ho(\mathrm{Spaces})$ denotes the homotopy category of spaces, on objects given by $V \mapsto BF(V)$. By construction the functor obtained when taking cohomology of this diagram, can be identified with the canonical functor which on objects is given by $V \mapsto H^*(B\breve{T}; \mathbf{Z}_p)^{W_V}$.

We want to lift this to a diagram in the category of spaces. The obstruction theory for doing this is described in [44, Thm. 1.1], noting that by [52, Lem. 11.15] our diagram is a so-called centric diagram so the assumptions of that theorem are satisfied.

By looking at their cohomology we see that all the spaces $F(V)$ are connected and hence by [52, Thm. 7.5] have center given by $\breve{T}^{W_V}$, since p is odd. In particular (see e.g. Lemma 10.2) the homotopy groups of $\mathcal{Z}F(V)$ are given by $\pi_0(\mathcal{Z}F(V)) = H^1(W_V; L)$ and $\pi_1(\mathcal{Z}F(V)) = L^{W_V}$. By [50, §8] (for details see Section 10) $\lim^*_{V \in \mathbf{A}} \pi_*(F(-)) = 0$, so by [44, Thm. 1.1] there exists a (unique) lift of our functor BF to a functor $\widetilde{BF}$ landing in Spaces. Set $BX = (\operatorname{hocolim}_{\mathbf{A}} \widetilde{BF})^\wedge_p$.

The spectral sequence for calculating the cohomology of a homotopy colimit [19, Ch. XII 4.5] has E_2-term given by $E_2^{i,j} = \lim^i_{V \in \mathbf{A}} H^j(B\breve{T}; \mathbf{Z}_p)^{W_V}$. But again by [50, §8] these groups vanish for $i > 0$ and for $i = 0$ give $\lim^0_{\mathbf{A}} H^*(B\breve{T}; \mathbf{Z}_p)^{W_V} \cong H^*(B\breve{T}; \mathbf{Z}_p)^W$. Hence the spectral sequence collapses onto the vertical axis, and we get $H^*(BX; \mathbf{Z}_p) \cong H^*(B\breve{T}; \mathbf{Z}_p)^W$.

Since $H^*(BX; \mathbf{Z}_p)$ is a polynomial algebra $H^*(X; \mathbf{Z}_p)$ will be an exterior algebra on odd generators (cf. Theorem 12.1), so X is indeed a p-compact group. The fact that BX is determined by $\mathcal{N}$ and satisfies $\Phi : \operatorname{Aut}(BX) \xrightarrow{\cong} \operatorname{Aut}(B\mathcal{N})$, also follows easily from the above—the details are given in the proof of Lemma 6.4. □

Remark 5.4. Note that Theorem 5.3 in itself does not quite give a stand-alone proof of the realization and uniqueness of all center-free p-compact groups with Weyl group satisfying that $\mathbf{Z}_p[L]^W$ is a polynomial algebra, since (W_V, L) is not center-free which prevents the obvious induction from working; the main problem is the unitary groups.

6. Proof of the main theorem using Sections 8, 9, 10, 11, and 12

The purpose of this section is to prove the main Theorems 1.1 and 1.4, but in the proofs referring forward to Sections 8 and 9 for information about elementary abelian p-subgroups of the simple center-free Lie groups and to Section 10 for the obstruction group calculations.

We start by explaining the strategy in general terms. Recall that the *centralizer* of an elementary abelian p-subgroup $\nu : E \to X$ of a p-compact group X is defined as the p-compact group $\mathcal{C}_X(\nu)$ with classifying space $B\mathcal{C}_X(\nu) = \operatorname{map}(BE, BX)_{B\nu}$. It is a theorem of Dwyer-Wilkerson [51, Prop. 5.1 and 5.2] that this actually is a p-compact group and that the evaluation map to X is a monomorphism. A theorem of Dwyer-Zabrodsky [41] [78, Thm. 3.2] says that if G is a compact Lie group with component group a p-group, then the map

$$BC_G(\nu(E))^\wedge_p \to B\mathcal{C}_{G^\wedge_p}(\nu) = \operatorname{map}(BE, BG^\wedge_p)_{B\nu}$$

induced by the adjoint of the canonical homomorphism $E \times C_G(\nu(E)) \to G$ is a homotopy equivalence. Note however that $\mathcal{C}_X(\nu)$ is not naturally a subobject of X, i.e., the map to X is defined in terms of ν, unlike in the Lie case.

For a p-compact group X, let $\mathbf{A}(X)$ denote the Quillen category of X. The objects of $\mathbf{A}(X)$ are conjugacy classes of monomorphisms $\nu \colon E \to X$ of non-trivial elementary abelian p-subgroups E into X. The morphisms $(\nu : E \to X) \to (\nu' : E' \to X)$ of $\mathbf{A}(X)$ consists of all group homomorphisms $\varphi \colon E \to E'$ such that ν and $\nu'\varphi$ are conjugate.

The centralizer construction gives a functor

$$B\mathcal{C}_X : \mathbf{A}(X)^{\mathrm{op}} \to \text{Spaces} \tag{6.1}$$

that takes the monomorphism $(\nu : E \to X) \in \mathrm{Ob}(\mathbf{A}(X))$ to its centralizer $B\mathcal{C}_X(\nu) = \mathrm{map}(BE, BX)_{B\nu}$ and a morphism φ to composition with $B\varphi : BE \to BE'$.

By a theorem of Dwyer-Wilkerson [52, §8], generalizing a theorem for compact Lie groups by Jackowski-McClure [75], the evaluation map

$$\mathrm{hocolim}_{\mathbf{A}(X)}\, B\mathcal{C}_X \to BX$$

induces an isomorphism on mod p homology. If X is connected and center-free, then for all ν, the centralizer $\mathcal{C}_X(\nu)$ is a p-compact group with smaller cohomological dimension setting the stage for a proof by induction. (The *cohomological dimension* of a p-compact group Y is defined as $\mathrm{cd}(Y) = \max\{n | H^n(Y; \mathbf{F}_p) \neq 0\}$; see [51, 6.13] and [53, 3.8].) To make use of this we need a way to construct a map from the elementary abelian p-subgroups and their centralizers in X, to any other p-compact group X' with the same maximal torus normalizer $\mathcal{N}$.

Suppose that $\mathcal{N}$ is embedded in connected p-compact groups X and X' via homomorphisms j and j' respectively. If $\nu : E \to X$ can be factored through a maximal torus $i : T \to X$, i.e., if there exists $\mu : E \to T$ such that $i\mu = \nu$, then μ is unique up to conjugation as a map to $\mathcal{N}$ by [53, Prop. 3.4], and furthermore by [52, Thm. 7.6], $\mathcal{C}_{\mathcal{N}}(\mu)$ is a maximal torus normalizer in $\mathcal{C}_X(\nu)$. In this case $j'\mu$ will be an elementary abelian p-subgroup of X', which we have assigned without making any choices, and $\mathcal{C}_{X'}(j'\mu)$ will have maximal torus normalizer $\mathcal{C}_{\mathcal{N}}(\mu)$. Elementary abelian p-subgroups which can be conjugated into T are called *toral* subgroups, and elementary abelian p-subgroups which do not have this property are called *non-toral* subgroups. The problem hence arises how to compare the centralizers in the case of non-toral elementary abelian p-subgroups. This problem was addressed by the third-named author in [92]:

Theorem 6.1. [92] *Let X be a p-compact group with maximal torus normalizer $\mathcal{N}$.*

If $\nu : E \to X$ is an elementary abelian p-subgroup of X, then there exists a lift $\mu : E \to \mathcal{N}$ with $\nu = j\mu$ such that $\mathcal{C}_{\mathcal{N}}(\mu) \to \mathcal{C}_X(\nu)$ is a maximal torus normalizer. Furthermore, if $E' \subseteq E$ and μ' is a lift of $\nu_{|E'}$ with the above property, then μ' can be extended to a lift μ of E which also has this property.

Assume now that X is connected. If $\nu : E \to X$ has rank one then ν is toral [51, Prop. 5.5.] *and the above lift μ is unique up to conjugation in $\mathcal{N}$. If ν has rank two, then the lift is unique if ν is toral and if ν is non-toral then there are precisely $p+1$ different lifts with the further property that $\pi_0(\mu) : E \to \pi_0(\mathcal{N})$ is not injective, corresponding to the $p+1$ rank one subgroups of E.* □

Remark 6.2. The analogous theorem of the above in the classical case of compact Lie groups does not seem to appear in the literature. However, as was explained by J.-P. Serre [115], this can be obtained by a modification of the proof of [123, Thm. II.5.16]—strictly speaking, we shall only need this theorem in the cases where either X is the p-completion of a compact Lie group, or where the mod p cohomology of BX is a polynomial algebra, the latter case being trivial.

If X is assumed connected then we can always arrange that the lift in Theorem 6.1 furthermore satisfies that the kernel of the map $\pi_0(\mu) : E \to \pi_0(\mathcal{N})$ is non-trivial, and such a map will be called a *preferred lift*. Note that if $\nu : E \to X$ can be factored through T then the corresponding map $\mu : E \to \mathcal{N}$ is a preferred lift and it is furthermore unique up to conjugacy in $\mathcal{N}$, i.e., the preferred lift is exactly the factorization through the maximal torus described earlier (see also [92, Prop. 4.10]).

Before proceeding with our discussion, let us recall the construction of adjoint maps, since these play a central role in what follows.

Remark 6.3 (Adjoint maps)**.** Let A be an abelian p-compact group, X a p-compact group, and $\nu : A \to X$ be a homomorphism. Suppose that E is a subgroup of A and note that we have a canonical map

$$B\psi : BA \times BE \xrightarrow{\text{mult}} BA \to BX$$

which only depends on the conjugacy class of ν. Since furthermore

$$\pi_0(\text{map}(BA \times BE, BX)) = \coprod_{\xi \in [BE,BX]} \pi_0(\text{map}(BA, \text{map}(BE, BX)_\xi))$$

we get that every element $\nu : A \to X$ gives rise to an element $\tilde{\nu} : A \to \mathcal{C}_X(\nu_{|E})$ making the diagram

$$\begin{array}{ccc} & & \mathcal{C}_X(\nu_{|E}) \\ & \overset{\tilde{\nu}}{\nearrow} & \downarrow \text{ev} \\ A & \xrightarrow{\nu} & X \end{array}$$

commutative. Here $\tilde{\nu}$ is well defined up to conjugacy, from the conjugacy class of ν. We will always use the notation $\widetilde{(\cdot)}$ for this construction.

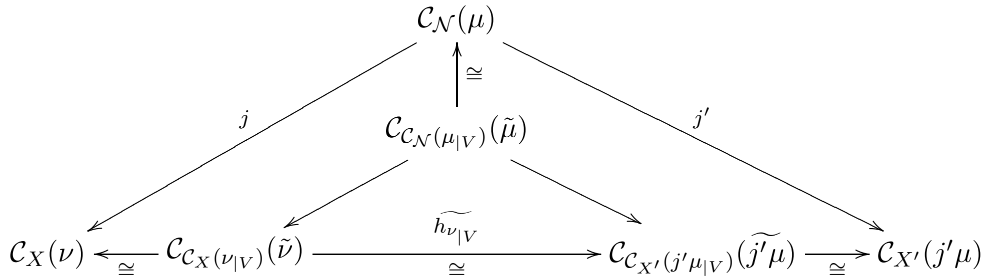
We now want to show how to use Theorem 6.1 to construct a map between the centralizer diagrams of connected p-compact groups X and X'.

Suppose that $\nu : V \to X$ has rank one and that $\mathcal{C}_X(\nu)$ is determined by $\mathcal{N}_{\mathcal{C}_X(\nu)}$ and satisfies $\Phi : \text{Aut}(B\mathcal{C}_X(\nu)) \xrightarrow{\cong} \text{Aut}(B\mathcal{N}_{\mathcal{C}_X(\nu)})$. Since ν has rank one and X is connected the unique preferred lift μ will in fact factor through T by [51, Prop. 5.5], and μ is a preferred lift of $j'\mu$ by [52, Thm. 7.6]. Hence $C_{\mathcal{N}}(\mu)$ is a maximal torus normalizer for both $C_X(\nu)$ and $C_{X'}(j'\mu)$, so by assumption there exists a homomorphism h_ν, unique up to conjugacy, making the diagram

(6.2)
$$\begin{array}{ccccc} & & \mathcal{C}_{\mathcal{N}}(\mu) & & \\ & \overset{j}{\swarrow} & & \overset{j'}{\searrow} & \\ \mathcal{C}_X(\nu) & & \xrightarrow[\cong]{h_\nu} & & \mathcal{C}_{X'}(j'\mu) \end{array}$$

commute. We can use the rank one case to get a map in general: For an arbitrary elementary abelian p-subgroup $\nu : E \to X$ with preferred lift $\mu : E \to \mathcal{N}$ we can by Theorem 6.1 choose a rank one subgroup V in the kernel of $E \to \pi_0(\mathcal{N})$. Hence $\mu_{|V}$ factors through T and we are in the rank one situation described above, i.e., we get a diagram as in (6.2) but with ν everywhere replaced by $\nu_{|V}$ and μ replaced by $\mu_{|V}$. The adjoint maps of ν, μ, and $j'\mu$ (see Remark 6.3) map into this diagram in a coherent way, which expresses $j'\mu$ in terms of ν and the rank one subgroup V, so it only depends on those parameters. Taking further adjoints with respect to ν, μ and $j'\mu$ produces a commutative diagram

(6.3)



where $\tilde{h}_{\nu_{|V}}$ is the map induced from $h_{\nu_{|V}}$ on the centralizers. But this means that $\mathcal{C}_{\mathcal{N}}(\mu)$ is a maximal torus normalizer also in $\mathcal{C}_X(j'\mu)$, e.g., by the characterizing property for maximal torus normalizers given in [92, Thm. 1.2(3)], so in particular μ is a preferred lift of $j'\mu$. Denote the bottom left-to-right composite in the above diagram by $h_{\nu,V}$, and note that the composition $E \xrightarrow{\tilde{\nu}} C_X(\nu) \xrightarrow{h_{\nu,V}} C_{X'}(j\mu) \to X'$ equals $j'\mu$, which we, by a slight abuse of notation, we will denote $h_V(\nu)$. We would like to see that $h_{V,\nu}$ does in fact not depend on the choice of V since this will allow us to construct a map in the homotopy category from the centralizer diagram of X to X'. Likewise we want see that this diagram can be rigidified to a diagram in the category of spaces, so as to get an induced map from the homotopy colimit of the centralizer diagram. The next lemma states precisely what needs to be checked—the calculations to verify that these conditions are indeed verified for all p-compact groups is essentially the contents of the rest of the paper.

Lemma 6.4. *Let X and X' be two connected p-compact groups with the same maximal torus normalizer $\mathcal{N}$ embedded via j and j' respectively. Assume that for all rank one elementary abelian p-subgroups $\nu : E \to X$ of X the centralizer $\mathcal{C}_X(\nu)$ is determined by $\mathcal{N}_{\mathcal{C}_X(\nu)}$ and that $\Phi : \mathrm{Aut}(B\mathcal{C}_X(\nu)) \xrightarrow{\cong} \mathrm{Aut}(B\mathcal{N}_{\mathcal{C}_X(\nu)})$ when ν is of rank one or two.*

1. *Assume that for every rank two non-toral subgroup $\nu : E \to X$ with a preferred lift μ, both the conjugacy class of $j'\mu : E \to X'$ and the conjugacy class of the induced map $h_{\nu,V} : C_X(\nu) \to C_{X'}(j'\mu)$ described above are independent of the choice of the subgroup V of E. Then there exists a map in the homotopy category of spaces from the centralizer diagram of BX to BX' (seen as a constant diagram), i.e., an element in $\lim^0_{\nu\in\mathbf{A}(X)}[B\mathcal{C}_X(\nu), BX']$, given via the maps $h_{\nu,V}$ described above.*
2. *Assume furthermore $\lim^i_{\nu\in\mathbf{A}(X)} \pi_j(B\mathcal{Z}\mathcal{C}_X(\nu)) = 0$ for $j = 1, 2$ and $i = j, j+1$, then there is a lift of this element in $\lim^0$ to a map in the (diagram) category of spaces. This produces an isomorphism $f : X \to X'$ under $\mathcal{N}$, unique up to conjugacy, and $\Phi : \mathrm{Aut}(BX) \xrightarrow{\cong} \mathrm{Aut}(B\mathcal{N})$.*

Proof. As explained before the proof there are no choices involved for rank one subgroups. If $\nu : E \to X$ has rank two and is toral then $h_{\nu,V}$ does not depend on the choice of V by diagram (6.3) since μ is uniquely determined from ν in this case, and we are assuming that Φ is an isomorphism for centralizers of rank two subgroups. If ν has rank two and is non-toral, then we are just simply assuming that $h_{\nu,V}$ does not depend on V.

We want to see that the rank two condition forces this to hold in general. Let V_1 and V_2 be two different rank one subgroups of E, and set $U = V_1 \oplus V_2$. Since $h_{\nu_{|U},V}$ does not depend on the choice of V the following diagram commutes up to conjugation

$$\begin{array}{ccccccc}
 & & \mathcal{C}_X(\nu_{|V_1}) & \xrightarrow{h_{\nu_{|V_1}}} & \mathcal{C}_{X'}(h(\nu_{|V_1})) & & \\
 & \nearrow^{\tilde{\nu}} & \uparrow & & \uparrow & \searrow & \\
E & \xrightarrow{\tilde{\nu}} & \mathcal{C}_X(\nu_{|U}) & \xrightarrow{h_{\nu_{|U}}} & \mathcal{C}_{X'}(h(\nu_{|U})) & \longrightarrow & X' \\
 & \searrow_{\tilde{\nu}} & \downarrow & & \downarrow & \nearrow & \\
 & & \mathcal{C}_X(\nu_{|V_2}) & \xrightarrow{h_{\nu_{|V_2}}} & \mathcal{C}_{X'}(h(\nu_{|V_2})) & &
\end{array}$$

But this shows that in general $h_{\nu,V}$ does not depend on the choice of V, since for any two choices of rank one subgroups V_1 and V_2 the above diagram shows that both maps will be centralizers of the common $h_{\nu_{|U}}$.

It is now easy to construct a map from the centralizer diagram of X into X', by assigning to ν the homomorphism $\mathcal{C}_X(\nu) \xrightarrow{h_\nu} \mathcal{C}_{X'}(h(\nu)) \to X'$. We have to check that this is really a natural transformation, i.e., that we get commutative diagrams on morphisms. Since all morphisms in $\mathbf{A}(X)$ are the composite of an isomorphism followed by an inclusion, it is enough to check the claim on these. Suppose $\varphi : (\nu : E \to X) \to (\nu' : E' \to X)$ is an isomorphism. If ν has rank one then by assumption $\mathcal{C}_X(\nu)$ has the same automorphisms as its maximal torus normalizer so the diagram

$$\begin{array}{ccc}
\mathcal{C}_X(\nu') & \xrightarrow[h_{\nu'}]{\cong} & \mathcal{C}_{X'}(j'\mu') \\
{\scriptstyle \mathcal{C}_X(\varphi)}\downarrow\cong & & \cong\downarrow{\scriptstyle \mathcal{C}_{X'}(\varphi)} \\
\mathcal{C}_X(\nu'\varphi) & \xrightarrow[\cong]{h_{\nu'\varphi}} & \mathcal{C}_{X'}(j'\mu\varphi)
\end{array}$$

commutes up to conjugation, since we can view the diagram of isomorphisms as taking place under $\mathcal{C}_{\mathcal{N}}(\mu') \to \mathcal{C}_{\mathcal{N}}(\mu'\varphi)$, for a preferred lift μ' of ν'. For a general isomorphism, if we restrict to a rank one subgroup V of E and its image $\varphi(V)$ in E', then the diagram above is, by construction of $h_{\nu'}$ and $h_{\nu'\varphi}$, just obtained by taking centralizers of the adjoint maps of $\nu'\varphi$ and ν' of the diagram in rank one case, and it hence has to commute up to conjugation as well. If φ is an inclusion of a subgroup then above diagram will also commute up to conjugation, by the same adjointness argument. This shows that for all maps φ in $\mathbf{A}(X)$ the diagram

$$\begin{array}{ccc}
\mathcal{C}_X(\nu') & \xrightarrow{\mathcal{C}_X(\varphi)} & \mathcal{C}_X(\nu) \\
 & \searrow \quad \swarrow & \\
 & X' &
\end{array}$$

commutes up to conjugation, where the diagonal maps are the natural composites $\mathcal{C}_X(\nu) \xrightarrow{h_\nu} \mathcal{C}_{X'}(h_\nu(\nu)) \to X'$. So we have constructed a map, of diagrams in the homotopy category, from the centralizer diagram of X to X' (seen as a constant diagram), or in other words we have given an element

$$[\rho] \in \lim_{\nu \in \mathbf{A}(X)}{}^{0} \pi_0(\mathrm{map}(B\mathcal{C}_X(\nu), BX'))$$

By [54, Rem. after Def. 6.3] [52, 11.15] (which says that the centralizer diagram of a p-compact group is "centric") we have that

$$\mathrm{map}(B\mathcal{C}_{X'}(h_\nu(\nu)), B\mathcal{C}_{X'}(h_\nu(\nu)))_1 \xrightarrow{\simeq} \mathrm{map}(B\mathcal{C}_{X'}(h_\nu(\nu)), BX')_e \xrightarrow{\simeq} \mathrm{map}(B\mathcal{C}_X(\nu), BX')_{eh_\nu}$$

where the first map is composition with the evaluation map $e : \mathcal{C}_{X'}(h_\nu(\nu)) \to X'$ and the second is precomposition with h_ν.

Likewise, by the homotopy equivalences established earlier in the proof

$$\mathrm{map}(B\mathcal{C}_{X'}(h_\nu(\nu)), B\mathcal{C}_{X'}(h_\nu(\nu)))_1 \simeq \mathrm{map}(B\mathcal{C}_X(\nu), B\mathcal{C}_X(\nu))_1$$

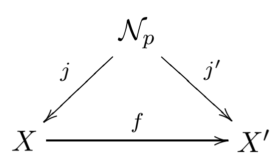
which by the definition of the center [52] equals $B\mathcal{Z}\mathcal{C}_X(\nu)$. Since these identifications are natural, this gives a canonical identification of the functor $\nu \mapsto \pi_i(\mathrm{map}(B\mathcal{C}_X(\nu), BX')_{[\rho]})$ with $\nu \mapsto \pi_i(B\mathcal{Z}\mathcal{C}_X(\nu))$.

By obstruction theory (see [135, Prop. 3] [77, Prop. 1.4]) the existence obstructions for lifting this to an element in $\pi_0(\mathrm{holim}_{\mathbf{A}(X)} \mathrm{map}(B\mathcal{C}_X(\nu), X')) \cong \pi_0(\mathrm{map}(BX, BX'))$ lie in

$$\lim_{\nu \in \mathbf{A}(X)}{}^{i+1} \pi_i(\mathrm{map}(B\mathcal{C}_X(\nu), BX')_{[\rho]}) \cong \lim_{\nu \in \mathbf{A}(X)}{}^{i+1} \pi_i(B\mathcal{Z}\mathcal{C}_X(\nu)),\ i \geq 1$$

But by assumption all these groups are identically zero, so our data lifts to a map $Bf : BX \to BX'$.

We now want to see that the construction of f forces it to be an isomorphism. Let $\mathcal{N}_p$ denote a p-normalizer of T, i.e., the union of components in $\mathcal{N}$ corresponding to a Sylow p-subgroup of W. Since $\mathcal{N}_p$ has non-trivial center (since the action of a p-group on $\breve{T}$ has to have a fixed point), we can find a rank one central subgroup $\mu : V \to \mathcal{N}_p$, and so we can view $\mathcal{N}_p$ as sitting inside $\mathcal{C}_{\mathcal{N}}(\mu)$. Hence by construction the diagram



commutes up to conjugation, and in particular $fj : \mathcal{N}_p \to X'$ is a monomorphism. This easily implies that f is a monomorphism as well: We have that $H^*(B\mathcal{N}_p; \mathbf{F}_p)$ is finitely generated over $H^*(BX'; \mathbf{F}_p)$ via $H^*(Bfj; \mathbf{F}_p)$ by [51, Prop. 9.11]. By an application of the transfer [51, Thm. 9.13] the map $H^*(Bj; \mathbf{F}_p) : H^*(BX; \mathbf{F}_p) \to H^*(B\mathcal{N}_p; \mathbf{F}_p)$ is a monomorphism, and since $H^*(BX'; \mathbf{F}_p)$ is noetherian by [51, Thm. 2.3] we conclude that $H^*(BX; \mathbf{F}_p)$ is finitely generated over $H^*(BX'; \mathbf{F}_p)$ as well, by the definition of noetherian. Hence $f : X \to X'$ is a monomorphism by another application of [51, Prop. 9.11]. Since we can identify the maximal tori of X' and X, the definition of the Weyl group defines a map between the Weyl groups $W_X \to W_{X'}$, which has to be injective since the Weyl groups act faithfully on T (by [51, Thm. 9.7]). But since we know that X and X' have the same maximal torus normalizer, the above map of Weyl groups has to be an isomorphism. By [52, Thm. 4.7] (or [93, Prop. 3.7] using [51, Thm. 9.7]) this means that f is indeed an isomorphism.

We now want to argue that this f is a map under $\mathcal{N}$. By Lemma 2.1 we know that there exists a $g \in \mathrm{Aut}(B\mathcal{N})$, unique up to conjugation, such that

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{g} & \mathcal{N} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j'} \\ X & \xrightarrow{f} & X' \end{array}$$

commutes up to conjugation. By covering space theory and Sylow's theorem we can restrict g to a self-map of g' making the diagram

$$\begin{array}{ccc} \mathcal{N}_p & \xrightarrow{g'} & \mathcal{N}_p \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j'} \\ X & \xrightarrow{f} & X' \end{array}$$

commute. Furthermore any other map $\mathcal{N}_p \to \mathcal{N}_p$ fitting in this diagram will be conjugate to g' in $\mathcal{N}$, by the proof of Lemma 2.1. However by construction f is a map under $\mathcal{N}_p$ so g' is conjugate in $\mathcal{N}$ to the canonical inclusion. But by Proposition 3.1 and 3.2 we get that if an outer automorphism of $\mathcal{N}$ restricts to the identity on $\mathcal{N}_p$ then it is in fact the identity. This shows that $g = 1$, i.e., that f is a map under $\mathcal{N}$. This also shows that $\pi_0(\mathrm{Aut}(BX)) \to \pi_0(\mathrm{Aut}(B\mathcal{N}))$ is surjective, since for any automorphism $g : \mathcal{N} \to \mathcal{N}$, jg is also a maximal torus normalizer in X by [92, Thm. 1.2(3)].

Note that if $\mathrm{Aut}_1(B\mathcal{N})$ is not contractible then we can find a rank one elementary abelian p-subgroup $\nu : V \to T$ such that $\mathcal{C}_{\mathcal{N}}(\nu) \xrightarrow{\cong} \mathcal{N}$ which by assumption means that $\Phi : \mathrm{Aut}(BX) \xrightarrow{\cong} \mathrm{Aut}(B\mathcal{N})$. So we can assume that $\mathrm{Aut}_1(B\mathcal{N})$ is contractible in which case $\mathrm{Aut}_1(BX)$ is as well by [52, Thm. 1.3 and Thm. 7.5].

The only remaining claim in the lemma is that the map $\pi_0(\Phi) : \pi_0(\mathrm{Aut}(BX)) \to \pi_0(\mathrm{Aut}(B\mathcal{N}))$ is injective under the additional assumption that $\lim^i_{\nu \in \mathbf{A}(X)} \pi_i(B\mathcal{Z}\mathcal{C}_X(\nu)) = 0$, $i \geq 1$. In other words we have to see that any self-equivalence f of X which, up to conjugacy, induces the identity on $\mathcal{N}$ is in fact conjugate to the identity. But if we examine the above argument with $X' = X$, the map on centralizers of rank one elements induced by f has to be the identity by the rank one uniqueness assumption. The maps for higher rank are centralizers of maps of rank one, so they as well have to be the identity. Hence f maps to the same element as the identity in $\lim^0_{\nu \in \mathbf{A}(X)} \pi_0(\mathrm{map}(B\mathcal{C}_X(\nu), BX))$, which means that f actually is the identity by the vanishing of the obstruction groups (again, see [135, Prop. 4] or [77, Prop. 1.4]). □

Remark 6.5. Note how the assumptions of the lemma fail (as they should) for the group SO(3) at the prime 2 which is not determined by its normalizer. In this case the element $\mathrm{diag}(-1, -1, 1)$ in the maximal torus $\mathrm{SO}(2) \times 1$ is fixed under the Weyl group action and has centralizer equal to the maximal torus normalizer O(2).

Recall that the Weyl group $W(\nu)$ of an elementary abelian p-subgroup $\nu : E \to X$ is defined as the subgroup of $\mathrm{Aut}(E)$ consisting of the elements α such that $\nu\alpha$ is homotopic to ν.

Remark 6.6. It is perhaps natural to ask whether the preferred lift of rank two elementary abelian p-subgroups is unique in the "subgroup of Lie group sense", i.e., whether the Weyl group of $\nu : E \to X$ acts transitively on the rank one subgroups of E. The lists in Section 8 and 9 reveals that this is *false* for the subgroup $E_{E_6}^{2a}$ of the group E_6 but true for all other center-free simple compact connected Lie groups at odd primes.

Lemma 6.7. *Let X and X' be two connected p-compact groups with the same maximal torus normalizer $\mathcal{N}$ embedded via j and j' respectively. Assume that for all rank one elementary abelian p-subgroups $\eta : E \to X$ of X the centralizer $\mathcal{C}_X(\eta)$ is determined by $\mathcal{N}_{\mathcal{C}_X(\eta)}$ and that $\Phi : \mathrm{Aut}(B\mathcal{C}_X(\eta)) \xrightarrow{\cong} \mathrm{Aut}(B\mathcal{N}_{\mathcal{C}_X(\eta)})$ when η is of rank one and two.*

Let $\nu : E \to X$ be a rank two elementary abelian p-subgroup of X and assume that:

1. *The Weyl group $W(\nu)$ of ν contains $\mathrm{SL}(E)$.*
2. *ν is up to isomorphism the only (non-toral) elementary abelian p-subgroup whose centralizer is isomorphic to $\mathcal{C}_X(\nu)$.*

Then the map $j'\mu : E \to X'$ constructed by picking a preferred lift $\mu : E \to \mathcal{N}$ of ν does not depend on the choice of μ (corresponding to the $p+1$ rank one subgroups of E; see Theorem 6.1 and Lemma 6.4).

Proof. Let $\{\nu\}$ denote the orbit of ν in $[BE, BX]$ under the natural $\mathrm{GL}(E)$-action and note that the size of the set $\{\nu\}$ equals the index of $W(\nu)$ in $\mathrm{GL}(E)$. Let $\{\mu\}$ denote the set of the preferred lifts in $[BE, B\mathcal{N}]$ which lifts elements in $\{\nu\}$. Since any ν has $p+1$ preferred lifts the map of $\mathrm{GL}(E)$-sets $\{\mu\} \to \{\nu\}$ induced by j is surjective and $(p+1)$-to-1.

By assumption $\mathrm{SL}(E) \subseteq W(\nu)$ so the Weyl group $W(\nu)$ acts transitively on the rank one subgroups of E. Hence if μ is a preferred lift of ν, any other preferred lift of μ can be obtained as $\mu\alpha$ for some $\alpha \in W(\nu)$. In other words also $\{\mu\}$ is a transitive $\mathrm{GL}(E)$-set.

We now want to argue that $\{\mu\}$ is also exactly the preferred lifts of the $\mathrm{GL}(E)$-orbit $\{j'\mu\}$ of $j'\mu$ where μ is any lift of ν. Namely, if μ' is any preferred lift of $j'\mu$ then by the first assumptions in the lemma (about centralizers of rank one and two subgroups) μ' is also a preferred lift of $j\mu'$ and $\mathcal{C}_X(j\mu')$ is isomorphic to $\mathcal{C}_{X'}(j'\mu)$ and $\mathcal{C}_X(\nu)$ (see the proof of Lemma 6.4). Hence by the assumption that there is only one object up to conjugacy in X with centralizer isomorphic to $\mathcal{C}_X(\nu)$ there exists $\alpha \in \mathrm{GL}(E)$ such that $\nu = j\mu'\alpha$. In other words $\mu' \in \{\mu\}$. So the surjective map of $\mathrm{GL}(E)$-sets $\{\mu\} \to \{j'\mu\}$ is $(p+1)$-to-1 as well.

Hence the set $\{j'\mu\}$ has the same cardinality as $\{\nu\}$. In particular the stabilizer of $j'\mu$ is a subgroup of $\mathrm{GL}(E)$ of index $[\mathrm{GL}(E) : W(\nu)]$. But since $\mathrm{SL}(E) \subseteq W(\nu)$ this means that this stabilizer equals $W(\nu)$, since $W(\nu)$ is the unique subgroup with this index (c.f. e.g. [74]). In particular $j'\mu = (j'\mu)\alpha = j'(\mu\alpha)$ and since $\mu\alpha$ exhausts the preferred lifts of ν, we conclude that $j'\mu$ does not depend on the choice of preferred lift. □

Lemma 6.8. *Let X and X' be two connected p-compact groups with the same maximal torus normalizer $\mathcal{N}$ embedded via j and j' respectively. Assume that for all rank one and two elementary abelian p-subgroups $\eta : E \to X$ of X the centralizer $\mathcal{C}_X(\eta)$ is determined by $\mathcal{N}_{\mathcal{C}_X(\eta)}$ and that $\Phi : \mathrm{Aut}(B\mathcal{C}_X(\eta)) \xrightarrow{\cong} \mathrm{Aut}(B\mathcal{N}_{\mathcal{C}_X(\eta)})$. Furthermore assume that for $\eta : E \to X$ of rank two, self-equivalences of $\mathcal{C}_X(\eta)_1$ are detected by its maximal torus (which is covered by the previous assumption if p is odd; cf. Proposition 3.2).*

Let $\nu : E \to X$ be a rank two non-toral elementary abelian p-subgroup of X. Assume the following:

1. *The Weyl group $W(\nu)$ of ν acts transitively on the rank one subgroups of E.*
2. *$E \times \mathcal{C}_X(\nu)_1 \xrightarrow{\cong} \mathcal{C}_X(\nu)$, where the identity component $\mathcal{C}_X(\nu)_1$ is a non-trivial p-compact group and $\alpha \in W(\nu)$ acts on $E \times \mathcal{C}_X(\nu)_1$ as $\alpha \times 1$.*

Then neither $j\mu$ nor the map $h_{\nu,V} : \mathcal{C}_X(\nu) \to \mathcal{C}_{X'}(j'\mu)$ of Lemma 6.4 depends on the choice of the rank one subgroup V of E (i.e., the assumptions of Lemma 6.4(1) are satisfied).

Proof. Fix a preferred lift μ of ν, and let T_E be the identity component of the space $\mathcal{C}_{\mathcal{N}}(\mu)$ (which will be a maximal torus in $\mathcal{C}_X(\nu)$ and by assumption non-trivial). Let V be the kernel of $\pi_0(\mu) : E \to \pi_0(\mathcal{N})$.

Since the map $\mu : E \to \mathcal{C}_{\mathcal{N}}(\mu)$ is central, we get an induced map $\bar{\mu} : E \times T_E = E \times \mathcal{C}_{\mathcal{N}}(\mu)_1 \to \mathcal{C}_{\mathcal{N}}(\mu) \to \mathcal{N}$.

The assumption about the action means that for $\alpha \in W(\nu)$ we have the following diagram which commutes up to conjugation:

$$\begin{array}{ccccccc} E \times T_E & \longrightarrow & \mathcal{C}_{\mathcal{N}}(\mu) & \longrightarrow & \mathcal{C}_X(j\mu) & & \\ \downarrow{\scriptstyle \alpha\times 1} & & & & \downarrow{\scriptstyle \alpha} & \searrow & \\ E \times T_E & \longrightarrow & \mathcal{C}_{\mathcal{N}}(\mu) & \longrightarrow & \mathcal{C}_X(j\mu) & \longrightarrow & X \end{array}$$

so $j\bar{\mu}(\alpha \times 1)$ is homotopic to $j\bar{\mu}$.

Let U be a rank one elementary abelian p-subgroup of T_E. By assumption we have a unique map $h_{\bar{\mu}_{|U}}$ making the following diagram commutative up to conjugation.

$$\begin{array}{ccccc} & & E \times T_E & & \\ & \swarrow{\scriptstyle \widetilde{j\bar{\mu}}} & \downarrow{\scriptstyle \widetilde{\bar{\mu}}} & \searrow{\scriptstyle \widetilde{j'\bar{\mu}}} & \\ & & \mathcal{C}_{\mathcal{N}}(\bar{\mu}_{|U}) & & \\ & \swarrow & & \searrow & \\ \mathcal{C}_X(j\bar{\mu}_{|U}) & & \xrightarrow[\cong]{h_{\bar{\mu}_{|U}}} & & \mathcal{C}_{X'}(j'\bar{\mu}_{|U}) \\ \downarrow & & & & \downarrow \\ X & & & & X' \end{array}$$

so $j'\mu(\alpha \times 1)$ is also homotopic to $j'\mu$. The same is hence true for the adjoint maps $E \times T_E \to \mathcal{C}_X(\nu)$ and $E \times T_E \to \mathcal{C}_{X'}(j'\mu)$.

$$\begin{array}{ccccccccc} & & \mathcal{C}_{\mathcal{N}}(\mu) & \xleftarrow{\widetilde{\bar{\mu}}} & E \times T_E & \xrightarrow{\widetilde{\bar{\mu}(\alpha\times 1)}} & \mathcal{C}_{\mathcal{N}}(\mu\alpha) & & \\ & \swarrow{\scriptstyle j} & & \searrow{\scriptstyle j'} & & \swarrow{\scriptstyle j'} & & \searrow{\scriptstyle j} & \\ \mathcal{C}_X(\nu) & & \xrightarrow[h_{\nu,V}]{\cong} & & \mathcal{C}_{X'}(j'\mu) & & \xleftarrow[h_{\nu,\alpha^{-1}(V)}]{\cong} & & \mathcal{C}_X(\nu) \end{array}$$

which is up to conjugation commutative by the above and where the two maps $E \times T_E \to \mathcal{C}_X(\nu)$ are identical.

Hence $h^{-1}_{\nu,\alpha^{-1}(V)} h_{\nu,V}$ is a self-equivalence of $\mathcal{C}_X(\nu) \cong E \times \mathcal{C}_X(\nu)_1$ which is the identity on $E \times T_E$. Since by assumption self-maps of $\mathcal{C}_X(\nu)_1$ are determined by their restriction to a maximal torus we hence conclude that $h^{-1}_{\nu,V} h_{\nu,\alpha^{-1}(V)}$ is indeed homotopic to the identity (cf. [52, Lem. 5.3]) and thus $h_{\nu,V}$ does not depend on the choice of subgroup V in E. □

The above two lemmas are in fact general enough to let us verify the conditions of Lemma 6.4 in all the situations we shall encounter, except the subgroup $E^{2a}_{E_6}$ of E_6 where we need a small variation of Lemma 6.8.

Lemma 6.9. *Let $\nu : E^{2a}_{E_6} \hookrightarrow X = (E_6)\hat{_3}$ be the rank two non-toral elementary abelian 3-subgroup of E_6 listed in Theorem 8.9. Assume that for all rank one elementary abelian 3-subgroups $\eta : V \to X$ of X the centralizer $\mathcal{C}_X(\eta)$ is determined by $\mathcal{N}_{\mathcal{C}_X(\eta)}$ and that $\Phi : \mathrm{Aut}(B\mathcal{C}_X(\eta)) \xrightarrow{\cong} \mathrm{Aut}(B\mathcal{N}_{\mathcal{C}_X(\eta)})$ when η is of rank one or two.*

Then the conjugacy class of $j'\mu : E^{2a}_{E_6} \to X'$ does not depend on the choice of preferred lift μ of ν and likewise the conjugacy class of $h_{\nu,V} : \mathcal{C}_X(\nu) \to \mathcal{C}_{X'}(j'\mu)$ does not depend on the choice of rank one subgroup V of $E^{2a}_{E_6}$.

Proof. The proof is almost identical to Lemma 6.8 but here we have to argue a bit more carefully since $W(\nu)$ does not act transitively on the rank one subgroups of $E = E^{2a}_{E_6}$.

Let $\mu_1, \ldots, \mu_4$ be the four preferred lifts. By Proposition 8.10 $T_E = \mathcal{C}_{\mathcal{N}}(\mu_i(E))_1$ is independent of the choice of μ. (Note that we are taking centralizers in the Lie group sense.) Set $\bar{\mu}_i : E \times T_E \xrightarrow{\mu_i \times 1} \mathcal{N}$. Also by Proposition 8.10 the conjugacy class of $j\bar{\mu}_i$ does not depend on μ_i. This brings us in exactly the same situation as in the proof of Lemma 6.8 (with the $\bar{\mu}(\alpha \times 1)$ replaced by the $\bar{\mu}_i$), which proves the lemma. □

Proof of Theorems 1.4, and 1.1 using Section 8, 9, 10, 11, and 12. We will show that Theorems 1.1 and 1.4 both hold by a simultaneous induction on the cohomological dimension of X. We will furthermore add to the induction hypothesis the statement that if X is connected and $\mathbf{Z}_p[L]^{W_X}$ is a polynomial ring, then $H^*(BX; \mathbf{Z}_p) \cong H^*(BT; \mathbf{Z}_p)^{W_X}$.

By Lemma 4.3 and 4.5, Theorem 1.4 holds for a p-compact group X if it holds for its identity component X_1, so we can assume that X is connected.

By [6, Thm. 1.2], if (W, L) is realized as the Weyl group of a p-compact group X, then $\mathcal{N}_X$ will be split, i.e., $\breve{\mathcal{N}}_X \cong \breve{T} \rtimes W_X$ (cf. also [128], [57], and [96] for the Lie case). So, to prove Theorems 1.4 and 1.1 we have to show that given any finite $\mathbf{Z}_p$-reflection group (W, L) there exists a unique connected p-compact group X realizing (W, L), with self-maps satisfying $\Phi : \mathrm{Aut}(BX) \xrightarrow{\cong} \mathrm{Aut}(B\mathcal{N}_X)$ (since this implies $\pi_0(\mathrm{Aut}(BX)) \xrightarrow{\cong} N_{\mathrm{GL}(L_X)}(W)/W$ by Proposition 3.1 and 3.2).

If (W, L) can be written as a non-trivial product, then there exists a p-compact group X realizing (W, L) by the induction hypothesis. The p-compact group is unique by [53, Thm. 1.4] and the induction hypothesis, i.e., X is determined by $\mathcal{N}_X$. Lemma 4.1 and the induction hypothesis show that X has the right space of automorphisms. Hence we can assume that (W, L) cannot be written as a product.

By the classification of finite $\mathbf{Z}_p$-reflection groups (Theorem 11.1), (W, L) is either of the form $(W_G, L_G \otimes \mathbf{Z}_p)$, for some compact connected Lie group G, or (W, L) is one of the exotic $\mathbf{Z}_p$-reflection groups.

In the first case (W, L) can of course be realized by $X = G\hat{_p}$. Note that, by Theorem 12.2 (and Theorem 12.1) $H^*(BX; \mathbf{Z}_p) \cong H^*(BT; \mathbf{Z}_p)^W$ if and only if $\mathbf{Z}_p[L]^W$ is a polynomial algebra.

In the latter case $\mathbf{Z}_p[L]^W$ is a polynomial algebra by Theorem 12.2 and (W, L) satisfies $\breve{T}^W = 0$ by Theorem 11.1. Hence the induction hypothesis shows that the assumptions of Theorem 5.3 are satisfied. So, by Theorem 5.3 there exists a unique p-compact group X with Weyl group (W, L) which is determined by $\mathcal{N}_X$, and satisfies that $\Phi : \mathrm{Aut}(BX) \xrightarrow{\cong} \mathrm{Aut}(B\mathcal{N}_X)$ and $H^*(BX; \mathbf{Z}_p) \cong H^*(BT; \mathbf{Z}_p)^{W_X}$.

We now want to show that X is uniquely determined by (W_X, L_X) and satisfies the properties about self-maps listed in Theorems 1.1 and 1.4. In fact, the statement about self-maps in Theorem 1.1 follows from the one in Theorem 1.4 by Proposition 3.2, so we can concentrate on Theorem 1.4.

By Lemma 4.7 we can assume that X is center-free. Likewise by the splitting theorem [53, Thm. 1.4 and 1.5] together with Lemma 4.1 we can assume that X is simple. By Theorem 12.2 (W, L) either has the property that $\mathbf{Z}_p[L]^W$ is a polynomial algebra, or (W, L) is one of the reflection groups $(W_{\mathrm{PU}(n)}, L_{\mathrm{PU}(n)} \otimes \mathbf{Z}_p)$ (with $p \,|\, n$), $(W_{E_8}, L_{E_8} \otimes \mathbf{Z}_5)$, $(W_{F_4}, L_{F_4} \otimes \mathbf{Z}_3)$, $(W_{E_6}, L_{E_6} \otimes \mathbf{Z}_3)$, $(W_{E_7}, L_{E_7} \otimes \mathbf{Z}_3)$, or $(W_{E_8}, L_{E_8} \otimes \mathbf{Z}_3)$.

If the $\mathbf{Z}_p$-reflection group (W, L) has the property that $\mathbf{Z}_p[L]^W$ is a polynomial algebra, then we have just seen that there exists a space X' such that $H^*(BX'; \mathbf{Z}_p) \cong H^*(B^2L; \mathbf{Z}_p)^W$. Hence all elementary abelian p-subgroups of X' are toral (cf. Lemma 7.8). In particular X' has no rank two non-toral elementary abelian p-subgroups so the assumptions of Lemma 6.4(1) (used with X and X' switched around) are satisfied. By Theorem 10.1 the assumptions of Lemma 6.4(2) also hold, and hence Lemma 6.4 implies that there exists an isomorphism of p-compact groups $X' \to X$, and that X satisfies the conclusion of Theorem 1.4.

Consider $(W, L) = (W_{\mathrm{PU}(n)}, L_{\mathrm{PU}(n)} \otimes \mathbf{Z}_p)$ where $p \mid n$. We divide up in two cases. If $n = pk$ with $k > 1$ then Theorem 9.1 shows that the assumptions of Lemma 6.8 are satisfied which implies that the assumptions of Lemma 6.4(1) are satisfied. By Theorem 10.1 the assumptions of Lemma 6.4(2) are also satisfied, so Lemma 6.4 produces an isomorphism of p-compact groups $\mathrm{PU}(n)\hat{_p} \to X$, and shows that X satisfies the conclusion of Theorem 1.4. If $n = p$ then we argue as above but replace the reference to Lemma 6.8 with a reference to Lemma 6.7, noting that the statement about $h_{\nu,V}$ in this case follows for free from that of $j'\mu$, since $E \cong C_{\mathrm{PU}(p)}(E)$.

If (W, L) equals $(W_{E_8}, L_{E_8} \otimes \mathbf{Z}_5)$, $(W_{F_4}, L_{F_4} \otimes \mathbf{Z}_3)$, $(W_{E_7}, L_{E_7} \otimes \mathbf{Z}_3)$, or $(W_{E_8}, L_{E_8} \otimes \mathbf{Z}_3)$ then by Theorem 8.2(3) the corresponding Lie group G does not have any non-toral rank 2 elementary abelian p-subgroups at the associated prime p. Furthermore the higher limits obstructions which feature in the assumptions for Lemma 6.4(2) vanish by Theorem 10.1. Hence Lemma 6.4 implies that there exists an equivalence of p-compact groups $G\hat{_p} \to X$, and that X satisfies the conclusion of Theorem 1.4.

Finally, if $(W, L) = (W_{E_6}, L_{E_6} \otimes \mathbf{Z}_3)$ then the rank 2 non-toral subgroup $E^{2b}_{E_6}$ of Theorem 8.9 satisfies the conditions of Lemma 6.8 so the assumptions of Lemma 6.4(1) are satisfied for this subgroup. Likewise for the subgroup $E^{2a}_{E_6}$ the custom made Lemma 6.9 shows that the assumptions of Lemma 6.4(1) are also satisfied for this subgroup. Since the assumptions of Lemma 6.4(2) are satisfied by Theorem 10.1 we conclude by Lemma 6.4 that X is homotopy equivalent to $(E_6)\hat{_3}$ and satisfies the conclusion of Theorem 1.4 also in this case. This concludes the proof of the main theorem. □

Remark 6.10. Note that taking the case $(W_{E_6}, L_{E_6} \otimes \mathbf{Z}_3)$ last in the above theorem is a bit misleading, since groups with adjoint form E_6 appear as centralizers in E_7 and E_8, so a separate inductive proof of uniqueness in those case would require knowledge of uniqueness of E_6.

Remark 6.11. The very careful reader might have noticed that the splitting result in [6], which we use in the above proof, to conclude the splitting for $(W_{\mathrm{PU}(3)}, L_{\mathrm{PU}(3)})$ refers to a uniqueness result in [21]. We now quickly sketch a more direct way to see this, which we were told by Dwyer-Wilkerson: We need to see that a 3-compact group with Weyl group $(W_{\mathrm{PU}(3)}, L_{\mathrm{PU}(3)} \otimes \mathbf{Z}_3)$ has to have split maximal torus normalizer $\mathcal{N}$. So, suppose that X is a hypothetical 3-compact group as above but with non-split maximal torus normalizer. By a transfer argument (cf. [51, 9.12]) $\mathcal{N}_3$ has to be non-split as well. Since every elementary abelian 3-subgroup in X can be conjugated into $\mathcal{N}_3$ (since $\chi(X/\mathcal{N}_p)$ is prime to p), this means that all elementary abelian 3-subgroups in X are toral. Furthermore by [53, Prop. 3.4] conjugation between toral elementary abelian p-subgroups is controlled by the Weyl group, so the Quillen category of X in fact agrees with the Quillen category of $\mathcal{N}$. The category has up to isomorphism one element of rank two and two of rank one. The centralizers $\mathcal{C}_{\mathcal{N}}(V)$ of these are respectively T, $T:\mathbf{Z}/2$, and $T\cdot\mathbf{Z}/3$. The unique 3-compact groups corresponding to these centralizers are in fact given by $B\mathcal{C}_{\mathcal{N}}(V)\hat{_3}$. Hence the map $B\mathcal{N}\to BX$ is an equivalence by the centralizer cohomology decomposition theorem [52, §8]. But since $\mathcal{N}$ is non-split, we can find a map $\mathbf{Z}/p^2\to\mathcal{N}$, which is not conjugate in $\mathcal{N}$ to a map into T. Hence the corresponding map $\mathbf{Z}/p^2\to X$ cannot be conjugated into T either, contradicting [51, Prop. 5.5].

7. Consequences of the main theorem

In this section we prove the theorems listed in the introduction which are consequences of the main theorem.

Proof of Theorem 1.2. The theorem follows directly from Theorem 1.1 together with the classification of finite $\mathbf{Z}_p$-reflection groups (Theorem 11.1), noting that by the proof of Theorem 1.1 (and Theorem 12.1) all the exotic p-compact groups have torsion free $\mathbf{Z}_p$-cohomology. □

Proof of Theorem 1.5. By [93, Thm. 1.4] X is isomorphic to a p-compact group of the form $(X'\times T'')/A$, where X' is a simply connected p-compact group, T'' is a p-compact torus, and A is a finite central subgroup of the product. Hence we have $X/T\cong X'/T'$, where T and T' are maximal tori of X and X' respectively. So we can without restriction assume that X is simply connected.

For compact connected Lie groups the statement of this theorem is the celebrated result of Bott [14, Thm. A]. Hence by Theorem 1.2 it is enough to prove the theorem when X is an exotic p-compact group. In that case $H^*(BX;\mathbf{Z}_p)$ is a polynomial algebra with generators in even degrees, and the number of generators equals the rank of X (by the proof of Theorem 1.4). The same is true over $\mathbf{F}_p$, and since $H^*(BT;\mathbf{F}_p)$ is finitely generated over $H^*(BX;\mathbf{F}_p)$ by [51, Prop. 9.11], $H^*(BX;\mathbf{F}_p)\to H^*(BT;\mathbf{F}_p)$ is injective by a Krull dimension consideration. But since they are both polynomial algebras it follows by e.g., [55, §11] that $H^*(BT;\mathbf{F}_p)$ is in fact free over $H^*(BX;\mathbf{F}_p)$. Hence the Eilenberg-Moore spectral sequence of the fibration $X/T\to BT\to BX$ collapses and

$$H^*(X/T;\mathbf{F}_p)\cong\mathbf{F}_p\otimes_{H^*(BX;\mathbf{F}_p)}H^*(BT;\mathbf{F}_p).$$

In particular $H^*(X/T;\mathbf{F}_p)$ is concentrated in even degrees so the rank equals the Euler characteristic $\chi(X/T)$ which again equals $|W_X|$ by [51, Prop. 9.5]. By the long exact sequence in cohomology and Nakayama's lemma we get that $H^*(X/T;\mathbf{Z}_p)$ is a free $\mathbf{Z}_p$-module of rank $|W_X|$ as wanted. □

Remark 7.1. Let $H^*_{\mathbf{Q}_p}(\cdot)=H^*(\cdot;\mathbf{Z}_p)\otimes\mathbf{Q}$. For any connected p-compact group X the natural map $X/T\to BT$ induces an isomorphism

$$H^*_{\mathbf{Q}_p}(X/T)\cong\mathbf{Q}_p\otimes_{H^*_{\mathbf{Q}_p}(BX)}H^*_{\mathbf{Q}_p}(BT)$$

since the Eilenberg-Moore spectral sequence of the fibration $X/T\to BT\to BX$ collapses by [51, Prop. 9.7] and [55, §11]. It then follows from [29] that the natural W_X-action on

$H^*(X/T;\mathbf{Z}_p)\otimes\mathbf{Q}_p = H^*_{\mathbf{Q}_p}(X/T)$ is isomorphic to the regular representation of W_X when ignoring the grading. Just as for compact connected Lie groups this is not true over $\mathbf{Z}_p$ when $p \mid |W_X|$.

Proof of Theorem 1.6. By Theorem 1.2 it is enough to prove the statement for the case where X is the p-completion of a compact connected Lie group and the case where X is an exotic p-compact group separately. The case where X is the p-completion of a compact connected Lie group of course follows directly from the classical Peter-Weyl theorem (cf. e.g. [20, Thm. III.4.1], so we can concentrate on the case where X is exotic. If p does not divide the order of the Weyl group the statement is also obvious: The inclusion $\breve{T}\to \mathrm{U}(r)$ induces a map $\breve{T}\rtimes W \to \mathrm{U}(r|W|)$ whose p-completion is a faithful representation. The remaining cases have been shown to have faithful representations by Castellana: If (W, L) is in the $2a$ family then this is carried out in [27] and if (W, L) is one of the pairs $(G_{12}, p = 3)$, $(G_{29}, p = 5)$, $(G_{31}, p = 5)$, or $(G_{34}, p = 7)$ this is carried out in [26]. □

We now turn to Theorem 1.7 which in fact follows easily from the classification. But let us first state the part which one can see by elementary means. (See also [93, Cor. 5.6] and [55, Lem. 9.3].)

Proposition 7.2. *Let X be a connected p-compact group then the natural composite map*

$$(L_X)_W \cong H_0(W; H_2(BT;\mathbf{Z}_p)) \to H_2(BX;\mathbf{Z}_p) \cong \pi_1(X)$$

induced by the inclusion $T \to X$ is surjective with finite kernel.

In particular if $(L_X)_W$ is torsion free then it is an isomorphism.

Proof. By [93, Thm. 1.4] X is isomorphic to a p-compact group of the form $(X' \times T'')/A$, where X' is a simply connected p-compact group, T'' is a p-compact torus, and A is a finite central subgroup of the product. Since the center of a connected p-compact group is contained in a maximal torus by [52, Thm. 7.5] we can assume A is a subgroup of $T' \times T''$, where T' is maximal torus for X', and hence $(T'\times T'')/A$ is a maximal torus for X. Therefore we get the following diagram of fibration sequences

$$\begin{array}{ccccc} BA & \longrightarrow & BT' \times BT'' & \longrightarrow & B((T' \times T'')/A) \\ \| & & \downarrow & & \downarrow \\ BA & \longrightarrow & BX' \times BT'' & \longrightarrow & BX \end{array}$$

The long exact sequence of homotopy groups and the five-lemma now shows that $\pi_2(B((T \times T'')/A)) \to \pi_2(BX)$ is surjective which is the first statement in the proposition. To see that the kernel is finite note that by [51, Thm. 9.7(3)] $H^2_{\mathbf{Q}_p}(BX) \to H^2_{\mathbf{Q}_p}(BT)^W$ is an isomorphism, which by dualizing to homology shows the claim.

That we get an isomorphism when $(L_X)_W$ is torsion free is obvious from the general statement. □

Remark 7.3. One easily shows that the image of the differential $d_3 : H_3(W;\mathbf{Z}_p) \to H_0(W; H_2(BT;\mathbf{Z}_p))$ in the Serre spectral sequence for the fibration $BT \to B\mathcal{N}_X \to BW$ is always in the kernel of the surjective map of Proposition 7.2. By standard group cohomology (cf. [28]) the image of this differential identifies with the image of the map given by capping with the k-invariant $\gamma \in H^3(W; H_2(BT;\mathbf{Z}_p))$ of the extension. If one knew that the double coset formula held for p-compact groups (more precisely that $H^*(B\mathcal{N};\mathbf{Z}_p) \xrightarrow{tr^*}$

$H^*(BX;\mathbf{Z}_p) \stackrel{res}{\to} H^*(BT;\mathbf{Z}_p)$ is the restriction map, cf. [59, Ex. VI.4]) then it would easily follow that this image is in fact equal to the kernel of the map in Proposition 7.2, which would give a conceptual proof of the formula for the fundamental group. Note that by a result of Tits [128] (see also [57, 96, 6]) the extension class γ is always of order 2 for compact connected Lie groups. The next proposition gives the complete answer in the Lie case.

Proposition 7.4. *Let G be a compact connected Lie group. Then the map $\pi_1(T)_W \to \pi_1(G)$ is surjective with kernel $(\mathbf{Z}/2)^s$, where s is the number of direct factors of G isomorphic to a symplectic group $\mathrm{Sp}(n)$, $n \geq 1$.*

Proof. That the map is surjective follows as in the p-compact case, so we just have to identify the kernel. By [85, Thm. 1.6], for any compact connected Lie group G, $T^W = Z(G) \oplus (\mathbf{Z}/2)^s$, where s is the number of direct factors of G isomorphic to an odd special orthogonal group $\mathrm{SO}(2n+1)$, $n \geq 1$.

Consider the dual group $G^\vee$ of G obtained as the taking the dual root diagram (see [17, §4, no. 8]). Then $G^\vee$ has fundamental group isomorphic to $\widehat{Z(G)}$, where the hat denotes the Poincaré dual group (see [17, §4, no. 9]). Likewise $\widehat{(L_{G^\vee})_W}$ is canonically isomorphic to T^W. Since duality is an involution on the set of compact connected Lie groups which sends direct summands to direct summands and $\mathrm{SO}(2n+1)$ to $\mathrm{Sp}(n)$ the claim about the fundamental group follows directly from the dual result about the center. □

Proof of Theorem 1.7. By Theorem 11.1 $(L_X)_W = 0$ for all the exotic p-compact groups, so Proposition 7.2 shows the formula in this case. By Theorem 1.2 we are hence reduced to showing the formula for X of the form $G\hat{_p}$ for some compact connected Lie group G. In this case the formula is well known and easy. Namely it follows from Remark 7.3 that the kernel of $(L_X)_W \to \pi_1(X)$ is an elementary abelian 2-group. Alternatively the same conclusion follows from the formula for the fundamental group of a compact connected Lie group (see [17, §4, no. 6, Prop. 11] or [1, Thm. 5.47], noting that in the notation of [1] $(1-\varphi_r)\gamma_r = 2\gamma_r$). □

We now start to prove Theorem 1.8 and 1.9.

Lemma 7.5. *Suppose X and X' are two connected p-compact groups both with maximal torus normalizer $\mathcal{N}$. Then all elementary abelian p-subgroups of X are toral if and only if all elementary abelian p-subgroups of X' are toral.*

Furthermore, if for all toral elementary abelian p-subgroups $V \to X$ the centralizer $\mathcal{C}_X(V)$ is connected then all elementary abelian p-subgroups in X are toral.

Proof. Suppose that X has a non-toral elementary abelian p-subgroup $V \to X$. We can assume that it is minimal, in the sense that any elementary abelian p-subgroup of smaller rank is toral. Write $V = V' \oplus V''$, where V' has rank one. We can assume that $V'' \to X$ factors through T (by the minimality) and that $V' \to X$ factors through $\mathcal{N}$ (by [52, Prop. 2.14]). Let $\mathcal{N}''$ denote the maximal torus normalizer of $\mathcal{C}_X(V'')_1$, which by [52, Thm. 7.6] can be described in terms of V'' and $\mathcal{N}$. The adjoint map $V' \to \mathcal{C}_{\mathcal{N}}(V'')$ cannot factor through $\mathcal{N}''$ since otherwise $V' \to \mathcal{C}_X(V'')$ would factor through T in $\mathcal{C}_X(V'')$ (by [51, Prop. 5.5]), contradicting that V is assumed not to be toral. Note that $\mathcal{N}''$ is normal in $C_{\breve{\mathcal{N}}}(V'')$ and $C_{\breve{\mathcal{N}}}(V)/\mathcal{N}'' \cong \pi_0(\mathcal{C}_X(V'')) \cong \pi_0(\mathcal{C}_{X'}(V''))$ (see [52, Rem. 2.11]). Hence $V' \to \pi_0(\mathcal{C}_{X'}(V''))$ is non-trivial so $V \to \mathcal{N} \to X'$ cannot be toral in X'.

The last part of the lemma is clear from the proof of the first part. □

Remark 7.6. Despite the above lemma it is not a priori clear how to determine whether a p-compact group X has the property that all elementary abelian p-subgroups are toral just from looking at $\mathcal{N}_X$ (but see [125, Thm. 2.28] for the Lie case). However, by a case-by-case analysis (Theorem 1.8), this is the case if and only if all toral elementary abelian p-subgroups have connected centralizers.

Remark 7.7. Note that by Lannes theory [81, Thm. 0.4] the property that every elementary abelian p-subgroup of X is toral is equivalent to that $H^*(BX; \mathbf{F}_p) \to H^*(BT; \mathbf{F}_p)^{W_X}$ is an F-isomorphism. (See also Theorem 12.1 and Remark 12.3.)

We state the following well known lemma for easy reference.

Lemma 7.8. *Suppose that X is a connected p-compact group such that $H^*(BX; \mathbf{Z}_p)$ is a polynomial algebra with generators concentrated in even degrees. Then all elementary abelian p-subgroups of X are toral.*

Proof. For every elementary abelian p-subgroup $\nu : E \to X$, $H^*(B\mathcal{C}_X(\nu); \mathbf{F}_p)$ is a polynomial algebra with generators concentrated in even degrees by [56, Thm. 1.3] (note that Lannes T-functor preserves objects concentrated in even degrees by [81, Prop. 2.1.3]). In particular $\mathcal{C}_X(\nu)$ is connected so by Lemma 7.5 all elementary abelian p-subgroups are toral. (Alternatively one can use Remark 12.3.) □

Proof of Theorem 1.8. First note that the implications (1) ⇒ (3) and (1) ⇒ (2) follows from Theorem 12.1. The implication (3) ⇒ (4) follows easily from Theorem 5.1. Namely for all toral elementary abelian p-subgroups $V \to X$, Theorem 5.1 implies that $W_{C_X(V)}$ is a reflection group, so by [52, Thm. 7.6] $C_X(V)$ is connected, using the assumption that p is odd. But this implies that all elementary abelian p-subgroups are toral by Lemma 7.5.

We now prove the implication (4) ⇒ (1). First note that by Theorem 11.1 and [53, Thm. 1.4] we can write $X \cong X' \times X''$ where X' has Weyl group $(W_G, L_G \otimes \mathbf{Z}_p)$, for some compact connected Lie group G, and (W'', L'') is a product of exotic finite $\mathbf{Z}_p$-reflection groups. Furthermore, since the normalizer of a connected p-compact group is split for p odd by [6, Thm. 1.2] we can in fact choose G such that $\mathcal{N}_{G\hat{_p}} \cong \mathcal{N}_{X'}$. Since by Lemma 7.5 the property of having all elementary abelian p-subgroups toral is a property which only depends on $\mathcal{N}$ we conclude that G has to have this property as well. But this implies that G has torsion free $\mathbf{Z}_p$-cohomology by [8, Thm. B] (see also [125, Thm. 2.28]). In the exotic case, we know by the proof of Theorem 1.4 that we can find a p-compact group $\tilde{X}''$ which has the same maximal torus normalizer as X'' and which has torsion free $\mathbf{Z}_p$-cohomology. Hence we have found a p-compact group $G\hat{_p} \times \tilde{X}''$ which has the same maximal torus normalizer as X and has torsion free $\mathbf{Z}_p$-cohomology. Since by Theorem 1.4 a p-compact group is determined by its normalizer we conclude that X in fact has torsion free $\mathbf{Z}_p$-cohomology. (Alternatively, one can appeal to Remark 7.11 which shows that the property of having torsion free $\mathbf{Z}_p$-cohomology only depends on $\mathcal{N}$.)

Finally we prove the implication (2) ⇒ (1), where we seem to need the full strength of Theorem 1.1. Note that by Theorem 1.2 we can write $X \cong G\hat{_p} \times X'$ where G is a compact connected Lie group and X' has torsion free $\mathbf{Z}_p$-cohomology. Likewise if $BG\hat{_p}$ has torsion free $\mathbf{Z}_p$-cohomology then $G\hat{_p}$ has torsion free $\mathbf{Z}_p$-cohomology by [9, p. 93]. □

Remark 7.9. Since the implication (1) ⇒ (4) follows from Lemma 7.8, we see that in the above theorem the implications (1) ⇒ (3), (1) ⇒ (2), and (1) ⇒ (4) follow by general arguments. In the case of compact connected Lie groups the implication (4) ⇒ (3) has a general proof, by combining [125, Thm. 2.28] with [38], and likewise (2) $\Rightarrow$ (1) has a general proof by [9, p. 93]. We do not know non-case-by-case proofs of these implications for p-compact groups. (The implication (2) $\Rightarrow$ (1) is stated in [94, Thm. 4.2] but the proof has a gap.) The remaining implications do not seem to have general proofs even for compact connected Lie groups. See also [125, §4].

Proof of Theorem 1.9. The statement is basically obvious by Theorem 1.2—we sketch a proof which only uses the classification in the torsion free case. As in the proof of Theorem 1.8 we can by Theorem 11.1 and [53, Thm. 1.4] write $X \cong X' \times X''$ where X' has Weyl group $(W_G, L_G \otimes \mathbf{Z}_p)$, for some compact connected Lie group G, and the Weyl group (W'', L'') of X'' is a product of exotic $\mathbf{Z}_p$-reflection groups.

By Theorem 1.1 and its proof we know that X'' is uniquely determined by its Weyl group (W'', L'') and has cohomology isomorphic to $H^*(B^2L''; \mathbf{Z}_p)^{W''}$. Since $H^2(B^2L''; \mathbf{Z}_p)^{W''} = (L''^*)^{W''} = 0$ we conclude by the universal coefficient theorem that $\pi_1(X'') = H_2(BX''; \mathbf{Z}_p) = 0$. Hence by the T-functor, as in the proof of Theorem 1.8, we conclude that X'' satisfies all three equivalent conditions of the theorem. Furthermore we can, again by [6, Thm. 1.2] since p is odd, choose G such that $\mathcal{N}_{G_{\hat{p}}} \cong \mathcal{N}_{X'}$. Hence by Lemma 7.5 and [52, Thm. 7.6] the theorem holds for X' if and only if it holds for G. But by [125, Thm. 2.27] the theorem is true for G, which finishes the proof of the theorem. □

Remark 7.10. The same arguments as above shows that the conjectural classification for $p = 2$ implies that Theorem 1.9 and Theorem 1.8 (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) $\Rightarrow$ (3) holds true for $p = 2$. However, in Theorem 1.8, (3) is not equivalent to the other conditions since $\mathbf{Z}_2[L_{\mathrm{SO}(2n+1)}]^W$ is a polynomial algebra, since this is true for $\mathrm{Sp}(n)$, despite $\mathrm{SO}(2n+1)$ having 2-torsion.

Remark 7.11. Notbohm states his classification of connected p-compact groups with $\mathbf{Z}_p[L]^W$ a polynomial algebra in the setup of spaces BX with polynomial cohomology (cf. [103]). This means that his uniqueness statement is a priori only uniqueness among p-compact groups with torsion free $\mathbf{Z}_p$-cohomology (cf. Theorem 12.1). We will here briefly sketch a direct but case-by-case way (following a line of argument given in a special case in [94, Pf. of Thm. 5.3]) to show that for a p-compact the property of having torsion free $\mathbf{Z}_p$-cohomology depends only on (W, L), which allows us to remove the extra assumption.

Assume that X is a connected p-compact group, p odd, such that $\mathbf{Z}_p[L_X]^{W_X}$ is a polynomial algebra. We want to show that $H^*(BX; \mathbf{Z}_p)$ is a polynomial algebra as well. By Theorem 12.2(1), $(L_X)_{W_X}$ is torsion free, so $\pi_1(X) = (L_X)_{W_X}$ by Proposition 7.2. By the Serre and Eilenberg-Moore spectral sequences $H^*(BX; \mathbf{Z}_p)$ is a polynomial algebra if and only if $H^*(B(X\langle 1\rangle); \mathbf{Z}_p)$ is a polynomial algebra. Furthermore by construction $L_{X\langle 1\rangle} = SL_X$ and by Theorem 12.2(1) $\mathbf{Z}_p[L_{X\langle 1\rangle}]^{W_X}$ is also a polynomial algebra, so we can without loss of generality assume that X is simply connected. By [53, Thm. 1.4 and Rem. 1.6] we can furthermore assume that X is a simple p-compact group.

By [6, Thm. 1.2] $\breve{\mathcal{N}}_X = \breve{T}_X \rtimes W_X$. Using Theorems 11.1 and 12.2 we first show that the cohomology of $\breve{\mathcal{N}}_X$ is detected on elementary abelian p-subgroups. More precisely we show that in each case there is a compact connected Lie group H such that $\breve{\mathcal{N}}_X$ contains a subgroup isomorphic to $\breve{\mathcal{N}}_{H_{\hat{p}}}$ with index prime to p having the required property. When $p \nmid |W|$ we take H to be a torus, and if (W_X, L_X) is in family 1 or family $2a$ we take $H = \mathrm{SU}(n)$ and $H = \mathrm{U}(n)$ respectively. If (W_X, L_X) is one of the exotic $\mathbf{Z}_p$-reflection groups $(G_{12}, p = 3)$, $(G_{29}, p = 5)$, $(G_{31}, p = 5)$, or $(G_{34}, p = 7)$ we take $H = \mathrm{SU}(p)$, cf. the proof of Theorem 5.1. The only remaining cases are the ones where $(W_X, L_X) =$

$(W_G, L_G \otimes \mathbf{Z}_p)$ for one of the following pairs (G, p): $(G_2, p = 3)$, $(3E_6, p = 5)$, $(2E_7, p = 5)$, $(2E_7, p = 7)$, and $(E_8, p = 7)$. In these cases we can by [78, Prop. 6.11] take $H = \mathrm{SU}(3)$, $\mathrm{SU}(2) \times_{C_2} \mathrm{SU}(6)$, $\mathrm{SU}(8)/C_2$, $\mathrm{SU}(8)/C_2$ and $\mathrm{SU}(9)/C_3$ respectively. Since both $\breve{\mathcal{N}}_{\mathrm{U}(n)\hat{_p}}$ and $\breve{\mathcal{N}}_{\mathrm{SU}(n)\hat{_p}}$ have cohomology which is detected on elementary abelian p-subgroups by [110, Prop. 3.4] (for $\mathcal{N}_{\mathrm{U}(n)\hat{_p}}$; $\mathcal{N}_{\mathrm{SU}(n)\hat{_p}}$ follows from this, cf. [98, Lem. 12.6]) we see that in all cases the cohomology of $\breve{\mathcal{N}}_{H\hat{_p}}$ is detected on elementary abelian p-subgroups. Hence, by a transfer argument, the mod p cohomology of BX is detected on elementary abelian p-subgroups.

Next, we want to show that all elementary abelian p-subgroups of X can be conjugated into a maximal torus. By Lemma 7.5 we just have to show that we can find *some* p-compact group X' with the same maximal torus normalizer which has that property. If (W_X, L_X) is of Lie type this follows from Borel's theorem [8, Thm. B]. If (W_X, L_X) is exotic this is also true since we know (by Theorem 5.3 or Notbohm's work [103]) that there exist a p-compact group with Weyl group (W_X, L_X) and classifying space having polynomial $\mathbf{Z}_p$-cohomology algebra.

The fact that all elementary abelian p-subgroups of X are toral combined with the fact that the cohomology is detected on elementary abelian p-subgroups implies that the mod p cohomology of BX is concentrated in even degrees. Hence $H^*(BX; \mathbf{Z}_p)$ is torsion free as wanted.

Proof of Theorem 1.10. Let X be a connected finite loop space with maximal torus $i : T \to X$. Note that $(X/T)\hat{_p} \simeq X\hat{_p}/T\hat{_p}$ by the fiber lemma [19, II 5.1], and consequently, by the definition of Euler characteristic, $\chi(X/T) = \chi(X\hat{_p}/T\hat{_p})$. Hence $T\hat{_p} \to X\hat{_p}$ will be a maximal torus for the p-compact group $X\hat{_p}$, for all primes p.

For our connected finite loop space X, define $W_X(T)$ to be the set of conjugacy classes of self-equivalences φ of T such that i and $i\varphi$ are conjugate. We obviously have an injective homomorphism $W_X \to W_{X\hat{_p}}$ for all primes p and we now want to see that this map is surjective as well, so that we can naturally identify $(W_X, \pi_1(T) \otimes \mathbf{Z}_p)$ with $(W_{X\hat{_p}}, L_{X\hat{_p}})$.

First note that by [51, Pf. of Thm. 9.7] we can view $W_{X\hat{_p}}$ as the Galois group of the extension of polynomial algebras $H^*_{\mathbf{Q}_p}(BX) \to H^*_{\mathbf{Q}_p}(BT)$. But, since BX has finitely many cells in each dimension and since BX is nilpotent, we can identify

$$\begin{array}{ccc} H^*(BX;\mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}_p & \longrightarrow & H^*(BT;\mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}_p \\ \Big| \cong & & \Big| \cong \\ H^*_{\mathbf{Q}_p}(BX) & \longrightarrow & H^*_{\mathbf{Q}_p}(BT) \end{array}$$

so the extensions $H^*(BX; \mathbf{Q}) \to H^*(BT; \mathbf{Q})$ and $H^*_{\mathbf{Q}_p}(BX) \to H^*_{\mathbf{Q}_p}(BT)$ have canonically isomorphic Galois groups. Hence any element in $W_{X\hat{_p}}$ lifts to a canonical element in the Galois group of the extension $H^*(BX; \mathbf{Q}) \to H^*(BT; \mathbf{Q})$. However, since $BX_{\mathbf{Q}}$ and $BT_{\mathbf{Q}}$ are products of Eilenberg-Mac Lane spaces (cf. e.g. [117, Ch. V Prop. 6]), this Galois group just identifies with the self-equivalences $BT_{\mathbf{Q}} \to BT_{\mathbf{Q}}$ which commute with the map $Bi_{\mathbf{Q}} : BT_{\mathbf{Q}} \to BX_{\mathbf{Q}}$ up to homotopy. Hence any element in $W_{X\hat{_p}}$ gives rise to a compatible family of self-equivalences of $BT_{\mathbf{Q}}$ and $BT\hat{_l}$, for all primes l. So by the arithmetic square [19, VI.8.1], we get a self-equivalence of BT which commutes with the map to BX up to homotopy, i.e., an element in W_X. The constructed element is a lift of the element in $W_{X\hat{_p}}$ we started with, so the map $W_X \to W_{X\hat{_p}}$ is surjective as well.

Likewise, the argument above showed that W_X is the Galois group of the extension $H^*(BX;\mathbf{Q}) \to H^*(BT;\mathbf{Q})$, so we have an isomorphism

$$H^*(BX;\mathbf{Q}) \xrightarrow{\cong} H^*(BT;\mathbf{Q})^{W_X}.$$

Since $H^*(BX;\mathbf{Q})$ is a polynomial algebra we get by the Shephard-Todd-Chevalley theorem (see [7, Thm. 7.2.1]) that $(W_X, \pi_1(T))$ is a $\mathbf{Z}$-reflection group. Hence, by the proof of Theorem 11.1, $(W_X, \pi_1(T))$ is the Weyl group of some compact connected Lie group G.

For each p we have an extension class $\gamma_p \in H^3(W_X; \pi_1(T) \otimes \mathbf{Z}_p)$ corresponding to the fibration sequence $BT^{\wedge}_p \to B\mathcal{N}_{X^{\wedge}_p} \to BW_X$. Since $H^3(W_X;\pi_1(T))$ is a finite abelian group, and hence given as a sum of its p-primary parts, these extension classes identify with a unique extension class $\gamma \in H^3(W_X;\pi_1(T))$. We define the loop space $\mathcal{N}_X$ to be the loop space of the total space in the fibration sequence $BT \to B\mathcal{N}_X \to BW_X$ with the canonical action of W_X on BT and extension class γ. Since the fiberwise $\mathbf{F}_p$-completion of $B\mathcal{N}_X$ with respect to this defining fibration identifies with $B\mathcal{N}_{X^{\wedge}_p}$, the arithmetic square produces a canonical morphism $\mathcal{N}_X \to X$. ($\mathcal{N}_X$ is, quite naturally, called the maximal torus normalizer of the finite loop space X [94, Def. 1.3].)

By [6] (see also [96], [57]) the extension classes defining $T \to \mathcal{N}_X \to W_X$ and $T \to N_G(T) \to W_G(T)$ are both 2-torsion. Let $\widetilde{BN}$ denote the fiber-wise $\mathbf{Z}[\frac{1}{2}]$-localization of the total space of the fibration $BT \to BN_G(T) \to BW_G(T)$ or equivalently the corresponding fibration with $B\mathcal{N}_X$. We hence have embeddings

$$\begin{array}{ccc} & \widetilde{BN} & \\ \swarrow & & \searrow \\ BX[\tfrac{1}{2}] & & BG[\tfrac{1}{2}] \end{array}$$

By the arithmetic square [19, VI.8.1], the following square is a pullback

$$\begin{array}{ccc} BX[\tfrac{1}{2}] & \longrightarrow & \prod_{p\neq 2} BX^{\wedge}_p \\ \downarrow & & \downarrow \\ BX_{\mathbf{Q}} & \longrightarrow & (\prod_{p\neq 2} BX^{\wedge}_p)_{\mathbf{Q}} \end{array}$$

and similarly for BG. By Theorem 1.4 we can construct unique maps between p-completions under $\widetilde{BN}$, and we obviously also have a unique map between the rationalizations. By construction (as maps under $\widetilde{BN}$) these maps agree on the rationalization of the product of the p-completions, so by the arithmetic square we get an induced map $BX[\frac{1}{2}] \to BG[\frac{1}{2}]$, which by construction is an $\mathbf{F}_p$-equivalence for all primes p. Since both spaces are one-connected this implies that the map is a homotopy equivalence. □

8. Non-toral elementary abelian p-subgroups of the exceptional groups

In this section we find all conjugacy classes of non-toral elementary abelian p-subgroups E, p odd, of any exceptional compact connected Lie group G, as well as their centralizers $C_G(E)$ and *Weyl groups* $W(E) = N_G(E)/C_G(E)$. (Recall that a subgroup of G is called *toral* if it is contained in a torus in G and *non-toral* otherwise.) We do this by expanding on the work of Griess [65], who found the maximal non-toral elementary abelian p-subgroups.

Our strategy is as follows. Using the work of Griess [65], we first find representatives of the conjugacy classes of maximal non-toral elementary abelian p-subgroups. We then get lower bounds for their Weyl groups by producing explicit elements in these. From this we are able to identify the non-maximal non-toral elementary abelian p-subgroups and get lower bounds for their Weyl groups. Finally we get exact results on the Weyl groups by computing centralizers.

To be compatible with the standard literature we will in this section state and prove all theorems in the context of linear algebraic groups over the complex numbers $\mathbf{C}$—we state in Proposition 8.4 why this is equivalent to considering compact Lie groups. (The results for $G(\mathbf{C})$ can furthermore be translated into results for $G(F)$ for any algebraically closed field F of characteristic prime to p, see [66, Thm. 1.22] and [62].)

This section is divided into four subsections. The first recalls some results from the theory of linear algebraic groups and discusses the relation to compact Lie groups. The remaining subsections deal with the elementary abelian 3-subgroups of the groups of type E_6, E_7 and E_8 respectively. (The remaining non-trivial cases $E_8(\mathbf{C}), p = 5$ and $F_4(\mathbf{C}), p = 3$ are treated completely in [65, Lem. 10.3 and Thm. 7.4].)

For some of our computations for the groups $3E_6(\mathbf{C})$ and $E_8(\mathbf{C})$ we have used the computer algebra system MAGMA [13], although this reliance on computers could if needed be replaced by some rather tedious hand calculations.

Notation 8.1. We use standard names for the linear algebraic groups we consider, e.g. $3E_6(\mathbf{C})$ denotes the simply connected group of type E_6 over $\mathbf{C}$ and $E_6(\mathbf{C})$ denotes its adjoint version. We let $\mathbf{T}_n$ denote an n-dimensional torus, i.e., $\mathbf{T}_n = (\mathbf{C}^\times)^n$.

To describe centralizers we follow standard notation for extensions of groups, cf. the ATLAS [34, p. xx]. Thus $A : B$ denotes a group which is the semidirect product of the normal subgroup A with the subgroup B, and $A \cdot B$ denotes a non-split extension of A with B. If p is a prime number, p^n denotes an elementary abelian p-group of rank n.

Whenever E is a concrete elementary abelian p-group of rank n we will always fix an ordered basis of E, so that $\mathrm{GL}(E)$ identifies with $\mathrm{GL}_n(\mathbf{F}_p)$. We make the standing convention that all matrices acts on columns.

We identify a permutation σ in the symmetric group Σ_n with its permutation matrix $A = [a_{ij}]$ given by $a_{ij} = \delta_{i,\sigma(j)}$ where δ is the Kronecker delta.

If K is a field, we let $M_n(K)$ denote the set of $n \times n$-matrices over K. For $a_1, \ldots, a_n \in K$ we let $\mathrm{diag}(a_1, \ldots, a_n) \in M_n(K)$ denote the diagonal matrix with the a_i's in the diagonal. For $1 \leq i, j \leq n$, $e_{ij} \in M_n(K)$ denotes the matrix whose only non-zero entry is 1 in position (i, j). Given matrices $A_1 \in M_{n_1}(K), \ldots, A_m \in M_{n_m}(K)$ we let $A_1 \oplus \ldots \oplus A_m$ denote the $n \times n$-block matrix with the A_i's in the diagonal, $n = n_1 + \ldots + n_m$. We also need the "blowup" homomorphism $\Delta_{n,m} : M_n(K) \longrightarrow M_{mn}(K)$ defined by replacing each entry a_{ij} by $a_{ij} I_m$, where $I_m \in M_m(K)$ is the identity matrix.

As $p = 3$ in all the cases we consider, we use some special notation. An arbitrary element of $\mathbf{F}_3$ is denoted by $*$, and ε denotes an element of the multiplicative group $\mathbf{F}_3^\times$. We let $\omega = e^{2\pi i/3}$ and $\eta = e^{2\pi i/9}$ and define elements $\beta, \gamma, \tau_1, \tau_2 \in \mathrm{SL}_3(\mathbf{C})$ by $\beta = \mathrm{diag}(1, \omega, \omega^2)$, $\gamma = (1, 2, 3)$,

$$\tau_1 = \frac{e^{-\pi i/18}}{\sqrt{3}} \begin{bmatrix} 1 & \omega^2 & 1 \\ 1 & 1 & \omega^2 \\ \omega^2 & 1 & 1 \end{bmatrix}$$

and $\tau_2 = \mathrm{diag}(\eta, \eta^{-2}, \eta)$. Note that $\beta^{\tau_1} = \beta\gamma$, $\gamma^{\tau_1} = \gamma$, $\beta^{\tau_2} = \beta$ and $\gamma^{\tau_2} = \beta\gamma$.

8.1. **Recollection of some results on linear algebraic groups.** Recall that a (not necessarily connected) linear algebraic group G is called *reductive* if the unipotent radical, i.e., the largest normal connected unipotent subgroup of G, is trivial.

Theorem 8.2. *Let G be a linear algebraic group over an algebraically closed field K.*

(1) *If A is a subgroup of G and S is some subset of A, then A is toral in G if and only if A is toral in $C_G(S)$.*
(2) *If H is a maximal torus of G, then two subsets of H are conjugate in G if and only if they are conjugate in $N_G(H)$. If A is a toral subgroup of G, then $W(A) = N_G(A)/C_G(A)$ is isomorphic to a subquotient of the Weyl group $W = N_G(H)/H$ of G.*
(3) *Assume that G is a connected reductive group such that G' is simply connected. Then the centralizer of any semisimple element in G is connected. In particular, if A is an abelian subgroup of G consisting of semisimple elements generated by at most two elements, then A is toral.*
(4) *If G is reductive and σ is a semisimple automorphism of G, then the fixed point subgroup G^σ is reductive and contains a regular element of G.*
(5) *Assume that G is a connected reductive group, let $Z \subseteq G$ be a central subgroup, and let $\pi : G \to G/Z$ be the quotient homomorphism. If A is a subgroup of G, then A is toral in G if and only if $\pi(A)$ is toral in G/Z.*
(6) *Assume* $\operatorname{char} K = 0$ *and let $\mathfrak{g}$ be the Lie algebra of G. If $S \subseteq G$ is a finite subset of G, then the Lie algebra of $C_G(S)$ is given by*

$$\mathfrak{c}_{\mathfrak{g}}(S) = \{x \in \mathfrak{g} \mid \operatorname{Ad}(s)(x) = x \text{ for all } s \in S\}.$$

In particular, if $S \subseteq G$ is a finite subgroup, then

$$\dim C_G(S) = \frac{1}{|S|} \sum_{s\in S} \operatorname{tr}\ \operatorname{Ad}(s)|_{\mathfrak{g}}.$$

Proof. (1): Obviously, if A is toral in $C_G(S)$ then A is toral in G. Conversely, if A is toral in G, then $A \subseteq H$ for a torus H in G. Since $S \subseteq A$ we get $H \subseteq C_G(S)$ and thus A is toral in $C_G(S)$.

(2): The first part follows by a Frattini argument. Assume that $A, A^g \subseteq H$ are conjugate subsets of H. Then H and $H^{g^{-1}}$ are maximal tori of $C_G(A)$ and thus conjugate in $C_G(A)$ (cf. [71, Cor. 21.3.A]). Thus we may write $H = H^{g^{-1}c}$ for some $c \in C_G(A)$ and we conclude that $n = g^{-1}c \in N_G(H)$. Then $A^{n^{-1}} = A^{c^{-1}g} = A^g$, which proves the first part. The second part follows similarly, cf. [83, Prop. 1.1(i)].

(3): The first part which is due to Steinberg is proved in [25, Thm. 3.5.6]. The second part follows from the first, cf. [123, II.5.1].

(4): We can assume G to be connected. In case G is semisimple and simply connected the first claim is proved in [124, Thm. 8.1] and the general case reduces to this one. Indeed we can find a finite cover $\tilde{G}$ of G which is a direct product of a semisimple simply connected group and a torus, and σ lifts to a semisimple automorphism of $\tilde{G}$ by [124, 9.16]. For the second claim see [134, Thms. 2 and 3] or [123, Pf. of Thm. II.5.16] in case G is semisimple; the general case clearly reduces to this one.

(5): By [71, Cor. 21.3.C] we know that if H is a maximal torus of G, then $\pi(H)$ is a maximal torus of G/Z, and all maximal tori of G/Z are of this form. Thus if A is toral in G, then $\pi(A)$ is toral in G/Z. Conversely, if H' is a maximal torus of G/Z containing

$\pi(A)$, then by the above we have $H' = \pi(H)$ for some maximal torus H of G. Thus we get $A \subseteq \langle H, Z\rangle$. However since G is connected and reductive, we get $Z \subseteq H$ by [71, Cor. 26.2.A(b)]. Thus $A \subseteq H$ and we are done.

(6): In case S consists of a single element, the first part follows from [71, Thm. 13.4(a)] (note that the connectivity assumption in [71, Thm. 13.4] is only used in part (b)). The general case follows from this by applying [71, Thm. 12.5(b)] to the centralizers $C_G(s)$, $s \in S$.

Now assume that $S \subseteq G$ is a finite subgroup, and let χ denote the character of the adjoint representation of G restricted to S. Then the dimension of

$$\mathfrak{c}_{\mathfrak{g}}(S) = \{x \in \mathfrak{g} \mid \mathrm{Ad}(s)(x) = x \text{ for all } s \in S\}.$$

equals the multiplicity of the trivial character in χ. By the orthogonality relations this is given by

$$(\chi \,|\, 1) = \frac{1}{|S|}\sum_{s\in S}\chi(s),$$

and we are done. □

We also need the following result whose proof is extracted from [115].

Theorem 8.3. *Let G be a reductive linear algebraic group, H a maximal torus of G and let $N = N_G(H)$. Let $U \subseteq N$ be a subgroup consisting of semisimple elements such that $U/(U \cap H)$ is cyclic. Let S be the identity component of H^U (the subgroup of H fixed by U), and assume that S is a maximal torus of $C_G(U)$. Then $C_N(U) = N_{C_G(U)}(S)$ and in particular $C_N(U)$ is a maximal torus normalizer in $C_G(U)$.*

Proof. As any element of $C_N(U)$ normalizes H^U and hence also its identity component S, the inclusion $C_N(U) \subseteq N_{C_G(U)}(S)$ is clear. Suppose conversely that $x \in N_{C_G(U)}(S)$. Let $C = U \cap H$. From [12, 2.15(d)] it follows that G^C is reductive. By assumption the cyclic group U/C acts by semisimple automorphisms on G^C. It now follows from Theorem 8.2(4) that $G^U = (G^C)^{U/C}$ is reductive and that every maximal torus of G^U is contained in a unique maximal torus of G^C. Since $C \subseteq H$, we see that H is the maximal torus of G^C containing S. As H^x is also a maximal torus of G^C and $H^x \supseteq S^x = S$ we conclude that $H^x = H$. Thus $x \in C_N(U)$ proving the result. □

We now explain the relationship between reductive complex linear algebraic groups and compact Lie groups. If G is a complex linear algebraic group then the underlying variety of G is an affine complex variety. By endowing this variety with the usual Euclidean topology instead of the Zariski topology we may view G as a complex Lie group since the group operations are given by polynomial maps.

Proposition 8.4. *Let G be a complex linear algebraic group.*

1. *Viewed as a Lie group, G contains a maximal compact subgroup which is unique up to conjugacy, and for any such subgroup K we have a diffeomorphism $G \cong K \times \mathbf{R}^s$ for some s.*
2. *Let K be a maximal compact subgroup of G, and let $S, S' \subseteq K$ be two subsets. If $S' = S^g$ for some $g \in G$, then there exists $k \in K$ such that $x^k = x^g$ for all $x \in S$.*
3. *Assume that G is reductive. If S is a finite subgroup of G, then $C_G(S)$ is also reductive. If K is a maximal compact subgroup of G containing S, then $C_K(S)$ is a maximal compact subgroup of $C_G(S)$.*

(4) *If G is reductive and K is a maximal compact subgroup of G, then we have a diffeomorphism $Z(G) \cong Z(K) \times \mathbf{R}^s$ for some s.*

Proof. Note first that the identity component G_1 of G seen as a Lie group coincides with the identity component of G seen as a linear algebraic group [108, Ch. 3, §3, no. 1]. Thus G/G_1 is finite by [71, Prop. 7.3(a)]. The first claim is now part of the Cartan-Chevalley-Iwasawa-Malcev-Mostow theorem [69, Ch. XV, Thm. 3.1] and the second claim also follows from this, cf. [11, Ch. V, §24, Prop. 2].

In case G is reductive it is possible to give a more explicit form of the decomposition above. By [71, Thm. 8.6] we may assume that G is a closed subgroup of $\mathrm{GL}(V)$ for some complex vector space V. From [108, Thm. 5.2.8] it follows that G has a compact real form K and we may thus choose a non-degenerate Hermitian inner product on V which is invariant under K (e.g. by [108, Thm. 3.4.2]). Let $\mathrm{U}(V)$ denote the set of operators in $\mathrm{GL}(V)$ which are unitary with respect to the chosen inner product. Using [108, Problems 5.2.3 and 5.2.4] we see that $G \subseteq \mathrm{GL}(V)$ is self-adjoint and that $K = G \cap \mathrm{U}(V)$. The last part now follows by combining [108, Cor. 2 of Thm. 5.2.2] with [108, Cor. 2 of Thm. 5.2.1].

If S is a subgroup of K, then S is selfadjoint since K consists of unitary operators. In particular $C_G(S)$ is also a selfadjoint subgroup of $\mathrm{GL}(V)$, so by [108, Cor. 3 of Thm. 5.2.1] it follows that $C_K(S) = C_G(S) \cap \mathrm{U}(V)$ is a maximal compact subgroup of $C_G(S)$.

It only remains to prove that $C_G(S)$ is reductive for a finite subgroup S of G. However by [108, Problem 6.11] and [108, Ch. 4, §1, no. 2] we see that a complex linear algebraic group is reductive if and only if its Lie algebra is reductive. Thus it suffices to prove that the Lie algebra of $C_G(S)$ is reductive. However by Theorem 8.2(6) this Lie algebra equals

$$\mathfrak{c}_{\mathfrak{g}}(S) = \{x \in \mathfrak{g} |\, \mathrm{Ad}(s)(x) = x \text{ for all } s \in S\}.$$

where $\mathfrak{g}$ denotes the Lie algebra of G. The claim now follows from [30, Ch. V, §2, no. 2, Prop. 8]. □

8.2. **The groups $E_6(\mathbf{C})$ and $3E_6(\mathbf{C})$, $p = 3$.** In this subsection we consider the elementary abelian 3-subgroups of the groups of type E_6 over $\mathbf{C}$. The group $3E_6(\mathbf{C})$ has two non-isomorphic faithful irreducible 27-dimensional representations. These have highest weight λ_1 and λ_6 respectively and are dual to each other. An explicit construction of $3E_6(\mathbf{C})$ based on one of these representations was originally given by Freudenthal [60]. This construction is described in more detail in [33, §2] from which we take most of our notation. In particular we let $\mathbf{K}$ be the 27-dimensional complex vector space consisting of triples $m = (m_1, m_2, m_3)$ of complex 3×3-matrices m_i, $1 \le i \le 3$ where addition and scalar multiplication is defined coordinatewise. We define a cubic form $\langle \cdot \rangle$ on $\mathbf{K}$ by

$$\langle m \rangle = \det(m_1) + \det(m_2) + \det(m_3) - \mathrm{tr}(m_1 m_2 m_3).$$

Then $3E_6(\mathbf{C})$ is the subgroup of $\mathrm{GL}(\mathbf{K})$ preserving the form $\langle \cdot \rangle$. Moreover the stabilizer in $3E_6(\mathbf{C})$ of the element $(I_3, 0, 0) \in \mathbf{K}$ is the group $F_4(\mathbf{C})$. For $g_1, g_2, g_3 \in \mathrm{SL}_3(\mathbf{C})$ we have the element s_{g_1,g_2,g_3} of $3E_6(\mathbf{C})$ given by

$$s_{g_1,g_2,g_3}(m_1, m_2, m_3) = \left(g_1 m_1 g_2^{-1}, g_2 m_2 g_3^{-1}, g_3 m_3 g_1^{-1}\right)$$

for $m = (m_1, m_2, m_3) \in \mathbf{K}$. This gives a representation of $\mathrm{SL}_3(\mathbf{C})^3$ which has kernel C_3 generated by $(\omega I_3, \omega I_3, \omega I_3)$, and we thus get an embedding of $\mathrm{SL}_3(\mathbf{C})^3/C_3$ in $3E_6(\mathbf{C})$. We will denote the element s_{g_1,g_2,g_3} by $[g_1, g_2, g_3]$.

We let $\{e^i_{j,k}\}$, $1 \le i, j, k \le 3$ be the natural basis of $\mathbf{K}$ consisting of the elements $e^i_{j,k}$ whose entries are all 0 except for the (j,k)-entry of the i'th matrix which equals 1. The elements of $3E_6(\mathbf{C})$ which acts diagonally with respect to this basis of $\mathbf{K}$ form a maximal torus H in $3E_6(\mathbf{C})$. Let $m_i^{j,k}$ denote the (j,k)-entry of the matrix m_i. We then have H-invariant subgroups

$$u_{\alpha_1}(t) = [I_3, I_3 + te_{1,3}, I_3], \qquad u_{-\alpha_1}(t) = [I_3, I_3 + te_{3,1}, I_3],$$
$$u_{\alpha_2}(t) = [I_3 + te_{2,1}, I_3, I_3], \qquad u_{-\alpha_2}(t) = [I_3 + te_{1,2}, I_3, I_3],$$
$$u_{\alpha_3}(t) = [I_3, I_3 + te_{2,1}, I_3], \qquad u_{-\alpha_3}(t) = [I_3, I_3 + te_{1,2}, I_3],$$

$$u_{\alpha_4}(t) : (m_i)_{i=1,2,3} \mapsto \left(m_i + t \cdot \begin{bmatrix} 0 & -m_{i+2}^{2,3} & 0 \\ 0 & 0 & 0 \\ 0 & m_{i+2}^{2,1} & 0 \end{bmatrix} \right)_{i=1,2,3}$$

$$u_{-\alpha_4}(t) : (m_i)_{i=1,2,3} \mapsto \left(m_i + t \cdot \begin{bmatrix} 0 & 0 & 0 \\ m_{i+1}^{3,2} & 0 & -m_{i+1}^{1,2} \\ 0 & 0 & 0 \end{bmatrix} \right)_{i=1,2,3}$$

$$u_{\alpha_5}(t) = [I_3, I_3, I_3 + te_{2,1}], \qquad u_{-\alpha_5}(t) = [I_3, I_3, I_3 + te_{1,2}],$$
$$u_{\alpha_6}(t) = [I_3, I_3, I_3 + te_{1,3}], \qquad u_{-\alpha_6}(t) = [I_3, I_3, I_3 + te_{3,1}].$$

Here, in the description of $u_{\pm\alpha_4}(t)$, the m_i's should be counted cyclicly mod 3, e.g. $m_{i+2} = m_1$ for $i = 2$.

The associated roots α_i, $1 \le i \le 6$, of these root subgroups form a simple system in the root system $\Phi(E_6)$ of $3E_6(\mathbf{C})$ (our numbering agrees with [15, p. 260–262]). For this simple system, the highest weight of $\mathbf{K}$ is λ_1. Furthermore the root subgroups $u_{\pm\alpha_i}$, $1 \le i \le 6$, have been chosen so that they satisfy [122, 8.1.1(i) and 8.1.4(i)], i.e. they form part of a *realization* ([122, p. 133]) of $\Phi(E_6)$ in $3E_6(\mathbf{C})$. For $\alpha = \pm\alpha_i$, $1 \le i \le 6$, and $t \in \mathbf{C}^\times$, we may then define the elements

$$n_\alpha(t) = u_\alpha(t) u_{-\alpha}(-1/t) u_\alpha(t), \qquad h_\alpha(t) = n_\alpha(t) n_\alpha(1)^{-1}.$$

Then the maximal torus consists of the elements $h(t_1,t_2,t_3,t_4,t_5,t_6) = \prod_{i=1}^6 h_{\alpha_i}(t_i)$ and the normalizer $N(H)$ of the maximal torus is generated by H and the elements $n_i = n_{\alpha_i}(1)$, $1 \le i \le 6$. It should be noted that this notation differs from the one used in [33]. More precisely, the element $h(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$ in [33] is $h(\delta,\alpha^{-1},\gamma^{-1},\beta,\varepsilon^{-1},\zeta)$ in our notation, and the elements n_1, n_2, n_3, n_4, n_5 and n_6 in [33] equal $n_1 h_{\alpha_1}(-1) h_{\alpha_3}(-1)$, $n_2 h(-1,1,1,-1,1,-1)$, $n_3 h_{\alpha_1}(-1)$, n_4, $n_5 h_{\alpha_6}(-1)$ and $n_6 h_{\alpha_5}(-1) h_{\alpha_6}(-1)$ respectively in our notation.

From the description of the root system of type E_6 in [15, p. 260–262] we see that the center Z of $3E_6(\mathbf{C})$ is cyclic of order 3 and is generated by the element $z = \left[I_3, \omega^2 I_3, \omega I_3\right]$. We consider also the element $a = [\omega I_3, I_3, I_3]$. A straightforward computation shows that the roots of the centralizer $C_{3E_6(\mathbf{C})}(a)$ are

$$\{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm\alpha_5, \pm\alpha_6, \pm\widetilde{\alpha}, \pm(\alpha_1+\alpha_3), \pm(\alpha_5+\alpha_6), \pm(\alpha_2-\widetilde{\alpha})\},$$

where $\widetilde{\alpha}$ is the longest root. The Dynkin diagram for this centralizer is the same as the extended Dynkin diagram for E_6 with the node α_4 removed. In particular it has type $A_2A_2A_2$ and a simple system of roots is given by $\{\alpha_1, \alpha_3, \alpha_5, \alpha_6, \alpha_2, -\widetilde{\alpha}\}$. Since $3E_6(\mathbf{C})$ is simply connected, Theorem 8.2(3) implies that the centralizer $C_{3E_6(\mathbf{C})}(a)$ is connected, and thus it is generated by the maximal torus H and the root subgroups $u_{\pm\alpha}(t)$ where α runs through the simple roots $\{\alpha_1, \alpha_3, \alpha_5, \alpha_6, \alpha_2, -\widetilde{\alpha}\}$. Now note that $u_{\widetilde{\alpha}}(t) = [I_3 + te_{3,1}, I_3, I_3]$

and $u_{-\widetilde{\alpha}}(t) = [I_3 + te_{1,3}, I_3, I_3]$ are root subgroups with associated roots $\widetilde{\alpha}$ and $-\widetilde{\alpha}$ respectively. Since these along with H and the root subgroups $u_{\pm\alpha_1}$, $u_{\pm\alpha_2}$, $u_{\pm\alpha_3}$, $u_{\pm\alpha_5}$ and $u_{\pm\alpha_6}$ generate the subgroup $\mathrm{SL}_3(\mathbf{C})^3/C_3$ of $3E_6(\mathbf{C})$ from above, we conclude that $C_{3E_6(\mathbf{C})}(a) = \mathrm{SL}_3(\mathbf{C})^3/C_3$.

To describe the conjugacy classes of elementary abelian 3-subgroups we need to introduce some more elements. Consider the following elements in $\mathrm{SL}_3(\mathbf{C})^3/C_3 \subseteq 3E_6(\mathbf{C})$:

$$x_1 = [I_3, \beta, \beta], \qquad x_2 = [\beta, \beta, \beta], \qquad y_1 = \left[I_3, \gamma, \gamma^2\right], \qquad y_2 = [\gamma, \gamma, \gamma].$$

We also need the following elements in $N(H)$:

$$\begin{aligned} s_1 &= n_1n_3n_4n_2n_5n_4n_3n_1n_6n_5n_4n_2n_3n_4n_5n_6, \\ s_2 &= n_1n_2n_3n_1n_4n_2n_3n_1n_4n_3n_5n_4n_2n_3n_1n_4n_3n_5n_4n_2n_6n_5n_4n_2n_3n_1n_4 \cdot \\ &\quad n_3n_5n_4n_2n_6n_5n_4n_3n_1 \end{aligned}$$

The action of these elements are as follows:

$$s_1(m_1, m_2, m_3) = (m_3, m_1, m_2), \qquad s_2(m_1, m_2, m_3) = (m_3^{\mathtt{T}}, m_2^{\mathtt{T}}, m_1^{\mathtt{T}}),$$

where $m_i^{\mathtt{T}}$ denotes the transpose of m_i. Thus these elements acts by conjugation on the subgroup $\mathrm{SL}_3(\mathbf{C})^3/C_3$ as follows

$$[g_1, g_2, g_3]^{s_1} = [g_2, g_3, g_1], \qquad [g_1, g_2, g_3]^{s_2} = \left[\left(g_1^{-1}\right)^{\mathtt{T}}, \left(g_3^{-1}\right)^{\mathtt{T}}, \left(g_2^{-1}\right)^{\mathtt{T}}\right].$$

Lemma 8.5. *We have*

$$\begin{aligned} z &= h(\omega, 1, \omega^2, 1, \omega, \omega^2), \quad a = h(\omega, 1, \omega^2, 1, \omega^2, \omega), \quad x_1 = h(\omega, 1, \omega, 1, \omega, \omega), \\ x_2 &= h(1, \omega^2, \omega^2, 1, \omega^2, 1), \quad y_1 = n_1n_3n_5n_6h_{\alpha_5}(-1), \\ y_2 &= n_1n_2n_3n_4n_3n_1n_5n_4n_2n_3n_4n_5n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_6n_5h_{\alpha_2}(-1). \end{aligned}$$

Moreover conjugation by the element

$$n_1n_4n_2n_3n_1n_4n_5n_4n_6n_5n_4n_2n_3n_1n_4 \cdot h_{\alpha_2}(-1)h_{\alpha_4}(-1)$$

acts as follows:

$$a \mapsto x_2, \qquad x_2 \mapsto a, \qquad y_1 \mapsto s_1, \qquad y_2 \mapsto y_2^2, \qquad x_2x_1^{-1} \mapsto h_{\alpha_4}(\omega) = [\tau_2, \tau_2, \tau_2].$$

Proof. Both parts of the lemma may be checked by direct computation. The second part also follows from the first by using the following relations in $N(H)$: The element n_i has image s_{α_i} in W ([122, 8.1.4(i)]), we have $n_i^2 = h_{\alpha_i}(-1)$ ([122, 8.1.4(ii)]) and

$$n_in_jn_i \ldots = n_jn_in_j \ldots$$

for $1 \le i, j \le 6$, where the number of factors on both sides equals the order of $s_{\alpha_i}s_{\alpha_j}$ in W ([122, 9.3.2]). □

Notation 8.6. For our calculations, we need some information on the conjugacy classes of elements of order 3 in $3E_6(\mathbf{C})$. These are given in [33, Table 2]: There are 7 such conjugacy classes, which we label **3A**, **3B**, **3B′**, **3C**, **3D**, **3E** and **3E′**, where **3B′** and **3E′** denotes the inverses of the classes **3B** and **3E**. This notation is almost identical to the notation in [33], but differs from [65]. We will need the following, which follows quickly from [33, Table 2] using the action of W on H: We have $z \in$ **3E**, $a, x_2, y_2 \in$ **3C**, $x_1, y_1 \in$ **3D** and $x_2x_1^{-1} \in$ **3A**. Multiplication by z acts as follows on the conjugacy classes:

$$\mathbf{3A} \mapsto \mathbf{3B},\ \mathbf{3B} \mapsto \mathbf{3B'},\ \mathbf{3B'} \mapsto \mathbf{3A},\ \mathbf{3C} \mapsto \mathbf{3C},\ \mathbf{3D} \mapsto \mathbf{3D},\ \mathbf{3E} \mapsto \mathbf{3E'},\ \mathbf{3E'} \mapsto \mathbf{1},$$

where **1** denotes the conjugacy class consisting of the identity element.

Theorem 8.7. *The conjugacy classes of non-toral elementary abelian* 3*-subgroups of* $3E_6(\mathbf{C})$ *are given by the following table.*

rank	*name*	*ordered basis*	$3E_6(\mathbf{C})$*-class distribution*	$C_{3E_6(\mathbf{C})}(E)$
3	$E^3_{3E_6}$	$\langle a, x_2, y_2\rangle$	$\mathbf{3C^{26}}$	$E^4_{3E_6}$
4	$E^4_{3E_6}$	$\langle z, a, x_2, y_2\rangle$	$\mathbf{3C^{78}3E^{1}3E'^{1}}$	$E^4_{3E_6}$

Their Weyl groups with respect to the given ordered bases are as follows:

$$W(E^3_{3E_6}) = \mathrm{SL}_3(\mathbf{F}_3),\ \ W(E^4_{3E_6}) = \left[\begin{array}{c|ccc} 1 & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbf{F}_3) & \\ 0 & & & \end{array}\right]$$

Proof. Non-toral subgroups: By [65, Thm. 11.13], there are two conjugacy classes of non-toral elementary abelian 3-subgroups in $3E_6(\mathbf{C})$, one non-maximal of rank three and one maximal of rank 4. We may concretely realize these as follows. Consider the subgroups

$$E^3_{3E_6} = \langle a, x_2, y_2\rangle \text{ and } E^4_{3E_6} = \langle z, a, x_2, y_2\rangle\,,$$

which are readily seen to be elementary abelian 3-subgroups of rank 3 and 4 respectively. In particular both groups are subsets of $C_{3E_6(\mathbf{C})}(a) = \mathrm{SL}_3(\mathbf{C})^3/C_3$, and since $\beta, \gamma \in \mathrm{SL}_3(\mathbf{C})$ does not commute, we see that the preimages of $E^3_{3E_6}$ and $E^4_{3E_6}$ under the projection $\mathrm{SL}_3(\mathbf{C})^3 \to \mathrm{SL}_3(\mathbf{C})^3/C_3$ are non-abelian. Thus by Theorem 8.2(5) $E^3_{3E_6}$ and $E^4_{3E_6}$ are non-toral in $\mathrm{SL}_3(\mathbf{C})^3/C_3 = C_{3E_6(\mathbf{C})}(a)$ and hence also non-toral in $3E_6(\mathbf{C})$ by Theorem 8.2(1). Thus by the above these two groups represent the conjugacy classes of non-toral elementary abelian 3-subgroups in $3E_6(\mathbf{C})$.

Lower bounds for Weyl groups: By [65, Thm. 7.4] there is a unique non-toral elementary abelian 3-subgroup of $F_4(\mathbf{C})$ of rank 3. Since we have an inclusion $F_4(\mathbf{C}) \subseteq 3E_6(\mathbf{C})$ this subgroup may also be considered as a subgroup of $3E_6(\mathbf{C})$. As its Weyl group in $F_4(\mathbf{C})$ is $\mathrm{SL}_3(\mathbf{F}_3)$, its Weyl group in $3E_6(\mathbf{C})$ must contain $\mathrm{SL}_3(\mathbf{F}_3)$. In particular it has order divisible by 13 and since $13 \nmid |W(E_6)|$, we conclude by Theorem 8.2(2) that E is non-toral in $3E_6(\mathbf{C})$ as well. Thus by the above E must be conjugate to $E^3_{3E_6}$, and we get that $W(E^3_{3E_6})$ contains $\mathrm{SL}_3(\mathbf{F}_3)$. From this we immediately see that $W(E^4_{3E_6})$ contains the group

$$\left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbf{F}_3) & \\ 0 & & & \end{array}\right]$$

Note that the element $\left[I_3, \beta, \beta^2\right]$ commutes with z, a and x_2 and conjugates y_2 to $y_2 z$. Thus it normalizes $E^4_{3E_6}$ and produces the element $I_4 + e_{1,4}$ in $W(E^4_{3E_6})$. As a result we see that $W(E^4_{3E_6})$ contains the group

$$\left[\begin{array}{c|ccc} 1 & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbf{F}_3) & \\ 0 & & & \end{array}\right]$$

Class distributions: Since $a \in \mathbf{3C}$ by 8.6 and $W(E^3_{3E_6})$ contains $\mathrm{SL}_3(\mathbf{F}_3)$ which acts transitively on $E^3_{3E_6} - \{1\}$, the class distribution of $E^3_{3E_6}$ follows immediately. Using this and the information given in 8.6 about multiplication by z, the class distribution of $E^4_{3E_6}$ is easily found.

Centralizers: We have already seen that $C_{3E_6(\mathbf{C})}(a) = \mathrm{SL}_3(\mathbf{C})^3/C_3$. From this we directly get

$$C_{3E_6(\mathbf{C})}(a, x_2) = C_{\mathrm{SL}_3(\mathbf{C})^3/C_3}(x_2) = \langle y_2, (\mathbf{T}_2 \times \mathbf{T}_2 \times \mathbf{T}_2)/C_3 \rangle,$$
$$C_{3E_6(\mathbf{C})}(a, x_2, y_2) = \langle x_2, y_2, (\langle \omega I_3\rangle \times \langle \omega I_3\rangle \times \langle \omega I_3\rangle)/C_3 \rangle = E^4_{3E_6},$$

proving that $C_{3E_6(\mathbf{C})}(E^3_{3E_6}) = C_{3E_6(\mathbf{C})}(E^4_{3E_6}) = E^4_{3E_6}$.

Exact Weyl groups: From the lower bounds above and the fact that z is central we get $\mathrm{SL}_3(\mathbf{F}_3) \subseteq W(E^3_{3E_6}) \subseteq \mathrm{GL}_3(\mathbf{F}_3)$ and

$$\left[\begin{array}{c|ccc} 1 & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbf{F}_3) & \\ 0 & & & \end{array}\right] \subseteq W(E^4_{3E_6}) \subseteq \left[\begin{array}{c|ccc} 1 & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{GL}_3(\mathbf{F}_3) & \\ 0 & & & \end{array}\right]$$

As $C_{3E_6(\mathbf{C})}(a, x_2) = \langle y_2, (\mathbf{T}_2 \times \mathbf{T}_2 \times \mathbf{T}_2)/C_3 \rangle$, we see that no element in $C_{3E_6(\mathbf{C})}(a, x_2)$ conjugates y_2 to y_2^{-1}. Hence $\mathrm{diag}(1,1,2) \notin W(E^3_{3E_6})$ and $\mathrm{diag}(1,1,1,2) \notin W(E^4_{3E_6})$ which shows that Weyl groups are the ones given in the theorem. □

We now turn to the group $E_6(\mathbf{C})$. As above we let Z be the center of $3E_6(\mathbf{C})$ and we let $\pi : 3E_6(\mathbf{C}) \to E_6(\mathbf{C}) = 3E_6(\mathbf{C})/Z$ denote the projection. For $g \in 3E_6(\mathbf{C})$ we write $\overline{g}$ instead of $\pi(g)$ and similarly we let $\overline{S} = \pi(S)$ for a subset $S \subseteq 3E_6(\mathbf{C})$.

Lemma 8.8. *Let E be a rank 2 non-toral elementary abelian 3-subgroup of $E_6(\mathbf{C})$. Then the Weyl group $W(E)$ is a subgroup of $\mathrm{SL}_2(\mathbf{F}_3)$.*

Proof. Let $E = \langle \overline{g_1}, \overline{g_2} \rangle$. By Theorem 8.2 part (5) and (3) the group $\langle g_1, g_2 \rangle \subseteq 3E_6(\mathbf{C})$ is non-abelian. Thus setting $z' = [g_1, g_2] \in Z$ we have $z' \neq 1$. Assume that $\sigma \in W(E)$ is represented by the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, i.e. we have $\sigma(\overline{g_1}) = (\overline{g_1})^{a_{11}}(\overline{g_2})^{a_{21}}$ and $\sigma(\overline{g_2}) = (\overline{g_1})^{a_{12}}(\overline{g_2})^{a_{22}}$. Since σ is given by a conjugation in $E_6(\mathbf{C})$, it lifts to a conjugation in $3E_6(\mathbf{C})$. Now the relation $[g_1, g_2] = z' \in Z$ shows $(z')^{a_{11}\cdot a_{22} - a_{12}\cdot a_{21}} = z'$, so since $z' \neq 1$ we have $\sigma \in \mathrm{SL}_2(\mathbf{F}_3)$. □

Theorem 8.9. *The conjugacy classes of non-toral elementary abelian 3-subgroups of $E_6(\mathbf{C})$ are given by the following table:*

rank	*name*	*ordered basis*	$3E_6(\mathbf{C})$*-class distribution*	$C_{E_6(\mathbf{C})}(E)$	$Z(C_{E_6(\mathbf{C})}(E))$
2	$E^{2a}_{E_6}$	$\langle \overline{y_1}, \overline{x_2} \rangle$	$\mathbf{3C^{18}3D^{6}3E^{1}3E'^{1}}$	$E^{2a}_{E_6} \times \mathrm{PSL}_3(\mathbf{C})$	$E^{2a}_{E_6}$
2	$E^{2b}_{E_6}$	$\langle \overline{y_1}, \overline{x_1} \rangle$	$\mathbf{3D^{24}3E^{1}3E'^{1}}$	$E^{2b}_{E_6} \times G_2(\mathbf{C})$	$E^{2b}_{E_6}$
3	$E^{3a}_{E_6}$	$\langle \overline{a}, \overline{y_1}, \overline{x_2} \rangle$	$\mathbf{3C^{60}3D^{18}3E^{1}3E'^{1}}$	$E^{3a}_{E_6} \circ_{\langle \overline{a} \rangle} (\mathbf{T}_2 : \langle \overline{y_2} \rangle)$	$E^{3a}_{E_6}$
3	$E^{3b}_{E_6}$	$\langle \overline{a}, \overline{x_2}, \overline{y_2} \rangle$	$\mathbf{3C^{78}3E^{1}3E'^{1}}$	$E^{3b}_{E_6} \cdot 3^3$	$E^{3b}_{E_6}$
3	$E^{3c}_{E_6}$	$\langle \overline{a}, \overline{y_1}, \overline{x_1} \rangle$	$\mathbf{3C^{6}3D^{72}3E^{1}3E'^{1}}$	$E^{3c}_{E_6} \circ_{\langle \overline{a} \rangle} \mathrm{SL}_3(\mathbf{C})$	$E^{3c}_{E_6}$
3	$E^{3d}_{E_6}$	$\left\langle \overline{x_2 x_1^{-1}}, \overline{y_1}, \overline{x_1} \right\rangle$	$\mathbf{3A^{2}3B^{2}3B'^{2}3C^{48}3D^{24}3E^{1}3E'^{1}}$	$E^{3d}_{E_6} \circ_{\left\langle \overline{x_2 x_1^{-1}} \right\rangle} \mathrm{GL}_2(\mathbf{C})$	$E^{3d}_{E_6} \circ_{\left\langle \overline{x_2 x_1^{-1}} \right\rangle} \mathbf{T}_1$
4	$E^{4a}_{E_6}$	$\langle \overline{a}, \overline{y_2}, \overline{y_1}, \overline{x_2} \rangle$	$\mathbf{3C^{186}3D^{54}3E^{1}3E'^{1}}$	$E^{4a}_{E_6}$	$E^{4a}_{E_6}$
4	$E^{4b}_{E_6}$	$\left\langle \overline{a}, \overline{x_2 x_1^{-1}}, \overline{y_1}, \overline{x_1} \right\rangle$	$\mathbf{3A^{6}3B^{6}3B'^{6}3C^{150}3D^{72}3E^{1}3E'^{1}}$	$E^{4b}_{E_6} \circ_{\left\langle \overline{a}, \overline{x_2 x_1^{-1}} \right\rangle} \mathbf{T}_2$	$E^{4b}_{E_6} \circ_{\left\langle \overline{a}, \overline{x_2 x_1^{-1}} \right\rangle} \mathbf{T}_2$

In particular we have ${}_3Z(C_{E_6(\mathbf{C})}(E)) = E$ for any non-toral elementary abelian 3-subgroup of $E_6(\mathbf{C})$. (In the table the $3E_6(\mathbf{C})$-class distribution of $E \subseteq E_6(\mathbf{C})$ denotes the class distribution of $\pi^{-1}(E) \subseteq 3E_6(\mathbf{C})$.)

The Weyl groups of these groups with respect to the given ordered bases are given as follows:

$$W(E^{2a}_{E_6}) = \begin{bmatrix} \varepsilon & * \\ 0 & \varepsilon \end{bmatrix}, \ W(E^{2b}_{E_6}) = \mathrm{SL}_2(\mathbf{F}_3), \ W(E^{3a}_{E_6}) = \begin{bmatrix} \varepsilon_1 & * & * \\ 0 & \varepsilon_2 & * \\ 0 & 0 & \varepsilon_2 \end{bmatrix}$$

$$W(E^{3b}_{E_6}) = \mathrm{SL}_3(\mathbf{F}_3), \ W(E^{3c}_{E_6}) = \left[\begin{array}{c|cc} \varepsilon & * & * \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathrm{SL}_2(\mathbf{F}_3)$}} \\ 0 & & \end{array}\right], \ W(E^{3d}_{E_6}) = \left[\begin{array}{c|cc} \varepsilon & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathrm{SL}_2(\mathbf{F}_3)$}} \\ 0 & & \end{array}\right]$$

$$W(E^{4a}_{E_6}) = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$\mathrm{GL}_2(\mathbf{F}_3)$}} & * & * \\ & & * & * \\ \hline 0 & 0 & \det & * \\ 0 & 0 & 0 & \det \end{array}\right], \ W(E^{4b}_{E_6}) = \left[\begin{array}{cc|cc} \varepsilon_1 & * & * & * \\ 0 & \varepsilon_2 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathrm{SL}_2(\mathbf{F}_3)$}} \\ 0 & 0 & & \end{array}\right]$$

where $\det$ *denotes the determinant of the matrix from* $\mathrm{GL}_2(\mathbf{F}_3)$ *in the description of* $W(E^{4a}_{E_6})$.

Proof. Maximal non-toral subgroups: By [65, Thm. 11.14], there are two conjugacy classes of maximal non-toral elementary abelian 3-subgroups in $E_6(\mathbf{C})$, both of rank 4. We may concretely realize these as follows. Consider the subgroups

$$E_a = \langle z, a, y_1, y_2, x_2 \rangle \text{ and } E_b = \left\langle z, a, x_2 x_1^{-1}, y_1, x_1 \right\rangle$$

of $C_{3E_6(\mathbf{C})}(a) = \mathrm{SL}_3(\mathbf{C})^3/C_3$. Since the commutator subgroup of both of these is Z, we see that $E^{4a}_{E_6} = \pi(E_a)$ and $E^{4b}_{E_6} = \pi(E_b)$ are elementary abelian 3-subgroups of rank 4 in $E_6(\mathbf{C})$. It follows from Theorem 8.2(5) that both $E^{4a}_{E_6}$ and $E^{4b}_{E_6}$ are non-toral in $E_6(\mathbf{C})$. We will see below that their class distributions are as given in the table. From this it follows that they are not conjugate and thus represents the two conjugacy classes of maximal elementary abelian 3-subgroups in $E_6(\mathbf{C})$.

Lower bounds for Weyl groups of maximal non-toral subgroups: We can find lower bounds for the Weyl groups of the maximal non-toral elementary abelian 3-subgroups by conjugating with elements coming from the centralizer $C_{3E_6(\mathbf{C})}(a) = \mathrm{SL}_3(\mathbf{C})^3/C_3$ and the normalizer $N(H)$ of the maximal torus.

The elements $\overline{[\beta^2, I_3, I_3]}$, $\overline{[I_3, \tau_1, \tau_1^2]}$, $\overline{s_1}$ and $\overline{s_2}$ normalize $E^{4a}_{E_6}$ and conjugation by these elements induce the automorphisms on $E^{4a}_{E_6}$ given by the matrices $I_4 + e_{1,2}$, $I_4 + e_{3,4}$, $I_4 + e_{2,3}$ and $\mathrm{diag}(2, 1, 2, 2)$. Moreover, by Lemma 8.5 we may conjugate the ordered basis of $E^{4a}_{E_6}$ into the ordered basis $\left\langle \overline{x_2}, \overline{y_2^2}, \overline{s_1}, \overline{a} \right\rangle$. Noting that the element $\overline{[\tau_1, \tau_1, \tau_1]}$ commutes with $\overline{y_2}$, $\overline{s_1}$ and $\overline{a}$ and conjugates $\overline{x_2}$ into $\overline{x_2 y_2}$, we see that $W(E^{4a}_{E_6})$ contains the element $I_4 + e_{2,1}$. The above matrices are easily seen to generate the group

$$W'(E^{4a}_{E_6}) = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$\mathrm{GL}_2(\mathbf{F}_3)$}} & * & * \\ & & * & * \\ \hline 0 & 0 & \det & * \\ 0 & 0 & 0 & \det \end{array}\right]$$

and thus $W^{4a}_{E_6}$ contains this group.

Now consider $E_{E_6}^{4b}$ and let $\sigma = -(2,3) \in \mathrm{SL}_3(\mathbf{C})$. We then see that the elements $\overline{[I_3,\tau_1,\tau_1^2]}$, $\overline{[I_3,\tau_2\beta,\tau_2^2]}$, $\overline{[\sigma,I_3,I_3]}$, $\overline{[\gamma,I_3,I_3]}$, $\overline{[I_3,\beta^2,I_3]}$ and $\overline{s_2}$ normalize $E_{E_6}^{4b}$, and conjugation by these elements induce the automorphisms on $E_{E_6}^{4b}$ given by the matrices $I_4+e_{3,4}$, $I_4+e_{4,3}$, $\mathrm{diag}(1,2,1,1)$, $I_4+e_{1,2}$, $I_4+e_{1,3}$ and $-I_4$. These matrices generate the group

$$W'(E_{E_6}^{4b}) = \left[\begin{array}{cc|cc} \varepsilon_1 & * & * & * \\ 0 & \varepsilon_2 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{\mathrm{SL}_2(\mathbf{F}_3)}} \\ 0 & 0 & & \end{array}\right]$$

and thus $W_{E_6}^{4b}$ contains this group.

Orbit computation: Any elementary abelian 3-subgroup of rank 1 is toral since $E_6(\mathbf{C})$ is connected. As we already know that $E_{E_6}^{4a}$ and $E_{E_6}^{4b}$ are representatives of the maximal non-toral elementary abelian 3-subgroups, we may find the conjugacy classes of non-toral elementary abelian 3-subgroups of rank 2 and 3 by studying subgroups of these.

Under the action of $W'(E_{E_6}^{4a})$, the set of rank 2 subgroups of $E_{E_6}^{4a}$ has orbit representatives

$$E_{E_6}^{2a} = \langle\overline{y_1},\overline{x_2}\rangle\,, \langle\overline{a},\overline{x_2}\rangle\,, \langle\overline{a},\overline{y_1}\rangle \text{ and } \langle\overline{a},\overline{y_2}\rangle\,,$$

and under the action of $W'(E_{E_6}^{4b})$, the set of rank 2 subgroups of $E_{E_6}^{4b}$ has orbit representatives

$$E_{E_6}^{2a} = \langle\overline{y_1},\overline{x_2}\rangle\,, E_{E_6}^{2b} = \langle\overline{y_1},\overline{x_1}\rangle\,, \langle\overline{a},\overline{x_2}\rangle\,, \langle\overline{a},\overline{y_1}\rangle\,, \left\langle\overline{a},\overline{x_2x_1^{-1}}\right\rangle \text{ and } \left\langle\overline{x_2x_1^{-1}},\overline{x_1}\right\rangle.$$

Similarly we find that under the action of $W'(E_{E_6}^{4a})$, the set of rank 3 subgroups of $E_{E_6}^{4a}$ has orbit representatives

$$E_{E_6}^{3a} = \langle\overline{a},\overline{y_1},\overline{x_2}\rangle\,, E_{E_6}^{3b} = \langle\overline{a},\overline{y_2},\overline{x_2}\rangle \text{ and } \langle\overline{a},\overline{y_1},\overline{y_2}\rangle\,,$$

and that under the action of $W'(E_{E_6}^{4b})$, the set of rank 3 subgroups of $E_{E_6}^{4b}$ has orbit representatives

$$E_{E_6}^{3a} = \langle\overline{a},\overline{y_1},\overline{x_2}\rangle\,, E_{E_6}^{3c} = \langle\overline{a},\overline{y_1},\overline{x_1}\rangle\,, E_{E_6}^{3d} = \left\langle\overline{x_2x_1^{-1}},\overline{y_1},\overline{x_1}\right\rangle \text{ and } \left\langle\overline{a},\overline{x_2x_1^{-1}},\overline{x_1}\right\rangle.$$

Other non-toral subgroups: We see directly that the subgroups $\langle\overline{a},\overline{x_2}\rangle$, $\left\langle\overline{a},\overline{x_2x_1^{-1}}\right\rangle$, $\left\langle\overline{x_2x_1^{-1}},\overline{x_1}\right\rangle$ and $\left\langle\overline{a},\overline{x_2x_1^{-1}},\overline{x_1}\right\rangle$ are toral. Noting that the elements β and γ are conjugate in $\mathrm{SL}_3(\mathbf{C})$ we see that the group $\langle\overline{a},\overline{y_1},\overline{y_2}\rangle$ is conjugate to the group $\left\langle\overline{a},\overline{[I_3,\beta,\beta^2]},\overline{x_2}\right\rangle$ which is obviously toral. Thus we see that the groups $\langle\overline{a},\overline{y_1},\overline{y_2}\rangle$, $\langle\overline{a},\overline{y_1}\rangle$ and $\langle\overline{a},\overline{y_2}\rangle$ are also toral. Using the fact that $[y_1,x_1]=[y_1,x_2]=z$ we see from Theorem 8.2(5) that both $E_{E_6}^{2a}$ and $E_{E_6}^{2b}$ are non-toral in $E_6(\mathbf{C})$. Since the groups $E_{E_6}^{3a}$, $E_{E_6}^{3c}$ and $E_{E_6}^{3d}$ all contain either $E_{E_6}^{2a}$ or $E_{E_6}^{2b}$ they are also non-toral. Using Theorem 8.2(5) we see that the group $E_{E_6}^{3b}$ is non-toral in $E_6(\mathbf{C})$, since we know that $\pi^{-1}(E_{E_6}^{3b}) = E_{3E_6}^{4}$ is non-toral in $3E_6(\mathbf{C})$ by Theorem 8.7.

Class distributions: Using 8.6 and the action of the groups $W'(E_{E_6}^{4a})$ and $W'(E_{E_6}^{4b})$ it is not hard to verify the class distributions in the table. As an example consider the group $E_{E_6}^{4b}$. From the action of $W'(E_{E_6}^{4b})$ we see that $E_{E_6}^{4b}-\{1\}$ contains 2 elements conjugate to $\overline{a}$, 6 elements conjugate to $\overline{x_2x_1^{-1}}$, 24 elements conjugate to $\overline{x_1}$ and 48 elements conjugate to $\overline{x_2}$. Thus by 8.6, the set $\pi^{-1}(E_{E_6}^{4b}-\{1\})$ contains 6 elements from each of the classes **3A**, **3B** and **3B**$'$, $3\cdot(2+48)=150$ elements from the class **3C** and $3\cdot 24=72$ elements from the class **3D**. Including the elements z and z^2 from the classes **3E** and **3E**$'$ respectively, we get the class distribution of $\pi^{-1}(E_{E_6}^{4b})-\{1\}$ given in the table. Similar computations give the remaining entries in the table. Since these distributions are different we see that the groups in the table are not conjugate and thus they provide a set of representatives for the conjugacy classes of non-toral elementary abelian 3-subgroups of $E_6(\mathbf{C})$.

Lower bounds for other Weyl groups: We now show that the other matrix groups in the table are all lower bounds for the remaining Weyl groups. To do this consider one of the non-maximal groups E from the table. We then have $E \subseteq E^{4a}_{E_6}$ or $E \subseteq E^{4b}_{E_6}$, and we get a lower bound on $W(E)$ by considering the action on E of the subgroup of $W'(E^{4a}_{E_6})$ or $W'(E^{4b}_{E_6})$ fixing E. As an example we see that $E^{2a}_{E_6} \subseteq E^{4a}_{E_6}$ and that the stabilizer of $E^{2a}_{E_6}$ inside $W'(E^{4a}_{E_6})$ is

$$\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{\mathrm{GL}_2(\mathbf{F}_3)}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & \det & x \\ 0 & 0 & 0 & \det \end{array}\right]$$

where det is the determinant of the matrix from $\mathrm{GL}_2(\mathbf{F}_3)$. The action of such a matrix on $E^{2a}_{E_6}$ is given by

$$\overline{y_1} \mapsto (\overline{y_1})^{\det}, \qquad \overline{x_2} \mapsto (\overline{y_1})^x(\overline{x_2})^{\det}.$$

Thus $W(E^{2a}_{E_6})$ contains the group

$$W'(E^{2a}_{E_6}) = \begin{bmatrix} \varepsilon & * \\ 0 & \varepsilon \end{bmatrix}$$

as claimed. Similar computations show that for the groups $E = E^{2b}_{E_6}$, $E^{3a}_{E_6}$, $E^{3c}_{E_6}$ and $E^{3d}_{E_6}$, the group $W'(E)$ occurring in the theorem is a lower bound for the Weyl group $W(E)$.

For the group $E^{3b}_{E_6} = \langle \overline{a}, \overline{x_2}, \overline{y_2} \rangle$ we know the structure of $W(\pi^{-1}(E^{3b}_{E_6})) = W(E^4_{3E_6})$ by Theorem 8.7. From this we immediately get $W(E^{3b}_{E_6}) = \mathrm{SL}_3(\mathbf{F}_3)$.

Exact Weyl groups: We now prove that the lower bounds on the Weyl groups established above are in fact equalities. By Lemma 8.8 the Weyl groups $W(E^{2a}_{E_6})$ and $W(E^{2b}_{E_6})$ are subgroups of $\mathrm{SL}_2(\mathbf{F}_3)$. From this we see that $W(E^{2b}_{E_6}) = \mathrm{SL}_2(\mathbf{F}_3)$ and that $W(E^{2a}_{E_6})$ is equal to either $W'(E^{2a}_{E_6})$ or $\mathrm{SL}_2(\mathbf{F}_3)$, since these are the only subgroups of $\mathrm{SL}_2(\mathbf{F}_3)$ containing $W'(E^{2a}_{E_6})$. We have $E^{2a}_{E_6} = \langle \overline{y_1}, \overline{x_2} \rangle$, and by 8.6 we see that the elements $\overline{y_1}$ and $\overline{x_2}$ are not conjugate in $E_6(\mathbf{C})$. In particular we see that $W(E^{2a}_{E_6})$ cannot act transitively on the non-trivial elements of $E^{2a}_{E_6}$, and we conclude that $W(E^{2a}_{E_6}) = W'(E^{2a}_{E_6})$ is the group from above.

For each of the remaining non-toral subgroups we now show that a strictly larger Weyl group contradicts the Weyl group results already established. The groups $E = E^{3a}_{E_6}$, $E^{3d}_{E_6}$, and $E^{4b}_{E_6}$ all contain $E^{2a}_{E_6}$. A direct computation shows that any proper overgroup of $W'(E)$ in $\mathrm{GL}(E)$ contains an element which normalizes the subgroup $E^{2a}_{E_6}$ and induces an automorphism which does not lie in $W(E^{2a}_{E_6})$. Hence $W(E) = W'(E)$. If $E = E^{3c}_{E_6}$ a similar argument, using the subgroup $E^{2b}_{E_6}$, again shows that $W(E) = W'(E)$. Consider finally $E = E^{4a}_{E_6}$. Each proper overgroup of $W'(E)$ contains an element which normalizes one of the subgroups $E^{2a}_{E_6}$ or $E^{3b}_{E_6}$ and induces an automorphism on it not contained in its Weyl group. Hence $W(E) = W'(E)$. This concludes the proof that the Weyl groups listed in the theorem are the correct ones.

Centralizers: Let $\Theta : \mathrm{SL}_3(\mathbf{C}) \longrightarrow \mathrm{SL}_3(\mathbf{C})^3/C_3 \subseteq 3E_6(\mathbf{C})$ denote the homomorphism given by $\Theta(g) = [g, g, g]$ for $g \in \mathrm{SL}_3(\mathbf{C})$. By Lemma 8.5 the group $E^{2a}_{E_6} = \langle \overline{x_2}, \overline{y_1} \rangle$ is conjugate to the group $\langle \overline{a}, \overline{s_1} \rangle$. Since $a^{s_1} = az^2$ we obtain $C_{E_6(\mathbf{C})}(\overline{a}) = \overline{\langle s_1, \mathrm{SL}_3(\mathbf{C})^3/C_3 \rangle}$, and hence

$$C_{E_6(\mathbf{C})}(\overline{a}, \overline{s_1}) = \overline{\langle a, s_1, z, \Theta(\mathrm{SL}_3(\mathbf{C})) \rangle} = \overline{\langle a, s_1, z \rangle \times \Theta(\mathrm{SL}_3(\mathbf{C}))} = \langle \overline{a}, \overline{s_1} \rangle \times \mathrm{PSL}_3(\mathbf{C}),$$

proving the claims for $E_{E_6}^{2a}$. By abusing the notation slightly, we let $\overline{g}$ denote the image of $g \in \mathrm{SL}_3(\mathbf{C})$ in the quotient $\mathrm{PSL}_3(\mathbf{C})$. From Lemma 8.5 we then see that the elements $\overline{a}$, $\overline{y_2}$ and $\overline{x_2 x_1^{-1}}$ in $C_{E_6(\mathbf{C})}(E_{E_6}^{2a})$ correspond to the elements $\overline{\beta}$, $\overline{\gamma^2}$ and $\overline{\tau_2}$ in the $\mathrm{PSL}_3(\mathbf{C})$ component of $C_{E_6(\mathbf{C})}(E_{E_6}^{2a})$. Thus we immediately get

$$C_{E_6(\mathbf{C})}(E_{E_6}^{3a}) = E_{E_6}^{2a} \times C_{\mathrm{PSL}_3(\mathbf{C})}(\overline{\beta}), \qquad C_{E_6(\mathbf{C})}(E_{E_6}^{3d}) = E_{E_6}^{2a} \times C_{\mathrm{PSL}_3(\mathbf{C})}(\overline{\tau_2}),$$
$$C_{E_6(\mathbf{C})}(E_{E_6}^{4a}) = E_{E_6}^{2a} \times C_{\mathrm{PSL}_3(\mathbf{C})}(\overline{\beta}, \overline{\gamma^2}), \qquad C_{E_6(\mathbf{C})}(E_{E_6}^{4b}) = E_{E_6}^{2a} \times C_{\mathrm{PSL}_3(\mathbf{C})}(\overline{\beta}, \overline{\tau_2}).$$

Note that $C_{\mathrm{PSL}_3(\mathbf{C})}(\overline{\beta}) = \overline{\mathbf{T}_2 : \langle \gamma \rangle}$, giving $C_{\mathrm{PSL}_3(\mathbf{C})}(\overline{\beta}, \overline{\gamma^2}) = \langle \overline{\beta}, \overline{\gamma} \rangle$ and $C_{\mathrm{PSL}_3(\mathbf{C})}(\overline{\beta}, \overline{\tau_2}) = \overline{\mathbf{T}_2}$. From this the results on $E_{E_6}^{3a}$, $E_{E_6}^{4a}$ and $E_{E_6}^{4b}$ follow directly. Note also that $C_{\mathrm{PSL}_3(\mathbf{C})}(\overline{\tau_2}) \cong \mathrm{GL}_2(\mathbf{C})$ from which we deduce the claims about $E_{E_6}^{3d}$.

Now consider the group $E_{E_6}^{3b}$. Since $C_{E_6(\mathbf{C})}(\overline{a}) = \overline{\langle s_1, \mathrm{SL}_3(\mathbf{C})^3/C_3 \rangle}$ we get

$$C_{E_6(\mathbf{C})}(\overline{a}, \overline{x_2}) = \overline{\langle s_1, y_1, y_2, (\mathbf{T}_2 \times \mathbf{T}_2 \times \mathbf{T}_2)/C_3 \rangle},$$
$$C_{E_6(\mathbf{C})}(\overline{a}, \overline{x_2}, \overline{y_2}) = \overline{\langle s_1, y_1, y_2, [I_3, \beta, \beta^2], x_2, (\langle \omega I_3 \rangle \times \langle \omega I_3 \rangle \times \langle \omega I_3 \rangle)/C_3 \rangle}$$

and thus $C_{E_6(\mathbf{C})}(E_{E_6}^{3b}) = \left\langle E_{E_6}^{3b}, \overline{s_1}, \overline{y_1}, \overline{[I_3, \beta, \beta^2]} \right\rangle$. It is now easy to check that $C_{E_6(\mathbf{C})}(E_{E_6}^{3b})$ has the structure $E_{E_6}^{3b} \cdot 3^3$ and that $Z(C_{E_6(\mathbf{C})}(E_{E_6}^{3b})) = E_{E_6}^{3b}$. For the group $E_{E_6}^{3c}$ we obtain

$$C_{E_6(\mathbf{C})}(\overline{a}, \overline{x_1}) = \overline{\langle y_1, (\mathrm{SL}_3(\mathbf{C}) \times \mathbf{T}_2 \times \mathbf{T}_2)/C_3 \rangle},$$
$$C_{E_6(\mathbf{C})}(\overline{a}, \overline{x_1}, \overline{y_1}) = \overline{\langle y_1, x_1, (\mathrm{SL}_3(\mathbf{C}) \times \langle \omega I_3 \rangle \times \langle \omega I_3 \rangle)/C_3 \rangle}.$$

Thus the centralizer $C_{E_6(\mathbf{C})}(E_{E_6}^{3c})$ equals the central product $E_{E_6}^{3c} \circ_{\langle \overline{a} \rangle} \mathrm{SL}_3(\mathbf{C})$ and we obtain the claims about $E_{E_6}^{3c}$.

Finally we consider the group $E_{E_6}^{2b} = \langle \overline{y_1}, \overline{x_1} \rangle$. If $g \in \pi^{-1}(C_{E_6(\mathbf{C})}(E_{E_6}^{2b}))$ then $[g, y_1], [g, x_2] \in Z$, and since $[y_1, x_2] = z$ it follows that $g \in \pi^{-1}(E_{E_6}^{2b}) \circ_Z C_{3E_6(\mathbf{C})}(\pi^{-1}(E_{E_6}^{2b}))$. Thus we have $C_{E_6(\mathbf{C})}(E_{E_6}^{2b}) = E_{E_6}^{2b} \times \overline{C_{3E_6(\mathbf{C})}(\pi^{-1}(E_{E_6}^{2b}))}$.

A direct computation shows that $C_{3E_6(\mathbf{C})}(x_1)$ has type $T_2 D_4$ and a system of simple roots of the centralizer is given by $\{\alpha_1 + \alpha_3 + \alpha_4, \alpha_2, \alpha_4 + \alpha_5 + \alpha_6, \alpha_3 + \alpha_4 + \alpha_5\}$. From this we see that the 2-dimensional torus consists of the elements $h(\alpha, 1, \gamma, 1, \alpha, \gamma)$ where $\alpha, \gamma \in \mathbf{C}^\times$. Moreover we see that $C_{3E_6(\mathbf{C})}(x_1) = \mathbf{T}_2 \circ_C \mathrm{Spin}(8, \mathbf{C})$, where the central product is over the group $C = Z(\mathrm{Spin}(8, \mathbf{C})) = C_2 \times C_2$ which consist of the elements $h(\alpha, 1, \gamma, 1, \alpha, \gamma)$, $\alpha, \gamma = \pm 1$.

Let σ denote the automorphism of $C_{3E_6(\mathbf{C})}(x_1)$ given by conjugation with y_1. A direct check shows that the map from C to C given by $x \mapsto x^{-1} x^\sigma$ is surjective. It then follows that

$$C_{3E_6(\mathbf{C})}(\pi^{-1}(E_{E_6}^{2b})) = (\mathbf{T}_2 \circ_C \mathrm{Spin}(8, \mathbf{C}))^\sigma = T_2^\sigma \circ_{C^\sigma} \mathrm{Spin}(8, \mathbf{C})^\sigma.$$

We have $T_2^\sigma = \langle z \rangle$, so $C_{3E_6(\mathbf{C})}(\pi^{-1}(E_{E_6}^{2b})) = \langle z \rangle \times \mathrm{Spin}(8, \mathbf{C})^\sigma$. Using the class distribution of $\pi^{-1}(E_{E_6}^{2b})$ found above together with [33, Table 2] and Theorem 8.2(6) we find

$$\dim C_{3E_6(\mathbf{C})}(\pi^{-1}(E_{E_6}^{2b})) = \frac{1}{3^3} \cdot \left(3 \cdot 78 + 24 \cdot (30 + 24\omega + 24\omega^2)\right) = 14.$$

Thus $\mathrm{Spin}(8,\mathbf{C})^\sigma$ has dimension 14 and since $Z(\mathrm{Spin}(8,\mathbf{C}))^\sigma = 1$ we also see that $\mathrm{Spin}(8,\mathbf{C})^\sigma$ has rank less than 4. From this it follows that the identity component of $\mathrm{Spin}(8,\mathbf{C})^\sigma$ must have type G_2. By [124, Thm. 8.1] we know that $\mathrm{Spin}(8,\mathbf{C})^\sigma$ is connected, so we get $\mathrm{Spin}(8,\mathbf{C})^\sigma = G_2(\mathbf{C})$ and hence $C_{3E_6(\mathbf{C})}(\pi^{-1}(E^{2b}_{E_6})) = \langle z\rangle \times G_2(\mathbf{C})$. Combining this with the computation from above we conclude $C_{E_6(\mathbf{C})}(E^{2b}_{E_6}) = E^{2b}_{E_6} \times G_2(\mathbf{C})$. □

For the proof of our main results we need the following auxiliary results about the two non-toral elementary abelian 3-subgroups of rank 2.

Proposition 8.10. *Let E be an elementary abelian 3-group of rank 2 with basis (e_1, e_2) and consider the 4 homomorphisms $\mu_i : E \to N_{E_6(\mathbf{C})}(\overline{H})$ (the maximal torus normalizer in $E_6(\mathbf{C})$), $1 \le i \le 4$ defined as follows:*

$$\begin{aligned}
\mu_1 &: e_1 \mapsto \overline{y_1}, \quad e_2 \mapsto \overline{x_2},\\
\mu_2 &: e_1 \mapsto \overline{y_1}, \quad e_2 \mapsto \overline{x_2 y_1},\\
\mu_3 &: e_1 \mapsto \overline{y_1}, \quad e_2 \mapsto \overline{x_2 y_1^{-1}},\\
\mu_4 &: e_1 \mapsto \overline{x_1}, \quad e_2 \mapsto \overline{x_2 y_1^{-1}}.
\end{aligned}$$

Then $\mu_i(E)$ is conjugate to $E^{2a}_{E_6}$ for all i and $\mu_i^{-1}(\overline{H})$ equals the 4 distinct rank 1 subgroups of E for $i = 1, \ldots, 4$. Moreover $T_E = (\overline{H}^{\mu_i(E)})_1$ is independent of i and $C_{N_{E_6(\mathbf{C})}(\overline{H})}(\mu_i(E))$ is a maximal torus normalizer in $C_{E_6(\mathbf{C})}(\mu_i(E))$ for all i. The canonical homomorphism $E \times T_E \xrightarrow{\mu_i \times 1} N(H) \to E_6(\mathbf{C})$ does not depend on i up to conjugacy in $E_6(\mathbf{C})$.

Proof. Obviously $\mu_1^{-1}(\overline{H}) = \langle e_2\rangle$, $\mu_2^{-1}(\overline{H}) = \langle e_1 - e_2\rangle$, $\mu_3^{-1}(\overline{H}) = \langle e_1 + e_2\rangle$ and $\mu_4^{-1}(\overline{H}) = \langle e_1\rangle$, so $\mu_i^{-1}(\overline{H})$ equals the 4 distinct rank 1 subgroups of E for $i = 1, \ldots, 4$.

Since $\overline{x_1}, \overline{x_2} \in \overline{H}$, it is clear that the identity component of $\overline{H}^{\mu_i(E)}$ equals the identity component of $\overline{H}^{\overline{y_1}}$. Hence $T_E = (\overline{H}^{\mu_i(E)})_1$ is independent of i and as $y_1 = [I_3, \gamma, \gamma^2]$, a direct computation shows that T_E consists of the elements of the form $\overline{[g, I_3, I_3]}$, $g \in \mathbf{T}_2$, where $\mathbf{T}_2 \subseteq \mathrm{SL}_3(\mathbf{C})$ denotes the maximal torus consisting of diagonal matrices.

We now prove that the homomorphisms $E \xrightarrow{\mu_i} C_{E_6(\mathbf{C})}(T_E)$ are conjugate in $C_{E_6(\mathbf{C})}(T_E)$. As $\overline{a} \in T_E$ we find that $C_{E_6(\mathbf{C})}(T_E)$ consists of the elements $\overline{[g_1, g_2, g_3]}$, where $g_1 \in \mathbf{T}_2$ and $g_2, g_3 \in \mathrm{SL}_3(\mathbf{C})$ are arbitrary. Note first that the conjugation by the element $\overline{[I_3, \tau_1, \tau_1^2]} \in C_{E_6(\mathbf{C})}(T_E)$ sends $\overline{y_1}$ to itself and $\overline{x_2}$ to $\overline{x_2 y_1}$. Hence the homomorphism $E \xrightarrow{\mu_i} C_{E_6(\mathbf{C})}(T_E)$ is conjugate to the homomorphism $E \xrightarrow{\mu_{i+1}} C_{E_6(\mathbf{C})}(T_E)$ for $i = 1, 2$. Letting

$$\tau_3 = -\frac{e^{\pi i/18}}{\sqrt{3}} \begin{bmatrix} 1 & \omega^2 & 1 \\ 1 & 1 & \omega^2 \\ 1 & \omega & \omega \end{bmatrix} \in \mathrm{SL}_3(\mathbf{C}),$$

we get $\beta^{\tau_3} = \beta\gamma$ and $\gamma^{\tau_3} = \beta^2$. Thus the conjugation by the element $\overline{[I_3, \tau_2\tau_1^{-1}, \tau_3]} \in C_{E_6(\mathbf{C})}(T_E)$ sends $\overline{y_1}$ to $\overline{x_1}$ and $\overline{x_2}$ to $\overline{x_2 y_1^{-1}}$ and hence the homomorphism $E \xrightarrow{\mu_1} C_{E_6(\mathbf{C})}(T_E)$ is conjugate to the homomorphism $E \xrightarrow{\mu_4} C_{E_6(\mathbf{C})}(T_E)$. This proves the claim. We conclude that $E \times T_E \xrightarrow{\mu_i \times 1} N(H) \to E_6(\mathbf{C})$ is independent of i up to conjugacy in $E_6(\mathbf{C})$ and also that $\mu_i(E)$ is conjugate to $\mu_1(E) = E^{2a}_{E_6}$ for all i.

Since $C_{E_6(\mathbf{C})}(E^{2a}_{E_6})$ has rank 2 it follows that T_E is a maximal torus in $C_{E_6(\mathbf{C})}(\mu_i(E))$ for all i. Since $\mu_i(E)$ is elementary abelian of rank 2 and $\mu_i(E) \cap \overline{H} \neq 1$ it now follows from Theorem 8.3 that $C_{N_{E_6(\mathbf{C})}(\overline{H})}(\mu_i(E))$ is a maximal torus normalizer in $C_{E_6(\mathbf{C})}(\mu_i(E))$ for all i. □

Proposition 8.11. *An element $\alpha \in W(E^{2b}_{E_6})$ acts up to conjugacy by $\alpha \times 1$ on $C_{E_6(\mathbf{C})}(E^{2b}_{E_6}) = E^{2b}_{E_6} \times G_2(\mathbf{C})$.*

Proof. Since $G_2(\mathbf{C})$ is connected and has trivial center it is clear that $\mathrm{Out}(E^{2b}_{E_6} \times G_2(\mathbf{C})) = \mathrm{Out}(E^{2b}_{E_6}) \times \mathrm{Out}(G_2(\mathbf{C}))$. The result now follows as $\mathrm{Out}(G_2(\mathbf{C})) = 1$ (e.g. by [71, Thm. 27.4]). □

8.3. **The group $E_8(\mathbf{C})$, $p = 3$.** In this section we consider the elementary abelian 3-subgroups of the group $E_8(\mathbf{C})$. By using [16, Table 2, p. 214] we see that the smallest faithful representation of $E_8(\mathbf{C})$ is the adjoint representation, i.e. the representation given by the action of $E_8(\mathbf{C})$ on its Lie algebra $\mathfrak{e}_8$, which has dimension 248. For our computations, we explicitly construct this representation on a computer by following the recipe in [24, Ch. 4]. (As explained in [24, Ch. 4] there is some ambiguity in choosing a Chevalley basis of $\mathfrak{e}_8$ and we fix a certain such choice; a different choice affects our formulas at only one point—see Remark 8.12.)

Letting $\Phi(E_8)$ denote the root system of type E_8 (we use the notation of [15, p. 268–270]), we have in particular a maximal torus H generated by the elements $h_{\alpha_i}(t)$, $1 \le i \le 8$, $t \in \mathbf{C}^\times$ ([24, p. 92, p. 97]) and root subgroups $u_\alpha(t)$, $\alpha \in \Phi(E_8)$, $t \in \mathbf{C}$. The normalizer $N(H)$ of the maximal torus, is generated by H and the elements $n_i = n_{\alpha_i}$, $1 \le i \le 8$ ([24, p. 93, p. 101]). We let

$$h(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) = \prod_{i=1}^{8} h_{\alpha_i}(t_i).$$

Note that by [24, p. 100 and Lem. 6.4.4] the root subgroups u_α form a *realization* ([122, p. 133]) of $\Phi(E_8)$ in $E_8(\mathbf{C})$. In particular we have the following relations. The element n_i has image s_{α_i} in $W = W(E_8)$ ([122, 8.1.4(i)]), we have $n_i^2 = h_{\alpha_i}(-1)$ ([122, 8.1.4(ii)]) and

$$n_i n_j n_i \ldots = n_j n_i n_j \ldots$$

for $1 \le i,j \le 8$, where the number of factors on both sides equals the order of $s_{\alpha_i}s_{\alpha_j}$ in W ([122, 9.3.2]).

Now let $\overline{a} = h_{\alpha_1}(\omega)h_{\alpha_2}(\omega)h_{\alpha_3}(\omega^2) \in E_8(\mathbf{C})$. Direct computation shows that for any root $\alpha \in \Phi(E_8)$ we have $\alpha(\overline{a}) = \omega^{2\langle\alpha,\lambda_2\rangle}$. From this we see that the Dynkin diagram of the centralizer $C_{E_8(\mathbf{C})}(\overline{a})$ is the same as the extended Dynkin diagram of E_8 with the node α_2 removed. Thus it has type A_8 and a simple system of roots is given by

$$\{\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8,-\widetilde{\alpha}\},$$

where $\widetilde{\alpha}$ is the longest root. As in [15, p. 250–251] we identify $\Phi(A_8)$ with the set of elements in $\mathbf{R}^9$ of the form $e_i - e_j$ with $i \neq j$ and $1 \le i,j \le 9$, where e_i denotes the i'th canonical basis vector in $\mathbf{R}^9$. We now consider $\mathrm{SL}_9(\mathbf{C})$, which is the simply connected group of type A_8 over $\mathbf{C}$. Given a root $\alpha' = e_i - e_j \in \Phi(A_8)$ we let $u'_{\alpha'}(t) = I_9 + te_{i,j}$ for $t \in \mathbf{C}$. With respect to the maximal torus consisting of the diagonal matrices, this is a root subgroup of $\mathrm{SL}_9(\mathbf{C})$ corresponding to the root α'. The roots $\alpha'_i = e_i - e_{i+1}$, $1 \le i \le 8$, is a simple system in $\Phi(A_8)$. From the above we then see that

$$u'_{\pm\alpha'_1}(t) \mapsto u_{\pm\alpha_1}(t), \quad u'_{\pm\alpha'_2}(t) \mapsto u_{\pm\alpha_3}(t), \quad u'_{\pm\alpha'_3}(t) \mapsto u_{\pm\alpha_4}(t), \quad u'_{\pm\alpha'_4}(t) \mapsto u_{\pm\alpha_5}(t),$$
$$u'_{\pm\alpha'_5}(t) \mapsto u_{\pm\alpha_6}(t), \quad u'_{\pm\alpha'_6}(t) \mapsto u_{\pm\alpha_7}(t), \quad u'_{\pm\alpha'_7}(t) \mapsto u_{\pm\alpha_8}(t), \quad u'_{\pm\alpha'_8}(t) \mapsto u_{\mp\widetilde{\alpha}}(t)$$

defines a homomorphism $\mathrm{SL}_9(\mathbf{C}) \to E_8(\mathbf{C})$ onto the centralizer $C_{E_8(\mathbf{C})}(\overline{a})$. It is easy to check that this map has kernel $C_3 = \langle \omega I_9 \rangle$ and thus we may make the identification $C_{E_8(\mathbf{C})}(\overline{a}) = \mathrm{SL}_9(\mathbf{C})/C_3$. For any $g \in \mathrm{SL}_9(\mathbf{C})$ we denote by $\overline{g}$ its image in $\mathrm{SL}_9(\mathbf{C})/C_3 = C_{E_8(\mathbf{C})}(\overline{a}) \subseteq E_8(\mathbf{C})$. In particular we see that $a = \eta I_9$ corresponds to the element $\overline{a}$ from above. We also define the following elements in $\mathrm{SL}_9(\mathbf{C})$:

$$x_1 = \mathrm{diag}(1,\omega,\omega^2,1,\omega,\omega^2,1,\omega,\omega^2), \qquad x_2 = \mathrm{diag}(1,1,1,\omega,\omega,\omega,\omega^2,\omega^2,\omega^2),$$
$$x_3 = \mathrm{diag}(1,1,1,1,1,1,\omega,\omega,\omega), \qquad y_1 = (1,2,3)(4,5,6)(7,8,9),$$
$$y_2 = (1,4,7)(2,5,8)(3,6,9).$$

From the explicit homomorphism above we easily find

$$\overline{a} = h_{\alpha_1}(\omega)h_{\alpha_2}(\omega)h_{\alpha_3}(\omega^2), \qquad \overline{x_1} = h_{\alpha_1}(\omega)h_{\alpha_5}(\omega)h_{\alpha_8}(\omega),$$
$$\overline{x_2} = h_{\alpha_1}(\omega)h_{\alpha_3}(\omega^2)h_{\alpha_5}(\omega^2)h_{\alpha_6}(\omega), \qquad \overline{x_3} = h_{\alpha_1}(\omega^2)h_{\alpha_3}(\omega)h_{\alpha_5}(\omega^2)h_{\alpha_6}(\omega),$$

and a direct computation in $E_8(\mathbf{C})$ shows that

$$n_{-\widetilde{\alpha}} = n_8n_7n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_2n_6n_5n_4n_3n_1n_7n_6n_5n_4n_2n_3n_4n_5n_6n_7n_8 \cdot$$
$$n_7n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_2n_6n_5n_4n_3n_1n_7n_6n_5n_4n_2n_3n_4n_5n_6n_7n_8.$$

Remark 8.12. A different choice of Chevalley basis for $\mathfrak{e}_8$ may effect the expression for $n_{-\widetilde{\alpha}}$ by an order two element in H. If a Chevalley basis is chosen such that the above formula holds then all further formulas will be independent of the choice.

From this and the explicit homomorphism above we find, either by direct computation or by using the relations in $N(H)$, that

$$\overline{y_1} = n_1n_3n_5n_6n_7n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_2n_6n_5n_4n_3n_1n_7n_6n_5 \cdot$$
$$n_4n_2n_3n_4n_5n_6n_7n_8n_7n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_2n_6n_5n_4n_3 \cdot$$
$$n_1n_7n_6n_5n_4n_2n_3n_4n_5n_6n_7n_8 \cdot h_{\alpha_1}(-1)h_{\alpha_2}(-1)h_{\alpha_7}(-1),$$

$$\overline{y_2} = n_2n_3n_1n_4n_2n_3n_4n_5n_4n_2n_3n_4n_6n_5n_4n_2n_3n_1n_4n_7n_6n_5n_4 \cdot$$
$$n_2n_3n_1n_4n_3n_5n_6n_7n_8n_7n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_2n_6n_5 \cdot$$
$$n_4n_3n_1n_7n_6n_5n_4n_2n_3n_4n_5n_8n_7n_6 \cdot h_{\alpha_2}(-1)h_{\alpha_5}(-1).$$

Notation 8.13. To distinguish subgroups of $E_8(\mathbf{C})$, we need some information on the conjugacy classes of elements of order 3. These are given in [65, Table VI] (which is taken from [32, Table 4]): There are 4 such conjugacy classes, which we label **3A**, **3B**, **3C** and **3D**. Moreover these classes may be distinguished by their traces on $\mathfrak{e}_8$. Since the trace of the element $h \in H$ is given by $8 + \sum_{\alpha\in\Phi(E_8)} \alpha(h)$ we get $\overline{a} \in$ **3A**, $\overline{x_1}, \overline{x_2}, \overline{x_3}, \overline{y_1}, \overline{y_2} \in$ **3B** and $\overline{x_3a^{-1}} \in$ **3D**.

Notation 8.14. If K is a field and n is a natural number, we define *the group of symplectic similitudes* as $\mathrm{CSp}_{2n}(K) = \{X \in \mathrm{GL}_{2n}(K) | X^t B X = cB, c \in K^\times\}$, where

$$B = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{n \text{ times}}$$

We define the homomorphism $\chi : \mathrm{CSp}_{2n}(K) \to K^\times$ by $\chi(X) = c$, where $X^t B X = cB$. The kernel of χ is *the symplectic group* $\mathrm{Sp}_{2n}(K)$. (The notation CSp is taken from [84].)

Theorem 8.15. *The conjugacy classes of non-toral elementary abelian 3-subgroups of $E_8(\mathbf{C})$ are given by the following table.*

rank	*name*	*ordered basis*	$E_8(\mathbf{C})$*-class dist.*	$C_{E_8(\mathbf{C})}(E)$	$Z(C_{E_8(\mathbf{C})}(E))$
3	$E^{3a}_{E_8}$	$\langle \overline{x_1}, \overline{y_1}, \overline{a} \rangle$	$\mathbf{3A^{18}3B^{8}}$	$E^{3a}_{E_8} \times \mathrm{PSL}_3(\mathbf{C})$	$E^{3a}_{E_8}$
3	$E^{3b}_{E_8}$	$\langle \overline{x_1}, \overline{y_1}, \overline{x_3} \rangle$	$\mathbf{3B^{26}}$	$E^{3b}_{E_8} \times G_2(\mathbf{C})$	$E^{3b}_{E_8}$
4	$E^{4a}_{E_8}$	$\left\langle \overline{x_1}, \overline{y_1}, \overline{x_3}, \overline{x_3 a^{-1}} \right\rangle$	$\mathbf{3A^{52}3B^{26}3D^{2}}$	$E^{4a}_{E_8} \circ_{\langle \overline{x_3 a^{-1}} \rangle} \mathrm{GL}_2(\mathbf{C})$	$E^{4a}_{E_8} \circ_{\langle \overline{x_3 a^{-1}} \rangle} \mathbf{T}_1$
4	$E^{4b}_{E_8}$	$\langle \overline{x_2}, \overline{x_1}, \overline{y_1}, \overline{a} \rangle$	$\mathbf{3A^{54}3B^{26}}$	$E^{4b}_{E_8} \circ_{\langle \overline{x_2} \rangle} (\mathbf{T}_2 : \langle \overline{y_2} \rangle)$	$E^{4b}_{E_8}$
4	$E^{4c}_{E_8}$	$\langle \overline{x_2}, \overline{x_1}, \overline{y_1}, \overline{x_3} \rangle$	$\mathbf{3B^{80}}$	$E^{4c}_{E_8} \circ_{\langle \overline{x_2} \rangle} \mathrm{SL}_3(\mathbf{C})$	$E^{4c}_{E_8}$
5	$E^{5a}_{E_8}$	$\left\langle \overline{x_2}, \overline{x_1}, \overline{y_1}, \overline{x_3}, \overline{x_3 a^{-1}} \right\rangle$	$\mathbf{3A^{156}3B^{80}3D^{6}}$	$E^{5a}_{E_8} \circ_{\langle \overline{x_2}, \overline{x_3 a^{-1}} \rangle} \mathbf{T}_2$	$E^{5a}_{E_8} \circ_{\langle \overline{x_2}, \overline{x_3 a^{-1}} \rangle} \mathbf{T}_2$
5	$E^{5b}_{E_8}$	$\langle \overline{x_1}, \overline{y_1}, \overline{x_2}, \overline{y_2}, \overline{a} \rangle$	$\mathbf{3A^{162}3B^{80}}$	$E^{5b}_{E_8}$	$E^{5b}_{E_8}$

In particular we have ${}_3Z(C_{E_8(\mathbf{C})}(E)) = E$ for any non-toral elementary abelian 3-subgroup of $E_8(\mathbf{C})$.

The Weyl groups of these groups with respect to the given ordered bases are given as follows:

$$W(E^{3a}_{E_8}) = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\mathrm{GL}_2(\mathbf{F}_3)} & \begin{matrix} * \\ * \end{matrix} \\ \hline 0 & 0 & \det \end{array}\right], \quad W(E^{3b}_{E_8}) = \mathrm{SL}_3(\mathbf{F}_3), \quad W(E^{4a}_{E_8}) = \left[\begin{array}{ccc|c} & & & 0 \\ & \mathrm{SL}_3(\mathbf{F}_3) & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & \varepsilon \end{array}\right]$$

$$W(E^{4b}_{E_8}) = \left[\begin{array}{c|cc|c} \varepsilon & * & * & * \\ \hline 0 & \multicolumn{2}{c|}{\mathrm{GL}_2(\mathbf{F}_3)} & * \\ 0 & & & * \\ \hline 0 & 0 & 0 & \det \end{array}\right], \quad W(E^{4c}_{E_8}) = \left[\begin{array}{c|ccc} \varepsilon & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbf{F}_3) & \\ 0 & & & \end{array}\right]$$

$$W(E^{5a}_{E_8}) = \left[\begin{array}{c|ccc|c} \varepsilon_1 & * & * & * & * \\ \hline 0 & & & & 0 \\ 0 & & \mathrm{SL}_3(\mathbf{F}_3) & & 0 \\ 0 & & & & 0 \\ \hline 0 & 0 & 0 & 0 & \varepsilon_2 \end{array}\right], \quad W(E^{5b}_{E_8}) = \left[\begin{array}{cccc|c} & & & & * \\ & \mathrm{CSp}_4(\mathbf{F}_3) & & & * \\ & & & & * \\ & & & & * \\ \hline 0 & 0 & 0 & 0 & \chi \end{array}\right]$$

where det *is the determinant of the matrix from $\mathrm{GL}_2(\mathbf{F}_3)$ in the description of $W(E^{3a}_{E_8})$ and $W(E^{4b}_{E_8})$. In the description of $W(E^{5b}_{E_8})$, χ denotes the value of the homomorphism $\chi : \mathrm{CSp}_4(\mathbf{F}_3) \to \mathbf{F}_3^\times$ defined in 8.14 evaluated on the matrix from $\mathrm{CSp}_4(\mathbf{F}_3)$.*

Remark 8.16. Note that our information on the rank five subgroup $E^{5a}_{E_8}$ corrects [65].

Proof of Theorem 8.15. Maximal non-toral subgroups: By [65, Lems. 11.7 and 11.9], any maximal non-toral elementary abelian 3-subgroup of $E_8(\mathbf{C})$ contains an element of type **3A**. We may thus find representatives in $C_{E_8(\mathbf{C})}(\overline{a}) = \mathrm{SL}_9(\mathbf{C})/C_3$. From [65, Cor. 11.10], it follows that there are two conjugacy classes of these maximal non-toral elementary abelian 3-subgroups: $E_{E_8}^{5a}$ and $E_{E_8}^{5b}$, both of rank 5. Moreover, by [65, Lem. 11.5], their preimages in $\mathrm{SL}_9(\mathbf{C})$ may be chosen to have the shapes $3^{1+2} \circ_{C_3} 9 \times 3 \times 3$ and $3^{1+4} \circ_{C_3} 9$. Using the representation theory of extraspecial 3-groups ([64, Ch. 5.5]) we find that $E_{E_8}^{5a}$ is represented by $\left\langle \overline{x_2}, \overline{x_1}, \overline{y_1}, \overline{x_3}, \overline{x_3 a^{-1}} \right\rangle$ and that $E_{E_8}^{5b}$ is represented by $\langle \overline{x_1}, \overline{y_1}, \overline{x_2}, \overline{y_2}, \overline{a} \rangle$.

Lower bounds for Weyl groups of maximal non-toral subgroups: We can find lower bounds for the Weyl groups of $E_{E_8}^{5a}$ and $E_{E_8}^{5b}$ by conjugating with elements in the centralizer $C_{E_8(\mathbf{C})}(\overline{a}) = \mathrm{SL}_9(\mathbf{C})/C_3$ and the normalizer $N(H)$ of the maximal torus.

Note that

$$a = \eta I_3 \oplus \eta I_3 \oplus \eta I_3, \qquad x_1 = \beta \oplus \beta \oplus \beta, \qquad x_2 = I_3 \oplus \omega I_3 \oplus \omega^2 I_3,$$
$$x_3 = I_3 \oplus I_3 \oplus \omega I_3, \qquad y_1 = \gamma \oplus \gamma \oplus \gamma$$

and $(A \oplus B \oplus C)^{y_2} = B \oplus C \oplus A$. Conjugation by $\tau_1 \oplus \tau_1 \oplus \tau_1$, $\tau_2 \oplus \tau_2 \oplus \tau_2$ and $I_3 \oplus \beta^2 \oplus \beta$ gives

$$\tau_1 \oplus \tau_1 \oplus \tau_1 : a \mapsto a, \quad x_1 \mapsto x_1 y_1, \quad x_2 \mapsto x_2, \quad x_3 \mapsto x_3, \quad y_1 \mapsto y_1, \quad y_2 \mapsto y_2. \tag{8.1}$$
$$\tau_2 \oplus \tau_2 \oplus \tau_2 : a \mapsto a, \quad x_1 \mapsto x_1, \quad x_2 \mapsto x_2, \quad x_3 \mapsto x_3, \quad y_1 \mapsto x_1 y_1, \quad y_2 \mapsto y_2. \tag{8.2}$$
$$I_3 \oplus \beta^2 \oplus \beta : a \mapsto a, \quad x_1 \mapsto x_1, \quad x_2 \mapsto x_2, \quad x_3 \mapsto x_3, \quad y_1 \mapsto x_2 y_1, \quad y_2 \mapsto x_1 y_2. \tag{8.3}$$

Now consider the group $E_{E_8}^{5a}$. From (8.1)—(8.3) we see that the elements $\overline{\tau_1 \oplus \tau_1 \oplus \tau_1}$, $\overline{\tau_2 \oplus \tau_2 \oplus \tau_2}$ and $\overline{I_3 \oplus \beta^2 \oplus \beta}$ normalize $E_{E_8}^{5a}$ and that conjugation by these elements induces the automorphisms on $E_{E_8}^{5a}$ given by the matrices $I_5 + e_{3,2}$, $I_5 + e_{2,3}$ and $I_5 + e_{1,3}$.

Letting $\sigma = -(1,4)(2,5)(3,6) \in \mathrm{SL}_9(\mathbf{C})$ we see that $(A \oplus B \oplus C)^\sigma = B \oplus A \oplus C$. Using this and the above we obtain that $\overline{\sigma}$, $\overline{y_2}$ and $\overline{I_3 \oplus I_3 \oplus \beta^2}$ normalize $E_{E_8}^{5a}$ and that conjugation by these elements induces the automorphisms on $E_{E_8}^{5a}$ given by the matrices $\mathrm{diag}(2,1,1,1,1)$, $I_5 + e_{1,4} + e_{1,5}$ and $I_5 + e_{4,3}$. By using the relations in $N(H)$ given above or by direct computation, it may be checked that conjugation by the element

$$n_1 n_2 n_4 n_2 n_3 n_5 n_4 n_2 n_3 n_1 n_4 n_3 n_5 n_4 n_6 n_5 n_4 n_2 n_3 n_4 n_7 n_6 n_5 n_4 n_8 n_7 n_6 \cdot$$
$$n_5 n_4 n_2 n_3 n_1 n_4 n_3 n_5 n_4 n_2 n_6 n_5 n_4 n_7 \cdot h(1,1,-1,-1,-1,1,-1,-1)$$

induces the automorphism on $E_{E_8}^{5a}$ represented by the matrix $\mathrm{diag}(1,1,1,1,2) + e_{2,4}$. It is easy to see that the above matrices generate the group

$$W'(E_{E_8}^{5a}) = \left[\begin{array}{c|ccc|c} \varepsilon_1 & * & * & * & * \\ \hline 0 & & & & 0 \\ 0 & & \mathrm{SL}_3(\mathbf{F}_3) & & 0 \\ 0 & & & & 0 \\ \hline 0 & 0 & 0 & 0 & \varepsilon_2 \end{array}\right]$$

and thus $W(E_{E_8}^{5a})$ contains this group.

Next consider the group $E_{E_8}^{5b}$. From (8.1)—(8.3) we see that the elements $\overline{\tau_1 \oplus \tau_1 \oplus \tau_1}$, $\overline{\tau_2 \oplus \tau_2 \oplus \tau_2}$ and $\overline{I_3 \oplus \beta^2 \oplus \beta}$ normalize $E_{E_8}^{5b}$ and that conjugation by these elements induces the automorphisms on $E_{E_8}^{5b}$ given by the matrices $I_5 + e_{2,1}$, $I_5 + e_{1,2}$ and $I_5 + e_{1,4} + e_{3,2}$. Now note that $a = \Delta_{3,3}(\eta I_3)$, $x_2 = \Delta_{3,3}(\beta)$ and $y_2 = \Delta_{3,3}(\gamma)$. Noting also that $\Delta_{3,3}(M_1)$

commutes with $M_2 \oplus M_2 \oplus M_2$ for any $M_1, M_2 \in M_3(\mathbf{C})$ we see that the elements $\overline{\Delta_{3,3}(\tau_1)}$ and $\overline{\Delta_{3,3}(\tau_2)}$ normalize $E_{E_8}^{5b}$. The automorphisms induced on $E_{E_8}^{5b}$ by conjugation with these elements have the matrices $I_5 + e_{4,3}$ and $I_5 + e_{3,4}$. The upper left 4×4-corner of these matrices are easily seen to generate the group $\mathrm{Sp}_4(\mathbf{F}_3)$ from 8.14. By using the relations in $N(H)$ given above or by direct computation, we get that conjugation by the element

$$n_2 n_8 n_7 n_6 n_5 n_4 n_2 n_3 n_1 n_4 n_3 n_5 n_4 n_2 n_6 n_5 n_4 n_3 n_1 n_7 n_6 \cdot \\ n_5 n_4 n_2 n_3 n_4 n_5 n_6 n_7 n_8 \cdot h(1, -1, -1, -1, -1, 1, 1, 1)$$

induces the automorphism on $E_{E_8}^{5b}$ represented by the matrix $\mathrm{diag}(1, 2, 1, 2, 2) + e_{3,5}$. It now follows that $W(E_{E_8}^{5b})$ contains the group

$$W'(E_{E_8}^{5b}) = \left[\begin{array}{c|c} \mathrm{CSp}_4(\mathbf{F}_3) & \begin{matrix} * \\ * \\ * \\ * \end{matrix} \\ \hline 0 \quad 0 \quad 0 \quad 0 & \chi \end{array}\right]$$

Lower bounds for other Weyl groups: We now show that the other Weyl groups in the table are all lower bounds. To do this consider one of the non-maximal groups E from the table. We then have $E \subseteq E_{E_8}^{5a}$, and we get a lower bound on $W(E)$ by considering the action on E of the subgroup of $W'(E_{E_8}^{5a})$ fixing E. As an example we find that the stabilizer of $E_{E_8}^{3a}$ inside $W'(E_{E_8}^{5a})$ is

$$\begin{bmatrix} \varepsilon_1 & 0 & 0 & x & x \\ 0 & a_{11} & a_{12} & a_{13} & 0 \\ 0 & a_{21} & a_{22} & a_{23} & 0 \\ 0 & 0 & 0 & \det & 0 \\ 0 & 0 & 0 & 0 & \det \end{bmatrix}$$

where $\det = a_{11} \cdot a_{22} - a_{12} \cdot a_{21} \neq 0$. The action of such a matrix on $E_{E_8}^{3a}$ is given by

$$\overline{x_1} \mapsto (\overline{x_1})^{a_{11}} (\overline{y_1})^{a_{21}}, \qquad \overline{y_1} \mapsto (\overline{x_1})^{a_{12}} (\overline{y_1})^{a_{22}}, \qquad \overline{a} \mapsto (\overline{x_1})^{a_{13}} (\overline{y_1})^{a_{23}} (\overline{a})^{\det}.$$

Thus $W(E_{E_8}^{3a})$ contains the group

$$W'(E_{E_8}^{3a}) = \left[\begin{array}{c|c} \mathrm{GL}_2(\mathbf{F}_3) & \begin{matrix} * \\ * \end{matrix} \\ \hline 0 \quad 0 & \det \end{array}\right]$$

as claimed. Similar computations show that for the remaining groups $E = E_{E_8}^{3b}$, $E_{E_8}^{4a}$, $E_{E_8}^{4b}$ and $E_{E_8}^{4c}$, the group $W'(E)$ occurring in the theorem is a lower bound for the Weyl group $W(E)$.

Orbit computation: Note first that all elementary abelian 3-subgroups of rank at most two are toral by Theorem 8.2(3). By using the lower bounds on the Weyl groups of $E_{E_8}^{5a}$ and $E_{E_8}^{5b}$ established above, we may find a set of representatives for the conjugacy classes of subgroups of $E_{E_8}^{5a}$ and $E_{E_8}^{5b}$ of rank 3 and 4.

Under the action of $W'(E_{E_8}^{5a})$, the set of rank 3 subgroups of $E_{E_8}^{5a}$ has orbit representatives

$$E_{E_8}^{3a} = \langle \overline{x_1}, \overline{y_1}, \overline{a} \rangle, E_{E_8}^{3b} = \langle \overline{x_1}, \overline{y_1}, \overline{x_3} \rangle, \langle \overline{x_1}, \overline{x_2}, \overline{y_1} \rangle, \\ \langle \overline{a}, \overline{x_1}, \overline{x_2} \rangle, \langle \overline{a}, \overline{x_1}, \overline{x_3} \rangle \text{ and } \langle \overline{a}, \overline{x_2}, \overline{x_3} \rangle,$$

and under the action of $W'(E_{E_8}^{5b})$, the set of rank 3 subgroups of $E_{E_8}^{5b}$ has orbit representatives

$$E_{E_8}^{3a} = \langle \overline{x_1}, \overline{y_1}, \overline{a} \rangle, \langle \overline{x_1}, \overline{x_2}, \overline{y_1} \rangle \text{ and } \langle \overline{a}, \overline{x_1}, \overline{x_2} \rangle.$$

Similarly we find that under the action of $W'(E_{E_8}^{5a})$, the set of rank 4 subgroups of $E_{E_8}^{5a}$ has orbit representatives

$$E_{E_8}^{4a} = \left\langle \overline{x_1}, \overline{y_1}, \overline{x_3}, \overline{x_3a^{-1}} \right\rangle, E_{E_8}^{4b} = \langle \overline{x_2}, \overline{x_1}, \overline{y_1}, \overline{a} \rangle,$$
$$E_{E_8}^{4c} = \langle \overline{x_2}, \overline{x_1}, \overline{y_1}, \overline{x_3} \rangle \text{ and } \langle \overline{a}, \overline{x_1}, \overline{x_2}, \overline{x_3} \rangle,$$

and that under the action of $W'(E_{E_8}^{5b})$, the set of rank 4 subgroups of $E_{E_8}^{5b}$ has orbit representatives

$$E_{E_8}^{4b} = \langle \overline{x_2}, \overline{x_1}, \overline{y_1}, \overline{a} \rangle \text{ and } E_0 = \langle \overline{x_1}, \overline{x_2}, \overline{y_1}, \overline{y_2} \rangle.$$

Class distributions: Recall that by 8.13, $\overline{a}$ is in the conjugacy class **3A**, $\overline{x_1}$ and $\overline{x_2}$ are in the class **3B** and $\overline{x_3a^{-1}}$ belongs to the class **3D**. Using the actions of $W'(E_{E_8}^{5a})$ and $W'(E_{E_8}^{5b})$ it is then straightforward to verify the class distributions given in the table. As an example consider the group $E_{E_8}^{5a}$. Under the action of $W'(E_{E_8}^{5a})$ it contains 156 elements conjugate to $\overline{a}$, 78 elements conjugate to $\overline{x_1}$, 2 elements conjugate to $\overline{x_2}$ and 6 elements conjugate to $\overline{x_3a^{-1}}$, which gives the class distribution in the table. Similar computations give the results for the remaining groups.

We also see that the group $E_0 = \langle \overline{x_1}, \overline{x_2}, \overline{y_1}, \overline{y_2} \rangle$ has class distribution $\mathbf{3B^{80}}$ and from the class distribution of $E_{E_8}^{5b}$ we get $E_0 = (E_{E_8}^{5b} \cap \mathbf{3B}) \cup \{1\}$. It then follows from [65, Lem. 11.5] that E_0 is toral.

Other non-toral subgroups: We see directly that the groups

$$\langle \overline{a}, \overline{x_1}, \overline{x_2}, \overline{x_3} \rangle, \langle \overline{a}, \overline{x_1}, \overline{x_2} \rangle, \langle \overline{a}, \overline{x_1}, \overline{x_3} \rangle \text{ and } \langle \overline{a}, \overline{x_2}, \overline{x_3} \rangle$$

are toral. Since the group $\langle \overline{x_1}, \overline{x_2}, \overline{y_1} \rangle$ is a subgroup of E_0 it is also toral. Alternatively, from the action of $W'(E_{E_8}^{5a})$ we see that it is conjugate to the group $\langle \overline{x_1}, \overline{x_2}, \overline{x_3} \rangle$, which is visibly toral. Thus any non-toral elementary abelian 3-subgroup of $E_8(\mathbf{C})$ is conjugate to a group in the table. Moreover, since their class distributions differ, none of the groups occurring in the table are conjugate.

To see that the groups in the table are actually non-toral we may proceed as follows. The group $E_{E_8}^{3a}$ contains the element $\overline{a}$, so by Theorem 8.2(1) it is toral if and only if it is toral in $C_{E_8(\mathbf{C})}(\overline{a}) = \mathrm{SL}_9(\mathbf{C})/C_3$. However this is not the case by Theorem 8.2(5), since its lift to $\mathrm{SL}_9(\mathbf{C})$ is non-abelian. The groups $E_{E_8}^{4a}$ and $E_{E_8}^{4b}$ are thus also non-toral since they contain $E_{E_8}^{3a}$. We saw above that the Weyl group of $E_{E_8}^{3b}$ contains $\mathrm{SL}_3(\mathbf{F}_3)$, which has order divisible by 13. Since $13 \nmid |W(E_8)|$ it follows from Theorem 8.2(2) that $E_{E_8}^{3b}$ is non-toral. Since $E_{E_8}^{4c}$ contains $E_{E_8}^{3b}$ it is also non-toral.

Centralizers: The subgroups $E = E_{E_8}^{3a}$, $E_{E_8}^{4a}$, $E_{E_8}^{4b}$, $E_{E_8}^{5a}$ and $E_{E_8}^{5b}$ are easy to deal with since they all contain $\overline{a}$, and hence we have $C_{E_8(\mathbf{C})}(E) = C_{\mathrm{SL}_9(\mathbf{C})/C_3}(E)$ for these. It is however notationally convenient first to change the representatives as follows. Define $x_4 = \tau_2^{-1} \oplus \tau_2^{-1} \oplus \tau_2^{-1} \in \mathrm{SL}_9(\mathbf{C})$, and note that conjugation by $(2,7,3,4)(5,8,9,6) \in \mathrm{SL}_9(\mathbf{C})$ acts as follows:

$$a \mapsto a, \qquad x_1 \mapsto x_2, \qquad x_2 \mapsto x_1^2, \qquad x_3a^{-1} \mapsto x_4, \qquad y_1 \mapsto y_2, \qquad y_2 \mapsto y_1^2.$$

In particular we see that $E_{E_8}^{3a}$ is conjugate to $\langle \overline{x_2}, \overline{y_2}, \overline{a} \rangle$. Moreover we have

$$C_{E_8(\mathbf{C})}(\overline{a}, \overline{x_2}) = \overline{\langle y_2, \{A \oplus B \oplus C \mid \det ABC = 1\} \rangle}.$$

From this we directly get

$$\begin{aligned} C_{E_8(\mathbf{C})}(\overline{a}, \overline{x_2}, \overline{y_2}) &= \overline{\left\langle x_2, y_2, \{A \oplus A \oplus A \mid (\det A)^3 = 1\} \right\rangle} \\ &= \overline{\langle x_2, y_2, a, \{A \oplus A \oplus A \mid \det A = 1\} \rangle} \\ &\cong \langle \overline{x_2}, \overline{y_2}, \overline{a} \rangle \times \mathrm{PSL}_3(\mathbf{C}). \end{aligned}$$

Thus $C_{E_8(\mathbf{C})}(E^{3a}_{E_8}) = E^{3a}_{E_8} \times \mathrm{PSL}_3(\mathbf{C})$ and $Z(C_{E_8(\mathbf{C})}(E^{3a}_{E_8})) = E^{3a}_{E_8}$. From the above we see that the elements $\overline{x_2}$, $\overline{x_3 a^{-1}}$ and $\overline{y_2}$ in $C_{E_8(\mathbf{C})}(E^{3a}_{E_8})$ correspond to the elements $\overline{\beta^2}$, $\overline{\tau_2^{-1}}$ and $\overline{\gamma^2}$ in the $\mathrm{PSL}_3(\mathbf{C})$ component of $C_{E_8(\mathbf{C})}(E^{3a}_{E_8})$. From this we easily compute the structure of $C_{E_8(\mathbf{C})}(E)$ for the representatives E which contain $E^{3a}_{E_8}$, cf. the proof of Theorem 8.9.

For the computation of the centralizers of $E^{3b}_{E_8}$ and $E^{4c}_{E_8}$ we consider the element $g = h_{\alpha_1}(\omega)h_{\alpha_3}(\omega^2) \in E_8(\mathbf{C})$. By using [32, Tables 4 and 6] we get that g belongs to the conjugacy class **3B** and that the centralizer $C_{E_8(\mathbf{C})}(g)$ has type E_6A_2. The precise structure of this centralizer may be found as follows. Since $E_8(\mathbf{C})$ is simply connected, Theorem 8.2(3) implies that $C_{E_8(\mathbf{C})}(g)$ is connected. Setting

$$\alpha' = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6,$$

we see that $\{\alpha_5, \alpha_8, \alpha_6, \alpha_7, \alpha', \alpha_2\} \cup \{\alpha_1, \alpha_3\}$ is a system of simple roots of $C_{E_8(\mathbf{C})}(g)$ (the simple systems of the components of type E_6 and A_2 have been ordered so that the numbering is consistent with [15, p. 250–251, 260–262]). From this we get an explicit homomorphism $3E_6(\mathbf{C}) \times \mathrm{SL}_3(\mathbf{C}) \to E_8(\mathbf{C})$ onto the centralizer $C_{E_8(\mathbf{C})}(g)$. The kernel is given by $\langle (z, \omega^2 I_3) \rangle$, where $z \in 3E_6(\mathbf{C})$ denotes the central element defined in Section 8.2. Thus $C_{E_8(\mathbf{C})}(g) = 3E_6(\mathbf{C}) \circ_{C_3} \mathrm{SL}_3(\mathbf{C})$, and we denote elements in this central product by $A \cdot B$ where $A \in 3E_6(\mathbf{C})$ and $B \in \mathrm{SL}_3(\mathbf{C})$. In particular we have $g = z \cdot I_3 = 1 \cdot \omega I_3$.

Now consider the subgroup $E = \langle z \cdot I_3, x_1 \cdot \beta, y_1 \cdot \gamma \rangle$ which is seen to be an elementary abelian 3-subgroup of rank 3 (here the elements $x_1, y_1 \in 3E_6(\mathbf{C})$ from Section 8.2 should not be confused with the elements $x_1, y_1 \in \mathrm{SL}_9(\mathbf{C})$ from above). We have

$$C_{E_8(\mathbf{C})}(z \cdot I_3, x_1 \cdot \beta) = C_{3E_6(\mathbf{C}) \circ_{C_3} \mathrm{SL}_3(\mathbf{C})}(x_1 \cdot \beta) = \left\langle y_1 \cdot \gamma, C_{3E_6(\mathbf{C})}(x_1) \circ_{C_3} C_{\mathrm{SL}_3(\mathbf{C})}(\beta) \right\rangle.$$

We note that $y_1 \cdot \gamma$ is not conjugate to its inverse in $C_{E_8(\mathbf{C})}(z \cdot I_3, x_1 \cdot \beta)$ since no element in $C_{\mathrm{SL}_3(\mathbf{C})}(\beta)$ conjugates γ into γ^{-1} times a power of ωI_3. Thus we have $\mathrm{diag}(1, 1, -1) \notin W(E)$ and in particular $W(E) \neq \mathrm{GL}_3(\mathbf{F}_3)$. From the above we also get

$$\begin{aligned} C_{E_8(\mathbf{C})}(E) &= \left\langle y_1 \cdot \gamma, x_1 \cdot \beta, C_{3E_6(\mathbf{C})}(x_1, y_1) \circ_{C_3} C_{\mathrm{SL}_3(\mathbf{C})}(\beta, \gamma) \right\rangle \\ &= \left\langle y_1 \cdot \gamma, x_1 \cdot \beta, C_{3E_6(\mathbf{C})}(\pi^{-1}(E^{2b}_{E_6})) \circ_{C_3} Z(\mathrm{SL}_3(\mathbf{C})) \right\rangle \\ &= \langle y_1 \cdot \gamma, x_1 \cdot \beta, (\langle z \rangle \times G_2(\mathbf{C})) \circ_{C_3} Z(\mathrm{SL}_3(\mathbf{C})) \rangle \\ &= E \times G_2(\mathbf{C}), \end{aligned}$$

using the computation of $C_{3E_6(\mathbf{C})}(\pi^{-1}(E^{2b}_{E_6}))$ from the last part of the proof of Theorem 8.9. Since the preimage of E in $3E_6(\mathbf{C}) \times \mathrm{SL}_3(\mathbf{C})$ is non-abelian it follows from Theorem 8.2(5) that E is non-toral in $3E_6(\mathbf{C}) \circ_{C_3} \mathrm{SL}_3(\mathbf{C})$. Now Theorem 8.2(1) shows that E is non-toral in $E_8(\mathbf{C})$ (alternatively one could also just observe that $C_{E_8(\mathbf{C})}(E)$ has rank less than 8). From what we have already proved we then see that E is conjugate to either $E^{3a}_{E_8}$ or $E^{3b}_{E_8}$ in $E_8(\mathbf{C})$. Since we already know $C_{E_8(\mathbf{C})}(E^{3a}_{E_8})$ we conclude that E must be conjugate to $E^{3b}_{E_8}$

(alternatively one could also compute the class distribution of E directly). In particular we have $C_{E_8(\mathbf{C})}(E^{3b}_{E_8}) = E^{3b}_{E_8} \times G_2(\mathbf{C})$ and $W(E^{3b}_{E_8}) \neq \mathrm{GL}_3(\mathbf{F}_3)$.

Using the inclusion $3E_6(\mathbf{C}) \subseteq 3E_6(\mathbf{C}) \circ_{C_3} \mathrm{SL}_3(\mathbf{C}) \subseteq E_8(\mathbf{C})$ we may also consider the subgroup $E^4_{3E_6} \subseteq 3E_6(\mathbf{C})$ from Theorem 8.7 as a subgroup of $E_8(\mathbf{C})$. Since $E^4_{3E_6}$ is non-toral in $3E_6(\mathbf{C})$, it is also non-toral in $3E_6(\mathbf{C}) \circ_{C_3} \mathrm{SL}_3(\mathbf{C})$, and hence also in $E_8(\mathbf{C})$ by Theorem 8.2(1). Thus $E^4_{3E_6}$ must be conjugate in $E_8(\mathbf{C})$ to one of the groups $E^{4a}_{E_8}$, $E^{4b}_{E_8}$ or $E^{4c}_{E_8}$. Comparing with the class distributions we can rule out $E^{4a}_{E_8}$ and $E^{4b}_{E_8}$, so we conclude that $E^4_{3E_6}$ is conjugate to $E^{4c}_{E_8}$. From Theorem 8.7 we have $C_{3E_6(\mathbf{C})}(E^4_{3E_6}) = E^4_{3E_6}$. Hence $C_{E_8(\mathbf{C})}(E^4_{3E_6}) = E^4_{3E_6} \circ_{C_3} \mathrm{SL}_3(\mathbf{C})$ from which we get $C_{E_8(\mathbf{C})}(E^{4c}_{E_8}) = E^{4c}_{E_8} \circ_{C_3} \mathrm{SL}_3(\mathbf{C})$. We determine the precise structure of the central product below after the computation of $W(E^{4c}_{E_8})$.

Exact Weyl groups: Recall from above that $E^{3a}_{E_8}$ is conjugate to $\langle \overline{x_2}, \overline{y_2}, \overline{a}\rangle$. If $W(E^{3a}_{E_8})$ was larger than the group $W'(E^{3a}_{E_8})$ from above, we then see that $W(E^{3a}_{E_8})$ would have to contain one of the groups

$$\left[\begin{array}{cc|c} \multicolumn{2}{c|}{\mathrm{GL}_2(\mathbf{F}_3)} & \begin{matrix} * \\ * \end{matrix} \\ \hline 0 & 0 & \varepsilon \end{array}\right] \quad \text{or} \quad \mathrm{SL}_3(\mathbf{F}_3)$$

which are the minimal overgroups of $W'(E^{3a}_{E_8})$ inside $\mathrm{GL}_3(\mathbf{F}_3)$. Thus $W(E^{3a}_{E_8})$ would have to contain one of the matrices $\mathrm{diag}(1, 2, 1)$ or $I_3 + e_{3,2}$. This would mean that inside $C_{E_8(\mathbf{C})}(\overline{x_2}, \overline{a})$ there would be an element which conjugates $\overline{y_2}$ into either $\overline{y_2}^2$ or $\overline{y_2 a}$. However from above we have

$$C_{E_8(\mathbf{C})}(\overline{x_2}, \overline{a}) = \overline{\langle y_2, \{A \oplus B \oplus C \mid \det ABC = 1\}\rangle},$$

and from this it is easily seen that no such element exists. Thus $W(E^{3a}_{E_8}) = W'(E^{3a}_{E_8})$ as claimed. For the group $E^{3b}_{E_8}$ we have $\mathrm{SL}_3(\mathbf{F}_3) \subseteq W(E^{3b}_{E_8}) \neq \mathrm{GL}_3(\mathbf{F}_3)$ and hence $W(E^{3b}_{E_8}) = \mathrm{SL}_3(\mathbf{F}_3)$.

As in the proof of Theorem 8.9 we show that the remaining Weyl groups equal the lower bounds already established, by looking at what a strictly larger Weyl group would imply for the subgroups $E^{3a}_{E_8}$ and $E^{3b}_{E_8}$. For $E = E^{4a}_{E_8}$, $E^{4b}_{E_8}$, $E^{5a}_{E_8}$, and $E^{5b}_{E_8}$, any proper overgroup of $W(E)$ contains an element which normalizes $E^{3a}_{E_8}$ but induces an automorphism on it not contained in its Weyl group. For $E^{4c}_{E_8}$ the result follows by considering the subgroup $E^{3b}_{E_8}$.

It remains only to determine the precise structure of the central product $C_{E_8(\mathbf{C})}(E^{4c}_{E_8}) = E^{4c}_{E_8} \circ_{C_3} \mathrm{SL}_3(\mathbf{C})$. From the structure of $W(E^{4c}_{E_8})$ we see that the subgroup $\langle\overline{x_2}\rangle$ is invariant under the action of $W(E^{4c}_{E_8})$. Thus a conjugation which sends to $E^{4c}_{E_8}$ to $E^4_{3E_6}$ must send $\langle\overline{x_2}\rangle$ to a $W(E^4_{3E_6})$-invariant subgroup of $E^4_{3E_6}$ of rank one. From the structure of $W(E^4_{3E_6})$ we see that there is only one such subgroup, namely $\langle z \cdot I_3\rangle = \langle 1 \cdot \omega I_3\rangle$. As this is exactly the center of the $\mathrm{SL}_3(\mathbf{C})$-component of $C_{E_8(\mathbf{C})}(E^4_{3E_6}) = E^4_{3E_6} \circ_{C_3} \mathrm{SL}_3(\mathbf{C})$, we see that $C_{E_8(\mathbf{C})}(E^{4c}_{E_8}) = E^{4c}_{E_8} \circ_{\langle\overline{x_2}\rangle} \mathrm{SL}_3(\mathbf{C})$. □

8.4. **The group $2E_7(\mathbf{C})$, $p = 3$.** In this section we consider the elementary abelian 3-subgroups of $2E_7(\mathbf{C})$. We let H be a maximal torus of $2E_7(\mathbf{C})$, $\Phi(E_7)$ be the root system relative to H, and choose a *realization* ([122, p. 133]) $(u_\alpha)_{\alpha\in\Phi(E_7)}$ of $\Phi(E_7)$ in $2E_7(\mathbf{C})$. By [122, 8.1.4(iv)] we may suppose that the root subgroups $(u'_\alpha)_{\alpha\in\Phi(E_6)}$ in $3E_6(\mathbf{C}) \subseteq 2E_7(\mathbf{C})$ coming from the choice of root subgroups for $3E_6(\mathbf{C})$ from Section 8.2 satisfy $u_\alpha = u'_\alpha$ for $\alpha \in \Phi(E_6)$.

For $\alpha = \alpha_i$, $1 \le i \le 7$, and $t \in \mathbf{C}^\times$ we define the elements

$$n_\alpha(t) = u_\alpha(t)u_{-\alpha}(-1/t)u_\alpha(t), \qquad h_\alpha(t) = n_\alpha(t)n_\alpha(1)^{-1}.$$

Then the maximal torus consists of the elements $\prod_{i=1}^{7} h_{\alpha_i}(t_i)$ and the normalizer $N(H)$ of the maximal torus is generated by H and the elements $n_i = n_{\alpha_i}(1)$, $1 \le i \le 7$.

As in Section 8.2 we define the following elements in $3E_6(\mathbf{C}) \subseteq 2E_7(\mathbf{C})$:

$$\begin{aligned} z &= h_{\alpha_1}(\omega)h_{\alpha_3}(\omega^2)h_{\alpha_5}(\omega)h_{\alpha_6}(\omega^2), \qquad a = h_{\alpha_1}(\omega)h_{\alpha_3}(\omega^2)h_{\alpha_5}(\omega^2)h_{\alpha_6}(\omega), \\ x_2 &= h_{\alpha_2}(\omega^2)h_{\alpha_3}(\omega^2)h_{\alpha_5}(\omega^2), \\ y_2 &= n_1n_2n_3n_4n_3n_1n_5n_4n_2n_3n_4n_5n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_6n_5h_{\alpha_2}(-1). \end{aligned}$$

Notation 8.17. The conjugacy classes of elements of order 3 in $2E_7(\mathbf{C})$ are given in [65, Table VI] and [32, Table 6] from which we take our notation. In particular, there are 5 such conjugacy classes, which we label **3A**, **3B**, **3C**, **3D** and **3E**. Moreover these classes may be distinguished by their traces on $\mathfrak{e}_7$, except for the classes **3A** and **3D** which have the same trace. Since the trace of the element $h \in H$ is given by $7 + \sum_{\alpha\in\Phi(E_7)} \alpha(h)$ we easily obtain the inclusions

$$\mathbf{3C}[3E_6] \subseteq \mathbf{3C}[2E_7], \qquad \mathbf{3E}[3E_6] \subseteq \mathbf{3B}[2E_7], \qquad \mathbf{3E}'[3E_6] \subseteq \mathbf{3B}[2E_7],$$

corresponding to the inclusion $3E_6(\mathbf{C}) \subseteq 2E_7(\mathbf{C})$.

Theorem 8.18. *The conjugacy classes of non-toral elementary abelian 3-subgroups of $2E_7(\mathbf{C})$ are given by the following table.*

rank	*name*	*ordered basis*	*$2E_7(\mathbf{C})$-class dist.*	$C_{2E_7(\mathbf{C})}(E)$	$Z(C_{2E_7(\mathbf{C})}(E))$
3	$E^3_{2E_7}$	$\langle a, x_2, y_2\rangle$	$\mathbf{3C^{26}}$	$E^3_{2E_7} \times \mathrm{SL}_2(\mathbf{C})$	$E^3_{2E_7} \times Z(2E_7(\mathbf{C}))$
4	$E^4_{2E_7}$	$\langle z, a, x_2, y_2\rangle$	$\mathbf{3B^2 3C^{78}}$	$E^4_{2E_7} \circ_{\langle z\rangle} \mathbf{T}_1$	$E^4_{2E_7} \circ_{\langle z\rangle} \mathbf{T}_1$

In particular we have ${}_3Z(C_{2E_7(\mathbf{C})}(E)) = E$ for any non-toral elementary abelian 3-subgroup of $2E_7(\mathbf{C})$.

The Weyl groups of these groups with respect to the given ordered bases are given as follows:

$$W(E^3_{2E_7}) = \mathrm{SL}_3(\mathbf{F}_3), \;\; W(E^4_{2E_7}) = \left[\begin{array}{c|ccc} \varepsilon & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbf{F}_3) & \\ 0 & & & \end{array}\right]$$

Remark 8.19. Note that our information on the rank 3 subgroup $E^3_{2E_7}$ corrects [65].

Proof of Theorem 8.18. Non-toral subgroups: From the way the realization $(u_\alpha)_{\alpha\in\Phi(E_7)}$ is chosen above, it follows from Theorem 8.7 that $E^3_{2E_7}$ and $E^4_{2E_7}$ are elementary abelian 3-subgroups of $2E_7(\mathbf{C})$ and that we have

$$W(E^3_{2E_7}) \supseteq \mathrm{SL}_3(\mathbf{F}_3), \;\; W(E^4_{2E_7}) \supseteq \left[\begin{array}{c|ccc} 1 & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbf{F}_3) & \\ 0 & & & \end{array}\right]$$

In particular we see that both $W(E^3_{2E_7})$ and $W(E^4_{2E_7})$ have orders divisible by 13 and since $13 \nmid |W(E_7)|$, we conclude by Theorem 8.2(2) that $E^3_{2E_7}$ and $E^4_{2E_7}$ are non-toral in $2E_7(\mathbf{C})$. By [65, Thm. 11.16] we know that there are precisely two conjugacy classes of non-toral elementary abelian 3-subgroups in $2E_7(\mathbf{C})$, and thus $E^3_{2E_7}$ and $E^4_{2E_7}$ represent these two conjugacy classes.

Class distributions: The class distributions follows directly from the class distributions of the groups $E^3_{3E_6}$ and $E^4_{3E_6}$ given in Theorem 8.7 and the information in 8.17 about the behavior of conjugacy classes in $3E_6(\mathbf{C})$ under the inclusion $3E_6(\mathbf{C}) \subseteq 2E_7(\mathbf{C})$.

Weyl groups: Using our realization $(u_\alpha)_{\alpha\in\Phi(E_7)}$ we may define a canonical map $\phi : W \to N(H)$ as follows ([122, 9.3.3]): If $w = s_{\alpha_{i_1}} \dots s_{\alpha_{i_r}}$ is a reduced expression for $w \in W$ we let $\phi(w) = n_{i_1} \dots n_{i_r}$ (by [122, 8.3.3 and 9.3.2] this does not depend on the reduced expression for w). Note that the element $\phi(w)$ is a representative in $N(H)$ for $w \in W$. Now let $w_0 \in W$ be the longest element in W, and let $n_0 = \phi(w_0)$. From [15, p. 264–266] it follows that w_0 equals the scalar transformation -1, and so conjugation by n_0 acts as inversion on H. Now let $w \in W$ and define w' by $ww' = w_0$. Since w_0 is central in our case, we have $(ww')\,w^{-1} = w^{-1}\,(ww') = w'$ so we conclude that $w'w = ww' = w_0$. Now let ℓ be the length function on W. By [73, p. 16] we have $\ell(w) + \ell(w') = \ell(w_0)$. In general the map ϕ is not a homomorphism, but we do have $\phi(w_1w_2) = \phi(w_1)\phi(w_2)$ if $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$ by [122, 9.3.4(i)]. From this it follows that $\phi(w)\phi(w') = \phi(w')\phi(w) = \phi(w_0) = n_0$, and we conclude that n_0 commutes with $\phi(w)$ for all $w \in W$.

Now consider the element

$$w = s_1s_2s_3s_4s_3s_1s_5s_4s_2s_3s_4s_5s_6s_5s_4s_2s_3s_1s_4s_3s_5s_4s_6s_5.$$

Using the fact that the length of an element is given by the number of positive roots it sends to negative roots ([73, Cor. 1.7]), we see that the above product is a reduced expression for w. Thus we have $y_2 = \phi(w)h_{\alpha_2}(-1)$. From the above we then conclude that conjugation by n_0 acts as follows:

$$z \mapsto z^2, \qquad a \mapsto a^2, \qquad x_2 \mapsto x_2^2, \qquad y_2 \mapsto y_2.$$

Thus n_0 normalizes $E^4_{2E_7}$ and gives the element $\operatorname{diag}(2,2,2,1)$ in $W(E^4_{2E_7})$. Combined with the above we conclude that

$$W(E^4_{2E_7}) \supseteq \left[\begin{array}{c|ccc} \varepsilon & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbf{F}_3) & \\ 0 & & & \end{array}\right]$$

From the inclusion $\Phi(E_7) \subseteq \Phi(E_8)$ we get the inclusion $2E_7(\mathbf{C}) \subseteq E_8(\mathbf{C})$, so we may consider $E^3_{2E_7}$ and $E^4_{2E_7}$ as subgroups of $E_8(\mathbf{C})$ as well. Since the orders of their Weyl groups in $2E_7(\mathbf{C})$ are divisible by 13 and $13 \nmid |W(E_8)|$, we see from Theorem 8.2(2) that $E^3_{2E_7}$ and $E^4_{2E_7}$ remain non-toral in $E_8(\mathbf{C})$. Using Theorem 8.15 and the class distributions from above we conclude that $E^3_{2E_7}$ and $E^4_{2E_7}$ are conjugate to $E^{3b}_{E_8}$ and $E^{4c}_{E_8}$ respectively in $E_8(\mathbf{C})$. Theorem 8.15 now shows that the lower bounds found above are indeed the Weyl groups of $E^3_{2E_7}$ and $E^4_{2E_7}$ in $2E_7(\mathbf{C})$.

Centralizers: For the computation of the centralizer of $E^3_{2E_7}$ we consider the element $g = h_{\alpha_1}(\omega)h_{\alpha_3}(\omega^2) \in 2E_7(\mathbf{C})$. By using [32, Table 6] we see that g belongs to the conjugacy class **3B** and that the centralizer $C_{2E_7(\mathbf{C})}(g)$ has type A_5A_2. The precise structure of this centralizer may be found as follows. Since $2E_7(\mathbf{C})$ is simply connected, Theorem 8.2(3) implies that $C_{2E_7(\mathbf{C})}(g)$ is connected. Setting

$$\alpha' = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6,$$

we see that $\{\alpha_5,\alpha_6,\alpha_7,\alpha',\alpha_2\}\cup\{\alpha_1,\alpha_3\}$ is a system of simple roots of $C_{2E_7(\mathbf{C})}(g)$ (the simple systems of the components of type A_5 and A_2 have been ordered so that the numbering is consistent with [15, p. 250–251]). From this we get an explicit homomorphism $\mathrm{SL}_6(\mathbf{C})\times\mathrm{SL}_3(\mathbf{C})\to 2E_7(\mathbf{C})$ onto the centralizer $C_{2E_7(\mathbf{C})}(g)$. The kernel is given by $\langle(\omega I_6,\omega^2 I_3)\rangle$. Thus $C_{2E_7(\mathbf{C})}(g)=\mathrm{SL}_6(\mathbf{C})\circ_{C_3}\mathrm{SL}_3(\mathbf{C})$, and we denote elements in this central product by $A\cdot B$ where $A\in\mathrm{SL}_6(\mathbf{C})$ and $B\in\mathrm{SL}_3(\mathbf{C})$. In particular we have $g=\omega I_6\cdot I_3=I_6\cdot\omega I_3$.

Now consider the subgroup $E=\langle\omega I_6\cdot I_3,(\beta\oplus\beta)\cdot\beta,(\gamma\oplus\gamma)\cdot\gamma^2\rangle$ which is seen to be an elementary abelian 3-subgroup of rank 3. We have

$$\begin{aligned}C_{2E_7(\mathbf{C})}(\omega I_6\cdot I_3,(\beta\oplus\beta)\cdot\beta)&=C_{\mathrm{SL}_6(\mathbf{C})\circ_{C_3}\mathrm{SL}_3(\mathbf{C})}((\beta\oplus\beta)\cdot\beta)\\&=\left\langle(\gamma\oplus\gamma)\cdot\gamma^2,C_{\mathrm{SL}_6(\mathbf{C})}(\beta\oplus\beta)\circ_{C_3}C_{\mathrm{SL}_3(\mathbf{C})}(\beta)\right\rangle.\end{aligned}$$

From this we get

$$\begin{aligned}C_{2E_7(\mathbf{C})}(E)&=\left\langle(\gamma\oplus\gamma)\cdot\gamma^2,(\beta\oplus\beta)\cdot\beta,C_{\mathrm{SL}_6(\mathbf{C})}(\beta\oplus\beta,\gamma\oplus\gamma)\circ_{C_3}C_{\mathrm{SL}_3(\mathbf{C})}(\beta,\gamma)\right\rangle\\&=\left\langle(\gamma\oplus\gamma)\cdot\gamma^2,(\beta\oplus\beta)\cdot\beta,C_{\mathrm{SL}_6(\mathbf{C})}(\beta\oplus\beta,\gamma\oplus\gamma)\circ_{C_3}Z(\mathrm{SL}_3(\mathbf{C}))\right\rangle.\end{aligned}$$

Here $C_{\mathrm{SL}_6(\mathbf{C})}(\beta\oplus\beta,\gamma\oplus\gamma)=\Delta_{2,3}(\{A\in\mathrm{GL}_2(\mathbf{C})|(\det A)^3=1\})$ is generated by $\Delta_{2,3}(\omega^2 I_2)=\omega^2 I_6$ and $\Delta_{2,3}(\mathrm{SL}_2(\mathbf{C}))$. From this we get

$$C_{2E_7(\mathbf{C})}(E)=\langle E,\Delta_{2,3}(\mathrm{SL}_2(\mathbf{C}))\rangle\cong E\times\mathrm{SL}_2(\mathbf{C}).$$

Since the preimage of E in $\mathrm{SL}_6(\mathbf{C})\times\mathrm{SL}_3(\mathbf{C})$ is non-abelian it follows from Theorem 8.2(5) that E is non-toral in $\mathrm{SL}_6(\mathbf{C})\circ_{C_3}\mathrm{SL}_3(\mathbf{C})$. Now Theorem 8.2(1) shows that E is non-toral in $2E_7(\mathbf{C})$ (alternatively one could also just observe that $C_{2E_7(\mathbf{C})}(E)$ has rank less than 7). It then follows that E is conjugate to $E^3_{2E_7}$ in $2E_7(\mathbf{C})$. In particular we have $C_{2E_7(\mathbf{C})}(E^3_{2E_7})=E^3_{2E_7}\times\mathrm{SL}_2(\mathbf{C})$. Hence $Z(C_{2E_7(\mathbf{C})}(E^3_{2E_7}))=E^3_{2E_7}\times Z(2E_7(\mathbf{C}))$ since the center of $2E_7(\mathbf{C})$ has order 2.

To compute the centralizer of $E^4_{2E_7}$ we note that $C_{2E_7(\mathbf{C})}(z)$ has centralizer type E_6T_1, and that the E_6 component corresponds to the subgroup $3E_6(\mathbf{C})\subseteq 2E_7(\mathbf{C})$. A computation shows that the T_1 component is given by $\mathbf{T}_1=\{h(t^2,t^3,t^4,t^6,t^5,t^4,t^3)|t\in\mathbf{C}^\times\}$, and thus we get $C_{2E_7(\mathbf{C})}(z)=3E_6(\mathbf{C})\circ_{\langle z\rangle}\mathbf{T}_1$. Theorem 8.7 now shows that $C_{2E_7(\mathbf{C})}(E^4_{2E_7})=C_{3E_6(\mathbf{C})}(E^4_{2E_7})\circ_{\langle z\rangle}\mathbf{T}_1=E^4_{2E_7}\circ_{\langle z\rangle}\mathbf{T}_1$. □

9. Non-toral elementary abelian p-subgroups of projective unitary groups

The purpose of this short section is to describe the non-toral elementary abelian subgroups of $\mathrm{PGL}_n(\mathbf{C})$, which by Theorem 8.4 is equivalent to finding them for its compact form $\mathrm{PU}(n)$, as well as to give information about centralizers and Weyl groups. The subgroups are easily determined and are described in [65, §3]—we here just add some extra information about centralizers and Weyl groups which we need in our proof of Theorem 1.1.

We first introduce a useful subgroup. If p^r divides n write $n=p^rk$ and consider the extraspecial group p_+^{1+2r} embedded in $\mathrm{GL}_n(\mathbf{C})$ by taking k copies of one of the $p-1$ faithful irreducible p^r-dimensional representations. (They all have the same image; see [74, Satz 16.14].) Note that this embedding maps the center of p_+^{1+2r} to the elements of order p in the center of $\mathrm{GL}_n(\mathbf{C})$. Let Γ_r denote the subgroup of $\mathrm{GL}_n(\mathbf{C})$ given by the subgroup generated by the image of p_+^{1+2r} and the center of $\mathrm{GL}_n(\mathbf{C})$. Note that as an abstract group Γ_r fits into an extension sequence

$$1\to\mathbf{C}^\times\to\Gamma_r\to\bar{\Gamma}_r\to 1$$

where $\mathbf{C}^\times$ identifies with the center of $\mathbf{C}^\times$ and $\bar{\Gamma}_r \cong (\mathbf{Z}/p)^{2r}$ identifies with the image of Γ_r in $\mathrm{PGL}_n(\mathbf{C})$. (The matrices for Γ_r are written explicitly for $k = 1$ in [107, p. 56] where it is called $\Gamma^U_{p^r}$.)

Theorem 9.1. *Suppose E is a non-toral elementary abelian p-subgroup of $\mathrm{PGL}_n(\mathbf{C})$ for an arbitrary prime p. Then, up to conjugacy E can be written $E = \bar{\Gamma}_r \times \bar{A}$, for some $r \geq 1$ and some abelian subgroup A of $C_{\mathrm{GL}_n(\mathbf{C})}(\Gamma_r) \cong \mathrm{GL}_k(\mathbf{C})$.*

For a given r the number of conjugacy classes of such subgroups E are in one-to-one correspondence with conjugacy classes of toral elementary abelian p-subgroups $\bar{A}$ of $\mathrm{PGL}_k(\mathbf{C}) \cong C_{\mathrm{PGL}_n(\mathbf{C})}(\bar{\Gamma}_r)_1$ (allowing the trivial subgroup), and the centralizer of E is given by $C_{\mathrm{PGL}_n(\mathbf{C})}(E) \cong \bar{\Gamma}_r \times C_{\mathrm{PGL}_k(\mathbf{C})}(\bar{A})$.

The Weyl group equals

$$W_{\mathrm{PGL}_n(\mathbf{C})}(E) = \begin{bmatrix} \mathrm{Sp}(\bar{\Gamma}_r) & 0 \\ * & W_{\mathrm{PGL}_k(\mathbf{C})}(\bar{A}) \end{bmatrix}$$

Here $\mathrm{Sp}(\bar{\Gamma}_r)$ is the symplectic group relative to the symplectic product coming from the commutator product $[\cdot,\cdot] : \bar{\Gamma}_r \times \bar{\Gamma}_r \to \mathbf{Z}/p \subseteq \mathbf{C}^\times$ and the symbol $*$ denotes a $\mathrm{rank}\,\bar{A} \times 2r$ matrix with arbitrary entries.

An element $\alpha \in \mathrm{Sp}(\bar{\Gamma}_r) \subseteq W_{\mathrm{PGL}_n(\mathbf{C})}(E)$ acts as up to conjugacy as $\alpha \times 1$ on $C_{\mathrm{PGL}_n(\mathbf{C})}(E) \cong \bar{\Gamma}_r \times C_{\mathrm{PGL}_k(\mathbf{C})}(\bar{A})$.

Sketch of proof: The existence of the decomposition $E = \bar{\Gamma}_r \times \bar{A}$ follows from Griess [65, Thm. 3.1] and the statements about uniqueness follow by representation theory of the extraspecial p-groups.

Since the image of p_+^{1+2r} is the sum of k identical irreducible representations we have by Schur's lemma that $C_{\mathrm{GL}_n(\mathbf{C})}(\Gamma_r) \cong \mathrm{GL}_k(\mathbf{C})$ (see also [107, Prop. 4]). From this the centralizer in $\mathrm{PGL}_n(\mathbf{C})$ can easily be worked out.

In the case where $\bar{A}$ is trivial the statement about Weyl groups is given in [107, Thm. 6] (and just uses elementary character theory). The general case follows similarly, again using character theory.

For the statement about the Weyl group action, first note that $\mathrm{Out}(\bar{\Gamma}_r \times \mathrm{PGL}_k(\mathbf{C})) \cong \mathrm{Aut}(\bar{\Gamma}_r) \times \mathrm{Out}(\mathrm{PGL}_k(\mathbf{C}))$. An element $\alpha \in \mathrm{Sp}(\Gamma_r) = W_{\mathrm{PGL}_n(\mathbf{C})}(\bar{\Gamma}_r)$ acts as an inner automorphism on $\mathrm{PGL}_k(\mathbf{C})$ since this is true for the action on $C_{\mathrm{GL}_n(\mathbf{C})}(\Gamma_r) \cong \mathrm{GL}_k(\mathbf{C})$ by character theory. Hence we can choose a representative $g \in N_{\mathrm{PGL}_n(\mathbf{C})}(\bar{\Gamma}_r)$ of α which acts as $\alpha \times 1$ on $C_{\mathrm{PGL}_n(\mathbf{C})}(\bar{\Gamma}_r) \cong \bar{\Gamma}_r \times \mathrm{PGL}_k(\mathbf{C})$. Hence g is also a representative of $\alpha \in \mathrm{Sp}(\bar{\Gamma}_r) \subseteq W_{\mathrm{PGL}_n(\mathbf{C})}(\bar{\Gamma}_r \times \bar{A})$. The claim now follows. □

10. Calculation of the obstruction groups

In this section we show that the existence and uniqueness obstructions to lifting our diagram in the homotopy category to a diagram in the category of spaces identically vanish. More precisely, we will show the following theorem.

Theorem 10.1. *Suppose that X is any of the following p-compact groups $(F_4)\hat{_3}$, $(E_6)\hat{_3}$, $(E_7)\hat{_3}$, $(E_8)\hat{_3}$, $(E_8)\hat{_5}$ or $\mathrm{PU}(n)\hat{_p}$ (any p), or suppose that p is odd and X is connected and center-free with $H^*(BX;\mathbf{Z}_p)$ a polynomial algebra. Then*

$$\lim_{\mathbf{A}(X)}{}^i \pi_j(B\mathcal{Z}(\mathcal{C}_X(-))) = 0, \text{ for all } i, j.$$

(See Theorem 12.2 for an explanation of why exactly these p-compact groups need attention.) Note that for the purpose of Theorem 1.4 we only need to calculate the above groups for $j = 1, 2$ and $i = j$ or $i = j + 1$.

We prove the theorem by filtering the functor $F_j = \pi_j(B\mathcal{Z}(\mathcal{C}_X(-)))$, and showing that all filtration quotients vanish (with a small twist for $\mathrm{PU}(2)\hat{_2}$). First we show that the quotient functor of F_j concentrated on the toral elementary abelian p-subgroups has vanishing limits, using a Mackey functor argument which first appeared in [50]. This takes care of the case where $H^*(BX; \mathbf{Z}_p)$ is a polynomial algebra since in this case all subgroups are toral by Lemma 7.8. For the exceptional compact connected Lie groups we then continue and filter the non-toral part of the functor by functors concentrated on only one non-toral subgroup, and use a formula of Oliver [106] to show that the higher limits of these subquotient functors all vanish. For $\mathrm{PU}(n)$ we use a variant of this technique by suitably grouping the non-toral subgroups and using a combination of Oliver's formula and the Mackey functor argument we used for the toral part.

We use the notation St_G to denote the Steinberg module over $\mathbf{Z}_p$ of a finite group of Lie type G of characteristic p, defined as the top homology group with $\mathbf{Z}_p$ coefficients of the Tits building of G (see e.g., [72]). In the special case of $\mathrm{GL}(E)$ we also write $\mathrm{St}(E)$ for the Steinberg module.

10.1. **The toral part.** Define a quotient functor F_j^{tor} of F_j by setting $F_j^{\mathrm{tor}}(V) = F_j(V)$ if V is toral and $F_j^{\mathrm{tor}}(V) = 0$ if V is non-toral. Let $\mathbf{A}^{\mathrm{tor}}(X)$ denote the full subcategory of $\mathbf{A}(X)$ consisting of toral subgroups. From the chain complex defining higher limits (see e.g., [63, 3.3]) it follows that

$$\lim_{\mathbf{A}(X)}{}^* F_j^{\mathrm{tor}} \cong \lim_{\mathbf{A}^{\mathrm{tor}}(X)}{}^* F_j^{\mathrm{tor}}$$

In order to use a Mackey functor argument on the right-hand side we need a more explicit description of the functor F_j^{tor}.

Lemma 10.2. *Fix a connected p-compact group X and let $\breve{T}$ be the discrete approximation to a maximal torus T in X. For a non-trivial elementary abelian p-subgroup $V \subseteq \breve{T}$, let $W_X(V)$ denote the Weyl group of $\mathcal{C}_X(V)$ and let $W_X(V)_1$ denote the Weyl group of $\mathcal{C}_X(V)_1$ (see* [52, Thm. 7.6]*).*

If $\breve{T}^{W_X(V)_1}$ is a discrete approximation to $\mathcal{Z}(\mathcal{C}_X(V)_1)$ then $\breve{T}^{W_X(V)}$ is a discrete approximation to $\mathcal{Z}(\mathcal{C}_X(V))$. In particular in this case $\pi_1(B\mathcal{Z}(\mathcal{C}_X(V))) = H^1(W_X(V); L_X)$ and $\pi_2(B\mathcal{Z}(\mathcal{C}_X(V)) = (L_X)^{W_X(V)}$, where $L_X = \pi_1(T)$.

Remark 10.3. For a connected p-compact group X and p odd, the fixed point set $\breve{T}^{W_X}$ always equals a discrete approximation to the center of X by [52, Thm. 7.6]. If X is the p-completion of a compact connected Lie group then this is likewise the case for $p = 2$ unless X contains a direct factor isomorphic to $\mathrm{SO}(2n+1)\hat{_2}$, by [85, Thm. 1.6].

Proof of Lemma 10.2. Set $Y = \mathcal{C}_X(V)$ and $\pi = \pi_0(Y)$ for short. Since V is toral, $\breve{T}$ is in a canonical way a discrete approximation to a maximal torus in Y.

First observe that the center of Y has discrete approximation in $\breve{T}$. Indeed, otherwise there would by [52, Thm. 6.4] exist a central homomorphism $f : \mathbf{Z}/p^n \to \breve{\mathcal{N}}_{p,Y}$ with image not in $\breve{T}$, which would produce a homomorphism $f' : \mathbf{Z}/p^n \to \breve{\mathcal{N}}_{p,X}$ commuting with $\breve{T}$ but not in $\breve{T}$, which contradicts the fact that T is self-centralizing in X by [51, Thm. 9.1], since X is connected.

Suppose that $\breve{T}^{W_X(V)_1}$ is a discrete approximation to $\mathcal{Z}(Y_1)$ and set $C = \breve{T}^{W_X(V)}$. We want to show that C is central in Y. Let $f : BC \to BY_1$ be the natural inclusion. We have an obvious diagram with horizontal maps fibrations

$$\begin{array}{ccccc}
\operatorname{map}(BC, BY_1)_{\{f\}} & \longrightarrow & \operatorname{map}(BC, BY)_f & \longrightarrow & \operatorname{map}(BC, B\pi)_0 \\
\downarrow & & \downarrow & & \Vert \\
BY_1 & \longrightarrow & BY & \longrightarrow & B\pi
\end{array}$$

where $\{f\}$ denotes the set of homotopy classes of maps $BC \to BY_1$ generated by f under the π-action on BY_1. If we can show that $\{f\}$ consists of just f then it follows from the five-lemma that the middle vertical map is a homotopy equivalence, since our assumption implies that C is central in Y_1.

To see that the action is trivial consider the following diagram:

$$\begin{array}{ccccc}
 & & B\mathcal{N}_{Y_1} & \xrightarrow{\tilde{g}} & B\mathcal{N}_{Y_1} \\
 & \overset{\tilde{f}}{\nearrow} & \downarrow & & \downarrow \\
BC & \xrightarrow{f} & BY_1 & \xrightarrow{g} & BY_1
\end{array}$$

where $\tilde{f}$ is the natural inclusion of BC in $B\mathcal{N}_{Y_1}$, g is an element in $\operatorname{Aut}(BY_1)$ induced by an element in π and $\tilde{g}$ is the corresponding self-map of $B\mathcal{N}_{Y_1}$ defined via Lemma 2.1. However by the definition of C, the composite $\tilde{g}\tilde{f}$ is homotopic to $\tilde{f}$ for all g induced by an element in π, so f is homotopic to gf as well. Hence we have shown that C is central in Y and since the center of Y has discrete approximation in $\breve{T}$ it is obviously the largest subgroup with this property. So C is a discrete approximation to $\mathcal{Z}Y$ as wanted.

The last statement about the homotopy groups now follows easily using the long exact sequence in group cohomology. □

Remark 10.4. The above lemma should be compared to Lemma 4.5 and Remark 4.6 which have slightly different assumptions and conclusions.

The following lemma is essentially contained in [50, §8].

Lemma 10.5. *Let X be a connected p-compact group, and assume that for each non-trivial toral elementary abelian p-subgroup $V \in \mathbf{A}(X)$ the fixed point set $\breve{T}^{W_X(V)_1}$ is a discrete approximation to $\mathcal{Z}(\mathcal{C}_X(V)_1)$. Then*

$$\lim_{\mathbf{A}(X)}{}^i F_j^{\mathrm{tor}} = \begin{cases} H^{2-j}(W_X; L_X) & \text{if } i = 0 \text{ and } j = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

where $H^{2-j}(W_X; L_X) \cong \pi_j(B\mathcal{Z}(X))$ if $\breve{T}^W$ is a discrete approximation to $\mathcal{Z}(X)$.

In particular, if p is odd, or more generally if for all reflections $s \in W_X$ the singular set $\sigma(s)$ equals the fixed point set $\breve{T}^{\langle s \rangle}$ then the assumptions above are satisfied and $\lim_{\mathbf{A}(X)}{}^i F_j^{\mathrm{tor}} = \pi_j(B\mathcal{Z}(X))$ if $i = 0$ and zero otherwise. (See [52, Def. 7.3] *and Remark 10.6 for the definition of $\sigma(s)$.)*

Proof. The first part of the proof consists of a translation of [50, §8] into the current notation. By [53, Prop. 3.4] all morphisms in $\mathbf{A}(X)$ between toral subgroups $V \to X$ and $V' \to X$ are induced by inclusions and action by elements of W_X. Hence we can identify $\mathbf{A}^{\mathrm{tor}}(X)$, up to equivalence of categories with a category which has objects non-trivial subgroups of ${}_p\breve{T} \cong (\mathbf{Z}/p)^r$ (where r is the rank of T) and morphisms the homomorphism between subgroups induced by inclusions and action by W_X. Also, by [52, Thm. 7.6], $W_X(V)$ consists of the elements in W_X which pointwise fixes V. Hence Lemma 10.2 shows that the functor F_2^{tor} on $\mathbf{A}^{\mathrm{tor}}(X)$ is isomorphic to the functor $\alpha^0_{\Gamma,M}$ on $\mathbf{A}_\Gamma$ from [50, §8], where $\Gamma = W_X$ and $M = L_X$. Likewise F_1^{tor} is isomorphic to $\alpha^1_{\Gamma,M}$. (Note that there is the slight difference from [50, §8] that M is a $\mathbf{Z}_p\Gamma$-module rather than an $\mathbf{F}_p\Gamma$-module, but this makes no difference.) Therefore [50, §8] (which is a Mackey functor argument, which can also be deduced from [49] or [75]) implies the first part of the lemma about obstruction groups.

To see the last part about the singular set recall that for an abelian subgroup $A \subseteq \breve{T}$ we have by [52, Thm. 7.6] that

$$\bigcap_{\substack{\text{reflections } s \in W_X \\ \text{such that } A \subseteq \sigma(s)}} \sigma(s)$$

is a discrete approximation to $\mathcal{Z}(\mathcal{C}_X(A)_1)$. Hence if $\sigma(s)$ equals $\breve{T}^{\langle s \rangle}$ then the assumptions of the first part are obviously satisfied since (again by [52, Thm. 7.6]) $W_X(A)_1$ is generated by reflections $s \in W_X$ with $A \subseteq \sigma(s)$. □

Remark 10.6. Let G be a compact connected Lie group with maximal torus T, and let α be a root of G relative to T with corresponding reflection s_α. In this case the singular set $\sigma(s_\alpha)$ is just the discrete approximation of the kernel U_α of α on T. (To see this note that by [17, §4, no. 5] the reflection s_α lifts to an element n_α (denoted by $\nu(\theta)$ in [17]) which satisfies $n_\alpha^2 = \exp(\alpha^\vee/2)$; the statement now follows—cf. [85, Pf. of Prop. 3.1(ii)].)

Explicit calculations [85, Prop. 3.1(ii)] (see also [58], [76, Prop. 3.2(vi)], and [109, §4]) show that for a compact connected Lie group G, $\sigma(s)$ in fact always equal to $\breve{T}^{\langle s \rangle}$ except when G contains a direct factor isomorphic to $\mathrm{SO}(2n+1)$. Combining this with Lemma 10.5, now gives the following calculation of the toral part of the obstruction groups, whose full strength at $p = 2$ we will however not use here.

Corollary 10.7. *Let G be a compact connected Lie group with no direct factors isomorphic to $\mathrm{SO}(2n+1)$ when $p = 2$. Set $X = G\hat{_p}$. Then*

$$\lim_{\mathbf{A}(X)}{}^i F_j^{\mathrm{tor}} = \begin{cases} \pi_j(B\mathcal{Z}(X)) & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

□

Proof of Theorem 10.1 when $H^*(BX; \mathbf{Z}_p)$ is polynomial, p odd. If $H^*(BX; \mathbf{Z}_p)$ is a polynomial algebra concentrated in even degrees then all elementary abelian p-subgroups are toral by Lemma 7.8, so $F = F^{\mathrm{tor}}$. Since p is odd the assumptions of Lemma 10.5 are satisfied and Theorem 10.1 follows. □

10.2. **The non-toral part for the exceptional groups.** In this subsection we prove Theorem 10.1 when X is the p-completion of one of the exceptional groups and p is odd. Let F_j^E denote the subquotient functor of F_j concentrated on a non-toral elementary abelian p-subgroup E. By Oliver's formula [106, Prop. 4]

$$\lim_{\mathbf{A}(X)}{}^i F_j^E = \begin{cases} \mathrm{Hom}_W(\mathrm{St}(E), F_j(E)) & \text{if } i = \mathrm{rk}\, E - 1 \\ 0 & \text{otherwise.} \end{cases}$$

We now embark on proving some lemmas which will be used to show that these obstructions groups identically vanish. (For the use in Theorem 1.4 we actually only need this when E has rank at most four.)

Since $\mathcal{Z}(\mathcal{C}_X(E))$ is the p-completion of an abelian compact Lie group (by [52, Thm. 1.1]), $F_j = 0$ unless $j = 1, 2$. The following lemma reduces the problem of showing that the obstruction groups vanish to showing that $\operatorname{Hom}_{W(E)}(\operatorname{St}(E), {}_p\breve{\mathcal{Z}}(\mathcal{C}_X(E))) = 0$, where ${}_p\breve{\mathcal{Z}}(\mathcal{C}_X(E))$ is the finite group of elements of order p in the discrete approximation $\breve{\mathcal{Z}}(\mathcal{C}_X(E))$.

Lemma 10.8. *Let A be an abelian compact Lie group, and let ${}_pA$ and A^p denote the kernel and the image of the pth power map on A (using multiplicative notation). Let P be a finitely generated projective $\mathbf{Z}_pW$-module for a finite group W, and assume that A has a module action of W.*

Then $\operatorname{Hom}_W(P, {}_pA) = 0$ *if and only if* $\operatorname{Hom}_W(P, \pi_1(A) \otimes \mathbf{Z}_p) = \operatorname{Hom}_W(P, \pi_0(A) \otimes \mathbf{Z}_p) = 0$.

Proof. The long exact sequence of homotopy groups associated to the exact sequence of groups $1 \to A^p \to A \to A/A^p \to 1$ shows that the inclusion $A^p \hookrightarrow A$ induces an isomorphism $\pi_1(A^p) \xrightarrow{\cong} \pi_1(A)$ and an injection $\pi_0(A^p) \hookrightarrow \pi_0(A)$.

Hence the exact sequence $1 \to {}_pA \to A \xrightarrow{p} A^p \to 1$ produces the following diagram, where the row, as well as the sequence going through $\pi_i(A)$ instead of $\pi_i(A^p)$, is exact.

$$\begin{array}{ccccccccc} \pi_1(A) & \xrightarrow{p} & \pi_1(A^p) & \longrightarrow & \pi_0({}_pA) & \longrightarrow & \pi_0(A) & \xrightarrow{p} & \pi_0(A^p) \\ & \searrow^{p} & \downarrow{\cong} & & & & & \searrow^{p} & \downarrow \\ & & \pi_1(A) & & & & & & \pi_0(A) \end{array}$$

Apply the exact functor $\operatorname{Hom}_W(P, - \otimes \mathbf{Z}_p)$ to this diagram. The lemma now follows from Nakayama's lemma, using that $\pi_0(A)$ is finite and $\pi_1(A)$ is finitely generated. □

The following elementary observation is so useful that it is worth stating explicitly.

Lemma 10.9. *Suppose that W is a subgroup of $\operatorname{GL}(E)$, p odd, such that $-1 \in W$. Then $\operatorname{Hom}_W(\operatorname{St}(E), E) = 0$.*

Proof. Set $Z = \langle -1 \rangle$. Since Z acts trivially on $\operatorname{St}(E)$ we have

$$\operatorname{Hom}_W(\operatorname{St}(E), E) \subseteq \operatorname{Hom}_Z(\operatorname{St}(E), E) = \operatorname{Hom}_Z(\operatorname{St}(E), E^Z) = 0.$$

□

We also need the following lemma, which is a special case of a theorem of Smith [121].

Lemma 10.10. *Let G be a finite group of Lie type of characteristic p, and let P be a parabolic subgroup of G with corresponding unipotent radical U and Levi subgroup $L \cong P/U$. Suppose that W is a subgroup $U \subseteq W \subseteq P$, and let M be an $\mathbf{F}_pW$-module.*

(1) *If U acts trivially on M, then* $\operatorname{Hom}_W(\operatorname{St}_G, M) = \operatorname{Hom}_{W/U}(\operatorname{St}_L, M)$.
(2) *If $\operatorname{St}_L \otimes \mathbf{F}_p$ is irreducible as an $\mathbf{F}_pW/U$-module and if M has a finite filtration as an $\mathbf{F}_pW$-module, with filtration quotients of $\mathbf{F}_p$-dimension strictly less than* $\operatorname{rank}_{\mathbf{Z}_p} \operatorname{St}_L$ *then* $\operatorname{Hom}_W(\operatorname{St}_G, M) = 0$.

Proof. Since U acts trivially on M, $\operatorname{Hom}_W(\operatorname{St}_G, M) = \operatorname{Hom}_{W/U}((\operatorname{St}_G)_U, M)$ where $(-)_U$ denotes coinvariants. But since the Steinberg module is self-dual, as is clear from its definition as a homology module, $(\operatorname{St}_G)_U \cong (\operatorname{St}_G)^U$. Now Smith's theorem [121] says that $(\operatorname{St}_G)^U \cong \operatorname{St}_L$, which proves the first part of the lemma.

For the second part, we can assume that the filtration quotients are simple $\mathbf{F}_pW$-modules. Since $U \subseteq O_p(W)$, U acts trivially on any irreducible $\mathbf{F}_pW$-module, by elementary representation theory. Hence the second part follows from the first together with a dimension consideration. □

The above lemma is usually used in conjunction with the following obvious observation.

Lemma 10.11. *Let E be a non-toral elementary abelian p-subgroup of a compact Lie group G. Then the $\mathbf{F}_p$-dimension of ${}_pZC_G(E)$ is at most equal to the maximal dimension of a non-toral elementary abelian p-subgroup of G, and E is a $W(E)$-submodule of ${}_pZC_G(E)$.* □

The last lemma we shall need is a concrete calculation.

Lemma 10.12. *Let E be a rank 4 elementary abelian 3-group, and let $W = \mathrm{SL}_3(\mathbf{F}_3) \times 1 \subseteq \mathrm{GL}(E)$. Then $\mathrm{Hom}_W(\mathrm{St}(E), E) = 0$.*

Proof. This most easily checked by computer, e.g., using MAGMA [13], but is indeed a sufficiently small calculation so that the computer's algorithm with a bit of effort can be redone by hand. Alternatively one can use some ad hoc Lie theoretic arguments. (We are grateful to A. Kleschev and H. H. Andersen for sketching a couple of such arguments to us—however, since these arguments are rather involved compared to the size of the calculation at hand we will not provide them here.) □

Before we start going through the exceptional groups, we need to introduce a bit of notation. For an $\mathbf{F}_p$-vector space $E = \langle e_1, \ldots, e_n\rangle$, we let $E_{ij\ldots}$ denote the subspace generated by e_i, e_j, ... Likewise we let $P_{ij\ldots}$ (resp. $U_{ij\ldots}$) denote the parabolic subgroup (resp. its unipotent radical) of $\mathrm{GL}(E)$ corresponding to the simple roots $\alpha_i, \alpha_j, \ldots$ in the standard basis and notation. For example in $\mathrm{GL}_3(\mathbf{F}_p)$, U_2 is the subgroup

$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Proof of Theorem 10.1 when $X = (E_8)\hat{_5}$, $(F_4)\hat{_3}$, $(E_6)\hat{_3}$, $(E_7)\hat{_3}$, or $(E_8)\hat{_3}$. By Lemma 10.8 it is enough to see that $\mathrm{Hom}_{W(E)}(\mathrm{St}(E), {}_pZC_G(E)) = 0$ for all non-toral elementary abelian p-subgroups of G. We proceed case-by-case.
$(E_8, 5)$ and $(F_4, 3)$: By [65, Lem. 10.3 and Thm. 7.4] G has, up to conjugacy, one non-toral elementary abelian p-subgroup E, which has rank 3, Weyl group $\mathrm{SL}(E)$, and (since E is necessarily maximal) $E = {}_pZC_G(E)$. Since $\mathrm{St}(E)$ is an irreducible $\mathrm{SL}(E)$-module of dimension p^3 we have that $\mathrm{Hom}_W(\mathrm{St}(E), E) = 0$.
$(E_7, 3)$: By Theorem 8.18 E_7 has, up to conjugacy, two non-toral elementary abelian 3-subgroups $E^3_{2E_7}$ and $E^4_{2E_7}$ of rank 3 and 4 respectively. Since $W(E^3_{2E_7}) = \mathrm{SL}_3(\mathbf{F}_3)$ a dimension consideration as above gives $\mathrm{Hom}_{W(E^3_{2E_7})}(\mathrm{St}(E^3_{2E_7}), {}_pZC_G(E^3_{2E_7})) = 0$. For $E^4_{2E_7}$ (whose Weyl group is listed in Theorem 8.18) we use Lemma 10.10(2), taking $U = U_{23}$, which immediately gives that also $\mathrm{Hom}_{W(E^4_{2E_7})}(\mathrm{St}(E^4_{2E_7}), {}_pZC_G(E^4_{2E_7})) = 0$.
$(E_6, 3)$: By Theorem 8.9 E_6 has eight non-toral elementary abelian 3-subgroups all of rank less than or equal to four. We follow the notation of this theorem. By Lemma 10.10(2), $\mathrm{Hom}_W(\mathrm{St}(E), {}_pZC_G(E)) = 0$ when $E = E^{2b}_{E_6}$, $E^{3b}_{E_6}$, $E^{3c}_{E_6}$ or $E^{4a}_{E_6}$ (taking $U = 1$, 1, U_2, and U_1 respectively). For $E = E^{2a}_{E_6}$, $E^{3a}_{E_6}$, $E^{3d}_{E_6}$, and $E^{4b}_{E_6}$ we use that by Theorem 8.9 $E = {}_pZC_G(E)$

in these cases (a fact that we did not need above), and also $-1 \in W(E)$ so Lemma 10.9 applies to show that $\mathrm{Hom}_W(\mathrm{St}(E), {}_pZC_G(E)) = 0$.
$(E_8, 3)$: By Theorem 8.15 E_8 has seven conjugacy classes of non-toral subgroups. If $E = E^{3a}_{E_8}$, $E^{3b}_{E_8}$, $E^{4b}_{E_8}$, or $E^{4c}_{E_8}$ then Lemma 10.10 shows that $\mathrm{Hom}_W(\mathrm{St}(E), {}_pZC_G(E)) = 0$ (taking $U = U_1$, 1, U_2, and U_{23} respectively). (Note that we do not need to know ${}_pZC_G(E)$ exactly since the rough bound from Lemma 10.11 will do.) We now consider $E = E^{4a}_{E_8}$. By Theorem 8.15 we have that ${}_pZ(C_G(E)) = E$, and by Lemma 10.12 $\mathrm{Hom}_{W(E)}(\mathrm{St}(E), E) = 0$. Suppose that $E = E^{5a}_{E_8}$. Then E has an invariant subspace E_1 upon which $U = U_{234}$ acts trivially. Now $\mathrm{Hom}_W(\mathrm{St}(E), E_1) = \mathrm{Hom}_{W/U}(\mathrm{St}(E_1) \otimes \mathrm{St}(E_{2345}), E_1) = 0$, where we use Lemma 10.10 (to see that $\mathrm{St}(E)^U \cong \mathrm{St}(E_1) \otimes \mathrm{St}(E_{2345})$) and Lemma 10.9 (using that ε_1 act trivially on $\mathrm{St}(E_1)$ but fixed point free on E_1). Now $\mathrm{Hom}_W(\mathrm{St}(E), E/E_1) = \mathrm{Hom}_{W/U}(\mathrm{St}(E_1) \otimes \mathrm{St}(E_{2345}), E/E_1) = 0$ by Lemma 10.12. Suppose that $E = E^{5b}_{E_8}$. Take $U = U_{123}$ and note that E_{1234} is an invariant subspace under W. Then $\mathrm{Hom}_W(\mathrm{St}(E), E_{1234}) = \mathrm{Hom}_{W/U}(\mathrm{St}(E_{1234}), E_{1234}) = 0$ since $-1 \in \mathrm{Sp}(E_{1234}) \subseteq W/U$. By [5] (or a direct calculation)

$$\mathrm{Hom}_{\mathrm{Sp}(E_{1234})}(\mathrm{St}(E_{1234}), E/E_{1234}) = 0,$$

which shows $\mathrm{Hom}_W(\mathrm{St}(E), E) = 0$ as well. This exhausts the list. □

10.3. **The non-toral part for the projective unitary groups.** We now embark in proving Theorem 10.1 for $X = \mathrm{PU}(n)\hat{_p}$. We will throughout this subsection use the notation for elementary abelian p-subgroups of X introduced in Section 9.

We first state the toral case.

Lemma 10.13. *Let $X = \mathrm{PU}(n)\hat{_p}$. Then*

$$\lim_{\mathbf{A}(X)}{}^i F^{\mathrm{tor}}_j = \begin{cases} \mathbf{Z}/2 & \text{if } n = p = 2, i = 0 \text{ and } j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $n \neq 2$ then it is immediate to check that $\breve{T}^{\langle s \rangle}$ is connected for an arbitrary reflection $s \in W_X$, so $\sigma(s) = \breve{T}^{\langle s \rangle}$ by the definition of $\sigma(s)$. Hence if $n \neq 2$ or p odd the lemma follows by Lemma 10.5.

Now suppose that $X = \mathrm{PU}(2)\hat{_2}$. Since for the non-trivial $V \subseteq \breve{T}$ we have that $W_X(V)_1$ is trivial and $C_X(V)_1 \cong T$ the first part of Lemma 10.5 still applies to finish the proof also in this case. □

We next record the following general lemma, which is obvious from the Künneth formula.

Lemma 10.14. *Suppose $\mathbf{D}_1$ and $\mathbf{D}_2$ are two categories with only finitely many morphisms. Let $C\mathbf{D}_i$ be "the cone on $\mathbf{D}_i$" i.e., the category constructed from $\mathbf{D}_i$ by adding an initial object e to $\mathbf{D}_i$, and let $\mathbf{D}_1 \star \mathbf{D}_2 = C\mathbf{D}_1 \times C\mathbf{D}_2 - (e, e)$, "the join of $\mathbf{D}_1$ and $\mathbf{D}_2$" (see* [112, §1]*). If $F_i : C\mathbf{D}_i \to \mathbf{Z}_p$-mod, $i = 1, 2$ are functors then*

$$C^*(C\mathbf{D}_1 \times C\mathbf{D}_2, \mathbf{D}_1 \star \mathbf{D}_2; F_1 \otimes F_2) \cong C^*(C\mathbf{D}_1, \mathbf{D}_1; F_1) \otimes C^*(C\mathbf{D}_2; \mathbf{D}_2; F_2).$$

In particular if one of the chain complexes has torsion free homology or if everything is defined over $\mathbf{F}_p$ then

$$H^*(C\mathbf{D}_1 \times C\mathbf{D}_2, \mathbf{D}_1 \star \mathbf{D}_2; F_1 \otimes F_2) \cong H^*(C\mathbf{D}_1, \mathbf{D}_1; F_1) \otimes H^*(C\mathbf{D}_2; \mathbf{D}_2; F_2).$$

□

The following result gives that certain filtration quotients have (almost) vanishing cohomology.

Theorem 10.15. *Set* $X = \mathrm{PU}(n)\hat{_p}$ *and fix an integer* $r > 0$. *Let* $F^r : \mathbf{A}(X) \to \mathbf{Z}_p\text{-mod}$ *denote the functor on objects given by* $F^r_j(E) = \pi_j(BZC_X(E))$ *if* E *is of the form* $\bar{\Gamma}_r \times \bar{A}$ *(in the notation of Section 9) and zero otherwise. Writing* $n = p^r k$ *we have that*

$$\lim_{\mathbf{A}(X)}{}^i F^r_j \quad = \quad \begin{cases} \mathbf{Z}/2 & \text{if } j = i = k = r = 1 \text{ and } p = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Define a functor $\tilde{F}^r_j$ by $\tilde{F}^r_j(E) = \pi_j(BZC_{\mathrm{PU}(k)}(\bar{A})\hat{_p})$ if $E = \bar{\Gamma}_r \times \bar{A}$ for a fixed r and zero otherwise. This is a subfunctor of F^r_j via the identification $\mathrm{PU}(k) \cong C_{\mathrm{PU}(n)}(\bar{\Gamma}_r)_1$. Set $\tilde{\tilde{F}}^r_j = F^r_j / \tilde{F}^r_j$ and observe that this is the trivial functor unless $j = 1$ where it is given by $\tilde{\tilde{F}}^r_j(E) = \bar{\Gamma}_r$ if E is of the form $\bar{\Gamma}_r \times \bar{A}$ and zero otherwise.

Consider the category

$$\mathbf{D} = \mathbf{A}^e(X)_{\subseteq \bar{\Gamma}_r} \times \mathbf{A}^e(\mathrm{PU}(k)\hat{_p}) - (e, e)$$

where the superscript e means that we do not exclude the trivial subgroup. We have a natural inclusion of categories $\iota : \mathbf{D} \to \mathbf{A}(X)$ on objects given by $(V, \bar{A}) \mapsto V \times \bar{A}$. *Step 1:* We claim that this map induces an isomorphism

$$\lim_{\mathbf{A}(X)}{}^* F^r_j \to \lim_{\mathbf{D}}{}^* F^r_j.$$

By filtering the functor and using Nakayama's lemma it is enough to show this for $\tilde{F}^r_j \otimes \mathbf{F}_p$ and $\tilde{\tilde{F}}^r_j \otimes \mathbf{F}_p$. We can furthermore replace these functors by the subquotient functors which are only concentrated on one subgroup $\bar{\Gamma}_r \times \bar{A}$.

Consider first such a subquotient of $\tilde{F}^r_j \otimes \mathbf{F}_p$. In this case the formula of Oliver [106, Prop. 4] together with Lemma 10.14 shows that the higher limits on both sides are only non-zero in a single degree, where the map identifies with the restriction map

$$\mathrm{Hom}_{W_X(\bar{\Gamma}_r \times \bar{A})}(\mathrm{St}(\bar{\Gamma}_r \times \bar{A}), \pi_j(BZC_{\mathrm{PU}(k)}(\bar{A})) \otimes \mathbf{F}_p) \to$$

$$\mathrm{Hom}_{\mathrm{Sp}(\bar{\Gamma}_r) \times W_{\mathrm{PU}(k)}(\bar{A})}(\mathrm{St}(\bar{\Gamma}_r) \otimes \mathrm{St}(\bar{A}), \pi_j(BZC_{\mathrm{PU}(k)}(\bar{A})) \otimes \mathbf{F}_p).$$

Now note that the elements U in $W_{\mathrm{PU}(n)}(\bar{\Gamma}_r \times \bar{A})$ which sends $\bar{\Gamma}_r$ to $\bar{A}$ act trivially on the target by Theorem 9.1. Furthermore we have by the theorem of Smith [121] (and self-duality of the Steinberg module) that $\mathrm{St}(\bar{\Gamma}_r \times \bar{A})_U \cong \mathrm{St}(\bar{\Gamma}_r \times \bar{A})^U \cong \mathrm{St}(\bar{\Gamma}_r) \otimes \mathrm{St}(\bar{A})$, where $(-)_U$ and $(-)^U$ denote coinvariants and invariants respectively. Hence this map is an isomorphism. The case of a subquotient of $\tilde{\tilde{F}}^r_j \otimes \mathbf{F}_p$ is completely analogous. This shows the isomorphism.

Step 2: We now proceed to calculate the higher limits over $\mathbf{D}$, which we do by calculating the limits of $\tilde{F}^r_j$ and $\tilde{\tilde{F}}^r_j$. We first consider $\tilde{\tilde{F}}^r_j$. We have already remarked that only $\tilde{\tilde{F}}^r_1 \neq 0$. Furthermore if $k \neq 1$ then $H^*(C\mathbf{A}^{\mathrm{tor}}(\mathrm{PU}(k)\hat{_p}), \mathbf{A}^{\mathrm{tor}}(\mathrm{PU}(k)\hat{_p}); \mathbf{Z}_p) = 0$ by [50, §8] so $\lim^*_{\mathbf{D}} \tilde{\tilde{F}}^r_j = 0$ by Lemma 10.14. If $k = 1$ we get $\lim^i_{\mathbf{D}} \tilde{\tilde{F}}^r \cong \mathrm{Hom}_{\mathrm{Sp}(\bar{\Gamma}_r)}(\mathrm{St}(\bar{\Gamma}_r), \bar{\Gamma}_r)$ if $i = 2r - 1$ and zero otherwise, by [106, Prop. 4].

Now consider $\tilde{F}^r_j$. By Lemma 10.14

$$\lim_{\mathbf{D}}{}^i \tilde{F}^r_j = \mathrm{Hom}_{\mathrm{Sp}(\bar{\Gamma}_r)}(\mathrm{St}(\bar{\Gamma}_r), \mathbf{Z}_p) \otimes \lim_{\bar{A} \in \mathbf{A}^{\mathrm{tor}}(\mathrm{PU}(k)\hat{_p})}{}^{i-2r} \pi_j(BZC_{\mathrm{PU}(k)\hat{_p}}(\bar{A})).$$

By Lemma 10.13 $\lim^{i-2r}_{\mathbf{A}^{\mathrm{tor}}(\mathrm{PU}(k)\hat{_p})} \pi_j(BZC_{\mathrm{PU}(k)\hat{_p}}(\bar{A})) = 0$ unless $p = k = 2$, $j = 1$ and $i - 2r = 1$ where we get $\lim^{1}_{\mathbf{A}^{\mathrm{tor}}(\mathrm{PU}(2)\hat{_2})} \pi_j(BZC_{\mathrm{PU}(2)\hat{_2}}(\bar{A})) = \mathbf{Z}/2$.

By an argument of H. H. Andersen and C. Stroppel [5] we have that

$$\mathrm{Hom}_{\mathrm{Sp}(\bar{\Gamma}_r)}(\mathrm{St}(\bar{\Gamma}_r), \mathbf{Z}_p) = 0$$

for all r and p.

To sum up we get that

$$\lim_{\mathbf{A}(X)}{}^{i} F_j^r \cong \lim_{\mathbf{D}}{}^{i} \tilde{\tilde{F}}_j^r \cong \mathrm{Hom}_{\mathrm{Sp}(\bar{\Gamma}_r)}(\mathrm{St}(\bar{\Gamma}_r), \bar{\Gamma}_r)$$

if $j = k = 1$ and $i = 2r - 1$ and zero otherwise.

However the same argument of H. H. Andersen and C. Stroppel [5] shows that

$$\mathrm{Hom}_{\mathrm{Sp}(\bar{\Gamma}_r)}(\mathrm{St}(\bar{\Gamma}_r), \bar{\Gamma}_r) = 0$$

unless $r = 1$ and $p = 2$ where it equals $\mathbf{F}_2$. (Note that this is obvious if p is odd by Lemma 10.9.) This shows the wanted formula. □

Remark 10.16. Note that slightly non-trivial statements from [5] are only used above for $p = 2$ and furthermore become trivial when $r = 1$, where $\mathrm{Sp}(\bar{\Gamma}_1) \cong \mathrm{SL}(\bar{\Gamma}_1)$, and this is in fact the only case which involve obstruction groups in the range needed for the proof of (the $p = 2$ version of) Theorem 1.4.

Remark 10.17. It is in fact possible to give a short proof of Smith's theorem in the special case used above, using the geometric definition of the Steinberg module $\mathrm{St}(E)$ via flags.

Proof of Theorem 10.1 for $X = \mathrm{PU}(n)\hat{_p}$. Lemma 10.15 and 10.13 directly shows the conclusion unless $n = 2$ and $p = 2$, so assume we are in this case. By writing down the definition it follows that $\lim^0_{\mathbf{A}(X)} F_1 = 0$. Now, consider the exact sequence of functors $0 \to F_1^1 \to F_1 \to F_1^{\mathrm{tor}} \to 0$. The long exact sequence of higher limits starts out as

$$0 \to 0 \to 0 \to \lim_{\mathbf{A}(X)}{}^0 F_1^{\mathrm{tor}} \to \lim_{\mathbf{A}(X)}{}^1 F_1^1 \to \lim_{\mathbf{A}(X)}{}^1 F_1 \to 0 \to \cdots$$

So, since $\lim^0_{\mathbf{A}(X)} F_1^{\mathrm{tor}} \cong \mathbf{Z}/2 \cong \lim^1_{\mathbf{A}(X)} F_1^1$ we get that $\lim^i_{\mathbf{A}(X)} F_1 = 0$ for $i > 0$ as well. This concludes the proof of this last case of Theorem 10.1. □

11. Appendix: The classification of finite $\mathbf{Z}_p$-reflection groups

The purpose of this appendix is to give a classification of finite $\mathbf{Z}_p$-reflection groups extending and simplifying work of Notbohm [100, 102] who gave a classification for odd primes p. See also [47, §5] for related earlier results.

We start by recalling some definitions. Let R be an integral domain with field of fractions K. An *R-reflection group* is a pair (W, L) where L is a finitely generated free R-module, and W is a subgroup of $\mathrm{Aut}(L)$ generated by elements α such that $1 - \alpha$ has rank one viewed as a matrix over K. Two finite R-reflection groups (W, L) and (W', L') are called *isomorphic*, if we can find an R-linear isomorphism $\varphi : L \to L'$ such that the group $\varphi W \varphi^{-1}$ equals W'. A finite R-reflection group (W, L) is said to be *irreducible* if the corresponding representation of W on $L \otimes_R K$ is irreducible. If R has characteristic zero we define the *character field* of an R-reflection group (W, L) as the field extension of $\mathbf{Q}$ generated by the values of the character of the representation $W \hookrightarrow \mathrm{Aut}(L)$. For $R = \mathbf{Z}_p$ or $\mathbf{Q}_p$ we define an *exotic* R-reflection group to be a finite irreducible R-reflection group with character field strictly larger than $\mathbf{Q}$.

The classification of finite $\mathbf{Z}_p$-reflection groups is based on the work of Clark and Ewing [31], which is again based on the classification of finite $\mathbf{C}$-reflection groups by Shephard and Todd [119]. Clark and Ewing showed that there is a bijection between finite $\mathbf{Q}_p$-reflection groups and finite $\mathbf{C}$-reflection groups whose character field may be embedded in $\mathbf{Q}_p$. The classification of finite complex reflection groups [119] is as follows: The irreducible ones fall into 3 infinite families (in the following called families 1, 2 and 3) and 34 sporadic cases (in the following labeled G_i, $4 \le i \le 37$). Moreover any finite complex reflection group can be written as a direct product of irreducible finite complex reflection groups, cf. [58, Rem. 2.3] (in fact this holds over any field of characteristic 0).

It is convenient to split family 2 further depending on the character field. The associated complex reflection group is the group $G(m,r,n)$ (where m, r and n are integers with $m,n \ge 2$, $r \ge 1$, $r\,|\,m$ and $(m,r,n) \ne (2,2,2)$) from [119, p. 277] which consists of monomial $n \times n$-matrices such that the non-zero entries are mth roots of unity and the product of the non-zero entries is an (m/r)th root of unity. Thus $G(m,r,n)$ is the semidirect product of its subgroup $A(m,r,n)$ of diagonal matrices with the subgroup of permutation matrices. Let $\zeta_m = e^{2\pi i/m}$. For $n \ge 3$ or $n = 2$ and $r \ne m$ the character field of $G(m,r,n)$ equals $\mathbf{Q}(\zeta_m)$, and for $n = 2$ and $r = m$ it equals $\mathbf{Q}(\zeta_m + \zeta_m^{-1})$ (see [31, p. 432–433]). We label the two cases family $2a$ and $2b$ respectively.

A complete list of the irreducible finite complex reflection groups, their character fields and the primes for which these embed in $\mathbf{Q}_p$ can be found in [79, p. 165] or [6, Table 1].

If (W,V) is a finite $\mathbf{Q}_p$-reflection group, then by [36, Prop. 23.16] we can find a (non-unique) finitely generated $\mathbf{Z}_pW$-submodule $L \subseteq V$ with $L \otimes \mathbf{Q} = V$. Thus any finite $\mathbf{Q}_p$-reflection group may be obtained from a finite $\mathbf{Z}_p$-reflection group by extension of scalars, but in general there are several non-isomorphic $\mathbf{Z}_p$-reflection groups which give rise to the same $\mathbf{Q}_p$-reflection group. The following result extends [102, Thm. 1.5 and Prop. 1.6] to all primes.

Theorem 11.1 (The classification of finite $\mathbf{Z}_p$-reflection groups)**.** *Let (W,L) be a finite $\mathbf{Z}_p$-reflection group. Then there exists a decomposition*

$$(W,L) = (W_1 \times W_2, L_1 \oplus L_2)$$

where $(W_1,L_1) \cong (W_G, L_G \otimes \mathbf{Z}_p)$, for some (non-unique) compact connected Lie group G with Weyl group W_G and integral lattice L_G, and (W_2,L_2) is a (up to permutation unique) direct product of exotic $\mathbf{Z}_p$-reflection groups.

The canonical map $(W,L) \mapsto (W, L\otimes\mathbf{Q})$ gives a one-to-one correspondence between exotic $\mathbf{Z}_p$-reflection groups up to isomorphism and exotic $\mathbf{Q}_p$-reflection groups up to isomorphism.

If (W,L) is any exotic $\mathbf{Z}_p$-reflection group, then $L \otimes \mathbf{F}_p$ is an irreducible $\mathbf{F}_pW$-module, and in particular we have $(L \otimes \mathbf{Z}/p^\infty)^W = 0$ and $H_0(W;L) = 0$.

Remark 11.2. For odd primes p the last two results says by definition that any exotic $\mathbf{Z}_p$-reflection group is respectively *center-free* and *simply connected*, cf. [100].

Note also that [102, Thm. 1.5] imposes the unnecessarily strong condition that the invariant ring $\mathbf{Z}_p[L]^W$ is a polynomial algebra, but this condition is not actually used in [102].

We recall the following elementary fact about elements of finite order in $\mathrm{GL}_n(\mathbf{Z}_p)$.

Lemma 11.3. *Let $G \subseteq \mathrm{GL}_n(\mathbf{Z}_p)$ be a finite subgroup. Then the mod p reduction $G \hookrightarrow \mathrm{GL}_n(\mathbf{Z}_p) \to \mathrm{GL}_n(\mathbf{F}_p)$ is injective if p is odd. For $p = 2$ the kernel of the composition is an* *elementary abelian* 2*-subgroup of rank at most* n^2*. In particular the kernel is contained in* $O_2(G)$*, the largest normal* 2*-subgroup of* G*.*

Proof. It is easy to see directly that any non-trivial finite order element in $\mathrm{GL}_n(\mathbf{Z}_p)$ has non-trivial reduction mod p if p is odd (cf. [120, Pf. of Lem. 10.7.1]). For $p = 2$ the same argument shows that this is true if we reduce mod 4. The result now follows. □

Proof of Theorem 11.1. We start by showing that for any exotic $\mathbf{Q}_p$-reflection group (W, V) we can find a finitely generated $\mathbf{Z}_pW$-submodule $L \subseteq V$ with $L \otimes \mathbf{Q} = V$, such that $L \otimes \mathbf{F}_p$ is an irreducible $\mathbf{F}_pW$-module.

Assume first that $p \nmid |W|$. By [36, Prop. 23.16] we can find a finitely generated $\mathbf{Z}_pW$-submodule $L \subseteq V$ with $L \otimes \mathbf{Q} = V$. It follows from [35, 75.6 and 76.15] that $L \otimes \mathbf{F}_p$ is automatically an irreducible $\mathbf{F}_pW$-module.

Assume now that W has order divisible by p. From Clark-Ewing's list we see that the only exotic $\mathbf{Q}_p$-reflection groups satisfying this condition are the groups $G(m, r, n)$ from family $2a$ or one of the groups G_{12} for $p = 3$, G_{24} for $p = 2$, G_{29} and G_{31} for $p = 5$ or G_{34} for $p = 7$.

In case $W = G(m, r, n)$ from family $2a$ we get the extra conditions $m \geq 3$, $p \equiv 1 \pmod{m}$ and $p \leq n$. Note in particular that $n \geq 3$. The description above directly gives a representation with entries in $\mathbf{Z}_p$ since the multiplicative group of $\mathbf{Z}_p$ contains the $(p-1)$th roots of unity. Let $L = (\mathbf{Z}_p)^n$ be the natural $\mathbf{Z}_pW$-module, i.e. the set of columns with entries in $\mathbf{Z}_p$. Assume that $0 \neq M \subseteq L \otimes \mathbf{F}_p$ is a $\mathbf{F}_pW$-submodule of $L \otimes \mathbf{F}_p$. Choose $x \in M$ with $x \neq 0$ and let $\theta \in \mathbf{F}_p$ be a primitive mth root of unity. Since W contains the permutation matrices and the diagonal matrix $\mathrm{diag}(\theta, \theta^{-1}, 1, \ldots, 1)$ we see that M contains an element of the form $x' = (x_1, x_2, 0, \ldots, 0)^{\mathrm{T}}$ with $x_1 \neq 0$. Since $n \geq 3$, W also contains the diagonal matrix $\mathrm{diag}(\theta, 1, \theta^{-1}, 1, \ldots, 1)$ and hence M contains $((1-\theta)x_1, 0, \ldots, 0)^{\mathrm{T}}$. As $\theta \neq 1$ and W contains all permutation matrices we conclude that $M = L \otimes \mathbf{F}_p$, proving the claim for the groups from family $2a$.

Next consider $W = G_{12}$ at $p = 3$. Since W is isomorphic to $\mathrm{GL}_2(\mathbf{F}_3)$, Lemma 11.3 shows that for any finitely generated $\mathbf{Z}_3W$-submodule $L \subseteq (\mathbf{Q}_3)^2$ of rank 2, we may identify $L \otimes \mathbf{F}_3$ as the natural $\mathbf{F}_3W$-module. In particular $L \otimes \mathbf{F}_3$ is an irreducible $\mathbf{F}_3W$-module.

For $W = G_{24}$ at $p = 2$ we have $W \cong \mathbf{Z}/2 \times \mathrm{GL}_3(\mathbf{F}_2)$. Hence Lemma 11.3 shows that for any finitely generated $\mathbf{Z}_2W$-submodule $L \subseteq (\mathbf{Q}_2)^3$ of rank 3, we may identify $L \otimes \mathbf{F}_2$ as the $\mathbf{F}_2(\mathbf{Z}/2 \times \mathrm{GL}_3(\mathbf{F}_2))$-module where $\mathbf{Z}/2$ acts trivially and $\mathrm{GL}_3(\mathbf{F}_2)$ acts naturally. In particular $L \otimes \mathbf{F}_2$ is an irreducible $\mathbf{F}_2W$-module.

Next consider the groups G_{29} and G_{31} at $p = 5$. Since G_{29} is contained in G_{31} it suffices to show the result for $W = G_{29}$. The representation in [119, p. 298] is defined over $\mathbf{Z}[\frac{1}{2}, i]$ and hence we get a representation over $\mathbf{Z}_5$ by mapping i to a primitive 4th root of unity in $\mathbf{Z}_5$. Let $L = (\mathbf{Z}_5)^4$ be the natural $\mathbf{Z}_5W$-module. There are 40 reflections in G_{29}: The 24 reflections in the hyperplanes of the form $x_j - i^\alpha x_k = 0$, $j \neq k$ and the 16 reflections in the hyperplanes of the form $\sum_{j=1}^4 i^{\alpha_j} x_j = 0$ with $\sum_{j=1}^4 \alpha_j \equiv 0 \pmod{4}$. In particular G_{29} contains the reflections in the hyperplanes $x_j - x_k = 0$ and thus G_{29} contains all permutation matrices. The product of the reflections in the hyperplanes $x_1 - ix_2 = 0$ and $x_1 - x_2 = 0$ equals the diagonal matrix $\mathrm{diag}(i, -i, 1, 1)$ and thus this element is also contained in G_{29}. Now the same argument used in the case of the groups from family $2a$ shows that $L \otimes \mathbf{F}_5$ is an irreducible $\mathbf{F}_5W$-module.

The argument for the group $W = G_{34}$ at $p = 7$ is similar. The representation given in [119, p. 298] is defined over $\mathbf{Z}[\frac{1}{3}, \omega]$, $\omega = \zeta_3$ and hence we get a representation over

$\mathbf{Z}_7$ by mapping ω to a primitive 3rd root of unity in $\mathbf{Z}_7$. Let $L = (\mathbf{Z}_7)^6$ be the natural $\mathbf{Z}_7W$-module. There are 126 reflections in G_{34}: The 45 reflections in the hyperplanes of the form $x_j - \omega^\alpha x_k = 0$, $j \neq k$ and the 81 reflections in the hyperplanes of the form $\sum_{j=1}^6 \omega^{\alpha_j} x_j = 0$ with $\sum_{j=1}^6 \alpha_j \equiv 0 \pmod 3$. In particular G_{34} contains all permutation matrices. The product of the reflections in the hyperplanes $x_1 - \omega x_2 = 0$ and $x_1 - x_2 = 0$ equals the diagonal matrix $\operatorname{diag}(\omega, \omega^2, 1, 1, 1, 1)$ and thus this element is also contained in G_{34}. As above we then see that $L \otimes \mathbf{F}_7$ is an irreducible $\mathbf{F}_7W$-module.

This proves the claim made above that for any exotic $\mathbf{Q}_p$-reflection group (W, V) we can find a finitely generated $\mathbf{Z}_pW$-submodule $L \subseteq V$ with $L \otimes \mathbf{Q} = V$, such that $L \otimes \mathbf{F}_p$ is an irreducible $\mathbf{F}_pW$-module. It now follows from [118, Ex. 15.3] that, up to scaling with a unit in $\mathbf{Q}_p$, L is the only such $\mathbf{Z}_pW$-submodule. This gives the bijection between exotic $\mathbf{Z}_p$-reflection groups and exotic $\mathbf{Q}_p$-reflection groups.

Since $L \otimes \mathbf{F}_p$ is an irreducible $\mathbf{F}_pW$-module we also conclude that $(L \otimes \mathbf{F}_p)^W = 0$ and $H_0(W; L \otimes \mathbf{F}_p) = 0$. Hence we get $(L \otimes \mathbf{Z}/p^\infty)^W = 0$ as claimed. We also see that multiplication by p is surjective on $H_0(W; L)$ and from this we obtain $H_0(W; L) = 0$ by Nakayama's lemma. This proves the part of the theorem pertaining to exotic $\mathbf{Z}_p$-reflection groups.

Now consider a finite $\mathbf{Z}_p$-reflection group (W, L) such that we have a direct sum decomposition $L \otimes \mathbf{Q} = V_1 \oplus V_2$ as $\mathbf{Q}_pW$-modules. Let W_1 (resp. W_2) be the subgroup of W which fixes V_2 (resp. V_1) pointwise. It it easy to see (cf. [53, Prop. 6.3]) that (W_i, V_i) is a $\mathbf{Q}_p$-reflection group and that we get the decomposition $(W, L \otimes \mathbf{Q}) = (W_1 \times W_2, V_1 \oplus V_2)$.

We now claim that if (W_2, V_2) is an exotic $\mathbf{Q}_p$-reflection group, then we have the decomposition $(W, L) = (W_1 \times W_2, L_1 \oplus L_2)$ with $L_i = L \cap V_i$. Let $\alpha : L_1 \oplus L_2 \longrightarrow L$ be the addition map. As in [53, Pf. of Thm. 1.5] it suffices to prove that $\alpha \otimes \mathbf{Z}/p^\infty : (L_1 \otimes \mathbf{Z}/p^\infty) \oplus (L_2 \otimes \mathbf{Z}/p^\infty) \longrightarrow L \otimes \mathbf{Z}/p^\infty$ is injective. Assume that (x_1, x_2) is in the kernel of $\alpha \otimes \mathbf{Z}/p^\infty$, $x_i \in L_i \otimes \mathbf{Z}/p^\infty$. Thus $x_1 + x_2 = 0$. If $s \in W_2$ is a reflection we have by definition $s \cdot x_1 = x_1$ and hence s also fixes $x_2 = -x_1$. Since W_2 is generated by reflections we get $x_2 \in (L_2 \otimes \mathbf{Z}/p^\infty)^W$ and hence $x_2 = 0$ by the results already proved for exotic $\mathbf{Z}_p$-reflection groups. Hence $x_1 = 0$ as well, and thus $\alpha \otimes \mathbf{Z}/p^\infty$ is injective proving the claim.

Since any finite $\mathbf{Q}_p$-reflection group may be decomposed into a (up to permutation unique) product of finite irreducible ones, we see by using the claim repeatedly that any finite $\mathbf{Z}_p$-reflection group (W, L) may be decomposed as a product $(W, L) \cong (W_1 \times W_2, L_1 \oplus L_2)$ where (W_1, L_1) is a $\mathbf{Z}_p$-reflection group with character field equal to $\mathbf{Q}$ and (W_2, L_2) is as in the theorem.

To finish the proof we thus need to show that for any finite $\mathbf{Z}_p$-reflection group (W, L) with character field equal to $\mathbf{Q}$ we may find a compact connected Lie group G such that (W, L) is isomorphic to $(W_G, L_G \otimes \mathbf{Z}_p)$. We start by reducing the problem to finite $\mathbf{Z}$-reflection groups. The representation $W \to \mathrm{GL}(L \otimes \mathbf{Q})$ is a reflection representation and hence has Schur index 1 by [31, Cor. p. 429]. Thus this representation is equivalent to a representation defined over $\mathbf{Q}$. Hence [36, Cor. 30.10] applied to $R = \mathbf{Z}_{(p)}$ shows that there exists a (unique) finitely generated $\mathbf{Z}_{(p)}W$-submodule $L' \subseteq L$ with $L' \otimes_{\mathbf{Z}_{(p)}} \mathbf{Z}_p = L$. Now [36, Cor. 23.14] applied to $R = \mathbf{Z}$ shows that L' contains a (non-unique) finitely generated $\mathbf{Z}W$-submodule $L'' \subseteq L'$ with $L' = L'' \otimes \mathbf{Z}_{(p)}$. We conclude in particular that $(W, L) \cong (W, L'' \otimes \mathbf{Z}_p)$.

We finish the proof by showing that there exists a (non-unique) compact connected Lie group G whose Weyl group (W_G, L_G) is isomorphic to (W, L''). For each reflection $s \in W$ the group $\{x \in L'' \mid s(x) = -x\}$ is an infinite cyclic group with two generators which we label $\pm\alpha_s$. Let $\Phi = \{\pm\alpha_s \mid s \text{ is a reflection in } W\}$ and $L_0'' = (L'')^W$. It then follows (cf. [109, p. 85]) that (L'', L_0'', Φ) is a reduced root diagram whose associated $\mathbf{Z}$-reflection group equals (W, L'') (see [17, §4, no. 8] for definitions). From the classification of compact connected Lie groups ([17, §4, no. 9, Prop. 16]) it then follows that there exists a compact connected Lie group G whose root diagram equals (L'', L_0'', Φ). In particular (W_G, L_G) is isomorphic to (W, L'') and we are done. □

The following result answers the question of when two compact connected Lie groups give rise to the same p-compact group. For a compact connected Lie group G let $G\langle 1\rangle$ denote the universal cover of G. Furthermore let H be the direct product of the identity component of the center, $Z(G)_1$, with the universal cover of the derived group of G. We have a canonical covering homomorphism $\varphi : H \to G$ with finite kernel (cf. [17, §1, no. 4, Prop. 4]). If p is a prime number, we let $\mathrm{Cov}^{p'}(G)$ denote the covering of G corresponding to the subgroup of $\pi_1(G)$ given as the preimage of the Sylow p-subgroup of $\pi_1(G)/\varphi(\pi_1(H))$.

Theorem 11.4 (Addendum to Theorem 1.1 and 11.1)**.** *Let G and G' be two compact connected Lie groups and p a prime number. Then*

1. (W_G, L_G) *and* $(W_{G'}, L_{G'})$ *are isomorphic if and only if G is isomorphic to G' up to the substitution of direct factors isomorphic to* $\mathrm{Sp}(n)$ *with direct factors isomorphic to* $\mathrm{SO}(2n+1)$.
2. $(W_G, L_G \otimes \mathbf{Z}_2)$ *and* $(W_{G'}, L_{G'} \otimes \mathbf{Z}_2)$ *are isomorphic if and only if* $\mathrm{Cov}^{2'}(G)$ *and* $\mathrm{Cov}^{2'}(G')$ *are isomorphic up to the substitution of direct factors isomorphic to* $\mathrm{Sp}(n)$ *with direct factors isomorphic to* $\mathrm{SO}(2n+1)$. *Moreover the following conditions are equivalent:*
   (a) $(W_G, L_G \otimes \mathbf{Z}_2, L_{G\langle 1\rangle} \otimes \mathbf{Z}_2)$ *and* $(W_{G'}, L_{G'} \otimes \mathbf{Z}_2, L_{G'\langle 1\rangle} \otimes \mathbf{Z}_2)$ *are isomorphic.*
   (b) $\mathrm{Cov}^{2'}(G)$ *is isomorphic to* $\mathrm{Cov}^{2'}(G')$.
   (c) $(BG)\hat{_2} \simeq (BG')\hat{_2}$.
3. *For p odd the following conditions are equivalent:*
   (a) $(W_G, L_G \otimes \mathbf{Z}_p)$ *and* $(W_{G'}, L_{G'} \otimes \mathbf{Z}_p)$ *are isomorphic.*
   (b) $\mathrm{Cov}^{p'}(G)$ *and* $\mathrm{Cov}^{p'}(G')$ *are isomorphic up to the substitution of direct factors isomorphic to* $\mathrm{Sp}(n)$ *with direct factors isomorphic to* $\mathrm{Spin}(2n+1)$.
   (c) $(BG)\hat{_p} \simeq (BG')\hat{_p}$.

Sketch of proof: By [109, §4] or [76, Prop. 3.2(vi)] we can recover the root datum of a compact connected Lie group from its integral lattice up to substitution of direct factors isomorphic to $\mathrm{Sp}(n)$ with direct factors isomorphic to $\mathrm{SO}(2n+1)$. Part (1) now follows.

Now, suppose that G is a compact connected Lie group of the form H/K where H is a direct product of a torus with a simply connected compact Lie group and K is a finite central p-group, i.e., that $\mathrm{Cov}^{p'}(G) = G$. Suppose moreover that G does not contain any direct factors isomorphic to $\mathrm{Sp}(n)$. By Proposition 7.4 the fundamental group of G equals the coinvariants $(L_G)_W$ and hence $L_H = (L_G)^W \oplus SL_G$. This shows that $(W, L_H \otimes \mathbf{Z}_p)$ can be reconstructed from $(W, L_G \otimes \mathbf{Z}_p)$. By the classification of simply connected compact Lie groups we can for $p = 2$ reconstruct H from $(W, L_H \otimes \mathbf{Z}_p)$. For p odd the only ambiguity arises from direct factors isomorphic to $\mathrm{Sp}(n)$ or $\mathrm{Spin}(2n+1)$. However, if p is odd then by the assumption on G, H cannot contain any direct factors isomorphic to $\mathrm{Sp}(n)$, and we conclude that in all cases we can reconstruct H from $(W, L_G \otimes \mathbf{Z}_p)$. Since K is the cokernel of the inclusion $L_H \otimes \mathbf{Z}_p \to L_G \otimes \mathbf{Z}_p$, we can also recover the inclusion $K \subseteq L_H \otimes \mathbf{Z}/p^\infty =$

$\breve{Z}(H) \subseteq Z(H)$ using [58, Thm. 1.4] for the middle equality. This shows that we can recover G from $(W, L_G \otimes \mathbf{Z}_p)$.

The above analysis directly shows the first claim in (2) as well as (3a) $\Leftrightarrow$ (3b). From the first claim in (2), (2a) $\Leftrightarrow$ (2b) follows, since $\mathrm{Sp}(n)$ and $\mathrm{SO}(2n+1)$ have different $\mathbf{Z}_2$-reflection data $(W_G, L_G \otimes \mathbf{Z}_2, L_{G\langle 1\rangle} \otimes \mathbf{Z}_2)$. The implications (2b) $\Rightarrow$ (2c) $\Rightarrow$ (2a) are clear.

To finish off we remark that (3c) $\Rightarrow$ (3a) is clear and (3b) $\Rightarrow$ (3c) follows since $B\,\mathrm{SO}(2n+1)\hat{_p}$ is homotopy equivalent to $B\,\mathrm{Sp}(n)\hat{_p}$ either by a very special case of our main Theorem 1.1 or by the original proof due to Friedlander [61]. □

12. Appendix: Invariant rings of finite $\mathbf{Z}_p$-reflection groups, p odd (following Notbohm)

The purpose of this appendix is to recall Notbohm's determination [102] of finite $\mathbf{Z}_p$-reflection groups (W, L), p odd, such that the invariant ring $\mathbf{Z}_p[L]^W$ is a polynomial algebra.

Before stating it let us however for easy reference recall the following 'classical' characterizations of being a 'p-torsion free' p-compact group, which has a proof by general arguments which we will sketch.

Theorem 12.1. *Let X be a connected p-compact group with maximal torus T and Weyl group W_X. The following statements are equivalent:*

1. *$H^*(X; \mathbf{Z}_p)$ is torsion free.*
2. *$H^*(X; \mathbf{Z}_p)$ is an exterior algebra over $\mathbf{Z}_p$ with generators in odd degrees (or equivalently with $\mathbf{F}_p$ instead of $\mathbf{Z}_p$).*
3. *$H^*(BX; \mathbf{Z}_p)$ is a polynomial algebra over $\mathbf{Z}_p$ with generators in even degree (or equivalently with $\mathbf{F}_p$ instead of $\mathbf{Z}_p$).*
4. *$H^*(BX; \mathbf{Z}_p)$ is a polynomial algebra and $H^*(BX; \mathbf{Z}_p) \xrightarrow{\cong} H^*(BT; \mathbf{Z}_p)^{W_X}$.*

We now give Notbohm's classification. The first part (which is a general argument reducing to the simply connected case) is [102, Thm. 1.3] and the second (which is a case-by-case argument in the simply connected case) is a slight extension of [102, Thm. 1.4]. For the benefit of the reader we give a streamlined proof of the second part. Recall that for a finite $\mathbf{Z}_p$-reflection group (W, L) we define SL to be the submodule of L generated by elements of the form $(1-w)x$ with $w \in W$ and $x \in L$. We call (W, L) *simply connected* if $L = SL'$ for some $\mathbf{Z}_pW$-lattice L' (note that for p odd this is equivalent to $SL = L$ since $S^2L' = SL'$, cf. the discussion of $\mathbf{Z}_p$-reflection data in the introduction).

Theorem 12.2 (Finite $\mathbf{Z}_p$-reflection groups with polynomial invariants, p odd)**.** *Let p be an odd prime and (W, L) a finite $\mathbf{Z}_p$-reflection group. Then we have the following statements:*

1. *$\mathbf{Z}_p[L]^W$ is a polynomial algebra if and only if $\mathbf{Z}_p[SL]^W$ is a polynomial algebra and the group of coinvariants L_W is torsion free.*
2. *Suppose (W, L) is irreducible and simply connected. The following conditions are equivalent:*
   - (a) *$\mathbf{Z}_p[L]^W$ is a polynomial algebra.*
   - (b) *$\mathbf{F}_p[L \otimes \mathbf{F}_p]^W$ is a polynomial algebra.*
   - (c) *(W, L) is not isomorphic to $(W_G, L_G \otimes \mathbf{Z}_p)$ for the following pairs (G, p): $(F_4, 3)$, $(3E_6, 3)$, $(2E_7, 3)$, $(E_8, 3)$ and $(E_8, 5)$.*

In particular, if X is an exotic p-compact group then $\mathbf{Z}_p[L_X]^{W_X}$ is a polynomial algebra and if $(W, L) = (W_G, L_G \otimes \mathbf{Z}_p)$ for a compact connected Lie group G then $\mathbf{Z}_p[L_G \otimes \mathbf{Z}_p]^{W_G}$ is a polynomial algebra if and only if $H^*(G; \mathbf{Z}_p)$ is torsion free.

Sketch of proof of Theorem 12.1. The equivalence of (1), (2), and (3) are old H-space and loop space arguments which we first very briefly sketch. By a Bockstein spectral sequence argument (cf. e.g., [79, §11-2]) $H^*(X;\mathbf{Z}_p)$ torsion free if and only if $H^*(X;\mathbf{Z}_p)$ is an exterior algebra on odd dimensional generators so (1) is equivalent to (2). This is again equivalent to that $H^*(BX;\mathbf{Z}_p)$ is a polynomial algebra on even dimensional generators (using the Eilenberg-Moore and the cobar spectral sequence; see e.g., [79, §7-4]), so (2) is equivalent to (3).

That (4) implies (3) is obvious. The fact that (1)–(3) also imply (4) requires more machinery and is probably first found in [47, Thm. 2.11]—we quickly sketch an argument. We want to show that the map $r : H^*(BX;\mathbf{Z}_p) \to H^*(BT;\mathbf{Z}_p)^W$ is an isomorphism. By [51, Thm. 9.7(3)]

$$H^*(BX;\mathbf{Z}_p)\otimes\mathbf{Q} \xrightarrow{\cong} H^*(BT;\mathbf{Z}_p)^W\otimes\mathbf{Q}. \tag{12.1}$$

This implies by comparing Krull dimensions that the number of polynomial generators equals the rank of T. Since $H^*(BT;\mathbf{F}_p)$ is finitely generated over $H^*(BX;\mathbf{F}_p)$ by [51, Prop. 9.11] it follows by comparing Krull dimensions again that $H^*(BX;\mathbf{F}_p)\to H^*(BT;\mathbf{F}_p)$ is injective. Hence $H^*(BX;\mathbf{Z}_p)\to H^*(BT;\mathbf{Z}_p)$ has to be injective by Nakayama's lemma. Likewise r has to be surjective: By (12.1) the cokernel of r has to be p-torsion. Since the reduction mod p of r is still injective (as seen above) the cokernel of r has to be p-torsion free as well (since $\mathrm{Tor}(\mathrm{coker}(r),\mathbf{F}_p)=0$). □

Remark 12.3. If p is odd then $\mathbf{F}_p$-coefficients can also be used in Theorem 12.1(4) by a Galois theory argument using Lemma 11.3. For $p=2$, this is not true as can be seen by taking $X=\mathrm{SU}(2)\hat{_2}$. See [48] for a $p=2$ version.

Remark 12.4. If (W,L) is a finite $\mathbf{Z}_p$-reflection group then $\mathbf{Z}_p[L]^W$ is a polynomial algebra if and only if $\mathbf{F}_p[L\otimes\mathbf{F}_p]^W$ is a polynomial algebra and the canonical monomorphism $\mathbf{Z}_p[L]^W\otimes\mathbf{F}_p\longrightarrow\mathbf{F}_p[L\otimes\mathbf{F}_p]^W$ is an isomorphism as shown in [102, Lem. 2.3]. Note that this can be reformulated as saying that $\mathbf{Z}_p[L]^W$ is a polynomial algebra if and only if $\mathbf{F}_p[L\otimes\mathbf{F}_p]^W$ is a polynomial algebra with generators in the same degrees as the generators of $\mathbf{Q}_p[L\otimes\mathbf{Q}]^W$, since $\dim_{\mathbf{Q}_p}(\mathbf{Q}_p[L\otimes\mathbf{Q}]^W)_n=\dim_{\mathbf{F}_p}(\mathbf{Z}_p[L]^W\otimes\mathbf{F}_p)_n\le\dim_{\mathbf{F}_p}(\mathbf{F}_p[L\otimes\mathbf{F}_p]^W)_n$ for any n.

Remark 12.5. The finite $\mathbf{Z}_3$-reflection group $(W,L)=(W_{\mathrm{PU}(3)},L_{\mathrm{PU}(3)}\otimes\mathbf{Z}_3)$ does not have invariant ring a polynomial ring (e.g., since $L_W\cong\mathbf{Z}/3$ is not torsion free). However a short calculation shows that $\mathbf{F}_3[L\otimes\mathbf{F}_3]^W$ *is* a polynomial ring with generators in degrees 1 and 6 (as opposed to the degrees over $\mathbf{Q}_3$ which are 2 and 3). (See also [47, Rem. 5.3].) It turns out that this example is essentially the only one since it can be proved that if (W,L) is a finite $\mathbf{Z}_p$-reflection group, p odd, such that $\mathbf{F}_p[L]^W$ is a polynomial algebra, then $\mathbf{Z}_p[L]^W$ is also a polynomial algebra unless $p=3$ and (W,L) contains $(W_{\mathrm{PU}(3)},L_{\mathrm{PU}(3)}\otimes\mathbf{Z}_3)$ as a direct factor. We omit the proof which is an extension of the technique used in the examples in Section 7 in a preprint version of [56], which can at the time of writing be found on Wilkerson's homepage.

Lemma 12.6. *Assume that L is a finitely generated free $\mathbf{Z}_p$-module and that W is a finite subgroup of $\mathrm{GL}(L)$. If $p\nmid|W|$ and $\mathbf{F}_p[L\otimes\mathbf{F}_p]^W$ is a polynomial algebra, then $\mathbf{Z}_p[L]^W$ is also a polynomial algebra.*

Proof. By assumption we have the averaging homomorphisms $\mathbf{Z}_p[L]\longrightarrow\mathbf{Z}_p[L]^W$ and $\mathbf{F}_p[L\otimes\mathbf{F}_p]\longrightarrow\mathbf{F}_p[L\otimes\mathbf{F}_p]^W$ given by $f\mapsto\frac{1}{|W|}\sum_{w\in W}w\cdot f$. These are obviously surjective and hence the commutative diagram

$$\begin{array}{ccc} \mathbf{Z}_p[L] & \longrightarrow & \mathbf{Z}_p[L]^W \\ \downarrow & & \downarrow \\ \mathbf{F}_p[L \otimes \mathbf{F}_p] & \longrightarrow & \mathbf{F}_p[L \otimes \mathbf{F}_p]^W \end{array}$$

shows that the reduction homomorphism $\mathbf{Z}_p[L]^W \to \mathbf{F}_p[L \otimes \mathbf{F}_p]^W$ is surjective. The result now follows easily from Nakayama's lemma (cf. [102, Lem. 2.3]). □

Proof of Theorem 12.2. Part (1) is contained in [102, Thm. 1.3]. To prove part (2) note that by Notbohm [100] (see also [102, Thm. 1.2(iii)] and Theorem 11.1), there is a unique finite irreducible simply connected $\mathbf{Z}_p$-reflection group for each group on the Clark-Ewing list. We now go through the list, verifying the result in each case.

If $p \nmid |W|$ the invariant ring $\mathbf{F}_p[L \otimes \mathbf{F}_p]^W$ is a polynomial algebra by the Shephard-Todd-Chevalley theorem ([7, Thm. 7.2.1] or [120, Thm. 7.4.1]), and thus Lemma 12.6 shows that $\mathbf{Z}_p[L]^W$ is a polynomial algebra.

Next, assume that (W, L) is an exotic $\mathbf{Z}_p$-reflection group. If (W, L) belongs to family number 2 on the Clark-Ewing list, the representing matrices with respect to the standard basis are monomial and so $\mathbf{Z}_p[L]^W$ is a polynomial algebra by [95, Thm. 2.4].

An inspection of the Clark-Ewing list now shows that only 4 exotic cases remain, namely $(G_{12}, p = 3)$, $(G_{29}, p = 5)$, $(G_{31}, p = 5)$ and $(G_{34}, p = 7)$. In the first case we have $G_{12} \cong \mathrm{GL}_2(\mathbf{F}_3)$ and Lemma 11.3 shows that the action on $L \otimes \mathbf{F}_3 = (\mathbf{F}_3)^2$ is the canonical one. The invariant ring $\mathbf{F}_3[L \otimes \mathbf{F}_3]^{\mathrm{GL}_2(\mathbf{F}_3)}$ was computed by Dickson [39]. In the remaining 3 cases the mod p invariant ring was calculated by Xu [136, 137] using computer, see also Kemper and Malle [80, Prop. 6.1]. The conclusion of these computations is that in all 4 cases the invariant ring $\mathbf{F}_p[L \otimes \mathbf{F}_p]^W$ is a polynomial algebra with generators in the same degrees as the generators of $\mathbf{Q}_p[L \otimes \mathbf{Q}]^W$. By Remark 12.4 we then see that $\mathbf{Z}_p[L]^W$ is a polynomial algebra in these cases.

The only remaining cases are the finite simply connected $\mathbf{Z}_p$-reflection groups which are not exotic. Since p is odd and $\pi_1(G)$ and $(L_G)_{W_G}$ only differ by an elementary abelian 2-group (cf. proof of Theorem 1.7 and Remark 7.4), we may assume that $(W, L) = (W_G, L_G \otimes \mathbf{Z}_p)$ for some simply connected compact Lie group G. In this case Demazure [38] shows that if p is not a torsion prime for the root system associated to G, then the invariant rings $\mathbf{Z}_p[L_G \otimes \mathbf{Z}_p]^{W_G}$ and $\mathbf{F}_p[L_G \otimes \mathbf{F}_p]^{W_G}$ are polynomial algebras.

By the calculation of torsion primes for the simple root systems, [38, Prop. 8], the excluded pairs (G, p) in the last part of the theorem are exactly the cases where the root system of G has p-torsion. In these cases Kemper and Malle [80, Prop. 6.1 and Pf. of Thm. 8.5] shows that $\mathbf{F}_p[L_G \otimes \mathbf{F}_p]^{W_G}$ is not a polynomial algebra. Hence in these cases $\mathbf{Z}_p[L_G \otimes \mathbf{Z}_p]^{W_G}$ is not a polynomial algebra by [102, Lem. 2.3(i)]. This proves the second claim.

Finally, let G be a compact connected Lie group with Weyl group W and integral lattice $L = L_G$. We now prove that $\mathbf{Z}_p[L \otimes \mathbf{Z}_p]^W$ is a polynomial algebra if and only if $H^*(G; \mathbf{Z}_p)$ is torsion free. (See also [103, Prop. 1.11].) One direction follows from Theorem 12.1, so assume now that $\mathbf{Z}_p[L \otimes \mathbf{Z}_p]^W$ is a polynomial algebra. From Theorem 12.2(1) we see that $\mathbf{Z}_p[S(L \otimes \mathbf{Z}_p)]^W$ is a polynomial algebra and that $(L \otimes \mathbf{Z}_p)_W$ is torsion free. Since p is odd, we have $(L \otimes \mathbf{Z}_p)_W = \pi_1(G) \otimes \mathbf{Z}_p$ and $S(L \otimes \mathbf{Z}_p) = L_{G\langle 1 \rangle} \otimes \mathbf{Z}_p$, cf. proof of Theorem 1.7 and Remark 7.4. From what we have proved above we conclude that $H^*(G\langle 1 \rangle; \mathbf{Z}_p)$ is torsion free. Since $\pi_1(G)$ does not contain p-torsion, it now follows easily from the Serre spectral sequence that $H^*(G;\mathbf{Z}_p)$ is torsion free. □

Remark 12.7. Let p be an odd prime and (W,L) a finite $\mathbf{Z}_p$-reflection group. We claim that the following conditions are equivalent:

(1) $\mathbf{Z}_p[L]^W$ is a polynomial algebra.
(2) $\mathbf{F}_p[L\otimes\mathbf{F}_p]^W$ is a polynomial algebra and L_W is torsion free.
(3) $\mathbf{F}_p[SL\otimes\mathbf{F}_p]^W$ is a polynomial algebra and L_W is torsion free.

Indeed we have (1) $\Leftrightarrow$ (3) by Theorem 12.2 since (W,SL) can be decomposed as a direct product of finite irreducible simply connected $\mathbf{Z}_p$-reflection groups by [100, Thm. 1.4]. The implication (1) $\Rightarrow$ (2) follows from [102, Thm. 1.3 and Lem. 2.3]. Finally (2) $\Rightarrow$ (3) follows from [95, Prop. 4.1] as L_W torsion free implies that $SL\otimes\mathbf{F}_p\to L\otimes\mathbf{F}_p$ is injective.

13. Appendix: Outer automorphisms of exotic finite $\mathbf{Z}_p$-reflection groups

Theorem 1.1 states that the outer automorphism group of a p-compact group, p odd, equals $N_{\mathrm{GL}(L)}(W)/W$, which makes it useful to have a complete case-by-case calculation of this group. Theorem 11.1 and Proposition 3.4 reduces the calculation to the case where $(W,L)=(W_G,L_G\otimes\mathbf{Z}_p)$ for some compact connected Lie group G and the case where (W,L) is exotic.

The purpose of this appendix section is to provide such a calculation when (W,L) is exotic based on calculations of Broué, Malle and Michel [23, Prop. 3.13] over the complex numbers. Information about the, perhaps more familiar, Lie case can be obtained similarly, or by a very close reading of [76].

For the statement of the result (which will take place in the theorem below as well as in the following elaborations), we fix the realizations $G(m,r,n)$ of the groups from family 2 as described in Section 11. Moreover we also fix the realization of the complex reflection groups G_i ($4\le i\le 37$) to be the one described in [119]. Finally we let μ_n denote the group of nth roots of unity.

Theorem 13.1 (Outer automorphisms of the exotic $\mathbf{Z}_p$-reflection groups)**.** *Let (W,L) be an exotic $\mathbf{Z}_p$-reflection group and let (W,V) be the associated complex reflection group. Then $N_{\mathrm{GL}(V)}(W)=\langle W,\mathbf{C}^\times\rangle$ and hence $N_{\mathrm{GL}(L)}(W)/W=\mathbf{Z}_p^\times/Z(W)$ and $N_{\mathrm{GL}(L)}(W)/\mathbf{Z}_p^\times W=1$ except in the following cases:*

(1) $W=G(m,r,n)$ from family 2 with $(m,r,n)\neq(4,2,2),(3,3,3)$: $N_{\mathrm{GL}(V)}(W)=\langle G(m,1,n),\mathbf{C}^\times\rangle$ and $N_{\mathrm{GL}(L)}(W)/\mathbf{Z}_p^\times W=C_{\gcd(r,n)}$, cf. 13.4.
(2) $W=G(4,2,2)$: $N_{\mathrm{GL}(V)}(W)=\langle G_8,\mathbf{C}^\times\rangle$ and $N_{\mathrm{GL}(L)}(W)/\mathbf{Z}_p^\times W=\Sigma_3$, cf. 13.5.
(3) $W=G(3,3,3)$: $N_{\mathrm{GL}(V)}(W)=\langle G_{26},\mathbf{C}^\times\rangle$ and $N_{\mathrm{GL}(L)}(W)/\mathbf{Z}_p^\times W=A_4$, cf. 13.6.
(4) $W=G_5$: $N_{\mathrm{GL}(V)}(W)=\langle G_{14},\mathbf{C}^\times\rangle$ and $N_{\mathrm{GL}(L)}(W)/\mathbf{Z}_p^\times W=C_2$, cf. 13.7.
(5) $W=G_7$: $N_{\mathrm{GL}(V)}(W)=\langle G_{10},\mathbf{C}^\times\rangle$ and $N_{\mathrm{GL}(L)}(W)/\mathbf{Z}_p^\times W=C_2$, cf. 13.8.

Lemma 13.2. *Let $K\subseteq K'$ be fields of characteristic zero, and $W\subseteq\mathrm{GL}_n(K)$ an irreducible reflection group. Then $N_{\mathrm{GL}_n(K')}(W)=\left\langle N_{\mathrm{GL}_n(K)}(W),K'^\times\right\rangle$.*

Proof. The inclusion "$\supseteq$" is clear, so suppose $g\in N_{\mathrm{GL}_n(K')}(W)$. Consider the system of equations $Xw=gwg^{-1}X$, $w\in W$ where X is an $n\times n$-matrix. Over K' this has the solution $X=g$. By [58, Lem. 2.10], the representation $W\to\mathrm{GL}_n(K')$ is irreducible, so the solution space is the 1-dimensional space spanned by g. Since the coefficients lie in K, the solution space over K is 1-dimensional as well, so we can write $g = \lambda g_1$ with $\lambda \in K'$ and $g_1 \in M_n(K)$. As $g \neq 0$ we get $\lambda \neq 0$ and $g_1 \in N_{\mathrm{GL}_n(K)}(W)$. □

We can now start the proof of Theorem 13.1. By [23, Prop. 3.13] we easily get the results on $N_{\mathrm{GL}(V)}(W)$ claimed above (note that $\langle G(4,1,2), G_6, \mathbf{C}^\times \rangle = \langle G_8, \mathbf{C}^\times \rangle$ and $G(3,1,3) \subseteq G_{26}$). Now assume that W does not belong to family 2 and $W \neq G_5, G_7$. Let n denote the rank of W and K the field extension of $\mathbf{Q}$ generated by the entries of the matrices representing W. Our assumption ensures that $N_{\mathrm{GL}(V)}(W) = \langle W, \mathbf{C}^\times \rangle$ and hence $N_{\mathrm{GL}_n(K)}(W) = \langle W, K^\times \rangle$. Lemma 13.2 now shows that $N_{\mathrm{GL}_n(\mathbf{Q}_p)}(W) = \langle W, \mathbf{Q}_p^\times \rangle$. Hence we get $N_{\mathrm{GL}(L)}(W) = \langle W, \mathbf{Z}_p^\times \rangle$ and since W is irreducible we have $W \cap \mathbf{Z}_p^\times = Z(W)$, cf. [58, Lem. 2.9].

This proves Theorem 13.1 in case W does not belong to family 2 and $W \neq G_5, G_7$. In the remaining cases we have also proved the statements concerning $N_{\mathrm{GL}(V)}(W)$, and we thus only need to find the structure of $N_{\mathrm{GL}(L)}(W)$ in these cases. This is done in Elaborations 13.4, 13.5, 13.6, 13.7 and 13.8 below.

To treat the dihedral group $G(m,m,2)$ from family 2 we need the following auxiliary result. Note also that the exotic groups from family 2*a* are also handled in [101, §6] (where the non-standard notation $G(q,r;n)$ for $G(q,q/r,n)$ is used).

Lemma 13.3. *Assume that $m \geq 3$ and $p \equiv \pm 1 \pmod m$ so that $\zeta_m + \zeta_m^{-1} \in \mathbf{Z}_p$. Then $2 + \zeta_m + \zeta_m^{-1}$ is a unit in $\mathbf{Z}_p$.*

Proof. It suffices to prove that the norm $N_{\mathbf{Q}(\zeta_m+\zeta_m^{-1})/\mathbf{Q}}(2 + \zeta_m + \zeta_m^{-1})$ is not divisible by p. Since its square equals the norm $N_{\mathbf{Q}(\zeta_m)/\mathbf{Q}}(2 + \zeta_m + \zeta_m^{-1})$ it is enough to see that this norm is not divisible by p. In $\mathbf{Q}(\zeta_m)$ we have $2 + \zeta_m + \zeta_m^{-1} = (1+\zeta_m)^2/\zeta_m$ and since ζ_m is a unit it is enough to see that $N_{\mathbf{Q}(\zeta_m)/\mathbf{Q}}(1 + \zeta_m)$ is not divisible by p. By definition

$$N_{\mathbf{Q}(\zeta_m)/\mathbf{Q}}(1+\zeta_m) = \prod_{\substack{0 \leq k \leq m \\ \gcd(k,m)=1}} (1+\zeta_m^k) = (-1)^{\phi(m)} \prod_{\substack{0 \leq k \leq m \\ \gcd(k,m)=1}} (-1-\zeta_m^k) = \Phi_m(-1).$$

The first claim now follows from [131, Lem. 2.9]. □

Elaboration 13.4 (Family 2, generic case)**.** Let $W = G(m,r,n)$ from family 2 and let p be a prime number such that W is an exotic $\mathbf{Z}_p$-reflection group. Thus if $n \geq 3$ or $n = 2$ and $r < m$ we have $m \geq 3$ and $p \equiv 1 \pmod m$, and for $n = 2$ and $m = r$ we have $m \geq 5, m \neq 6$ and $p \equiv \pm 1 \pmod m$. Assume moreover that $(m,r,n) \neq (4,2,2), (3,3,3)$ (these two cases are dealt with in Elaborations 13.5 and 13.6 below).

Assume first that $p \equiv 1 \pmod m$. The realizations of the groups $G(m,r,n)$ and $G(m,1,n)$ from above are both defined over the ring $\mathbf{Z}[\zeta_m]$ which embeds in $\mathbf{Z}_p$. Lemma 13.2 shows that $N_{\mathrm{GL}_n(\mathbf{Z}_p)}(W) = \langle G(m,1,n), \mathbf{Z}_p^\times \rangle$ whence the natural homomorphism $(A(m,1,n)/A(m,r,n)) \times \mathbf{Z}_p^\times \longrightarrow N_{\mathrm{GL}_n(\mathbf{Z}_p)}(W)/W$ is surjective. The kernel is the cyclic group generated by the element $([\zeta_m I_n], \zeta_m^{-1})$ (here $[\zeta_m I_n] \in A(m,1,n)/A(m,r,n)$ denotes the coset of $\zeta_m I_n$) and thus $N_{\mathrm{GL}_n(\mathbf{Z}_p)}(W)/W = (A(m,1,n)/A(m,r,n)) \circ_{C_m} \mathbf{Z}_p^\times$. Note that $A(m,1,n)/A(m,r,n)$ is cyclic of order r generated by the element $x = [\mathrm{diag}(1,\dots,1,\zeta_m)]$ and that $[\zeta_m I_n] = x^n$.

If the assumption $p \equiv 1 \pmod m$ is not satisfied, then $W = G(m,m,2)$ is the dihedral group of order $2m$ with $m \geq 5, m \neq 6$ and $p \equiv -1 \pmod m$. Conjugating the realization of $G(m,m,2)$ from above with the element

$$g = \begin{bmatrix} 1 & -\zeta_m^{-1} \\ 1 & -\zeta_m \end{bmatrix}$$

gives a realization $G(m,m,2)^g$ defined over the character field $\mathbf{Q}(\zeta_m+\zeta_m^{-1})$. Note that if m is odd, then $N_{\mathrm{GL}_2(\mathbf{C})}(G(m,m,2) = \langle G(m,1,2),\mathbf{C}^\times\rangle = \langle G(m,m,2),\mathbf{C}^\times\rangle$ and hence $N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G(m,m,2)^g)/G(m,m,2)^g = \mathbf{Z}_p^\times$, so we may assume m to be even. Since $G(m,1,2)$ is generated by $G(m,m,2)$ and $\mathrm{diag}(1,\zeta_m)$ we find

$$N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G(m,m,2)^g) = \left\langle G(m,m,2)^g, \begin{bmatrix} 1 & 1 \\ -1 & 1+\zeta_m+\zeta_m^{-1} \end{bmatrix}, \mathbf{Q}_p^\times \right\rangle \cap \mathrm{GL}_2(\mathbf{Z}_p)$$

using Lemma 13.2. From Lemma 13.3 we see that the above matrix is invertible over $\mathbf{Z}_p$ and hence

$$N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G(m,m,2)^g) = \left\langle G(m,m,2)^g, \begin{bmatrix} 1 & 1 \\ -1 & 1+\zeta_m+\zeta_m^{-1} \end{bmatrix}, \mathbf{Z}_p^\times \right\rangle$$

Thus the homomorphism $\mathbf{Z}\times(\mathbf{Z}_p^\times/\mu_2) \longrightarrow N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G(m,m,2)^g)/G(m,m,2)^g$ which maps $(k,[\lambda])$ to the coset of $\lambda \begin{bmatrix} 1 & 1 \\ -1 & 1+\zeta_m+\zeta_m^{-1} \end{bmatrix}^k$ is surjective. The kernel is easily seen to be the infinite cyclic group generated by the element $(-2,[1+\zeta_m+\zeta_m^{-1}])$ and thus we get $N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G(m,m,2)^g)/G(m,m,2)^g \cong \mathbf{Z}\circ_{\mathbf{Z}}(\mathbf{Z}_p^\times/\mu_2)$. It easily checked that $[2+\zeta_m+\zeta_m^{-1}]$ has a square root in $\mathbf{Z}_p^\times/\mu_2$ if and only if either $m\equiv 0 \pmod 4$ or $m\equiv 2 \pmod 4$ and $p\equiv -1 \pmod{2m}$. In this case we have $N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G(m,m,2)^g)/G(m,m,2)^g \cong C_2\times(\mathbf{Z}_p^\times/\mu_2)$.

Elaboration 13.5 ($G(4,2,2)$)**.** The realization of the group $G(4,2,2)$ from above and the realization of the group G_8 from [119, p. 280–281] are both defined over their common character field $\mathbf{Q}(i)$. Thus the relevant primes p are the ones satisfying $p\equiv 1 \pmod 4$. More precisely the representations are defined over $\mathbf{Z}[\frac{1}{2},i]$ and as this ring embeds in $\mathbf{Z}_p$ for all p as above, we see that $N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G(4,2,2)) = \langle G_8,\mathbf{Z}_p^\times\rangle$. It is easily checked that $G_8=\langle G(4,2,2),H\rangle$, where H is the group of order 24 generated by the elements

$$\begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix}, \frac{1+i}{2}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

Since $G(4,2,2)\cap\langle H,\mathbf{Z}_p^\times\rangle = Z(H) = \mu_4$ we conclude that $N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G(4,2,2))/G(4,2,2) \cong (H/Z(H))\times(\mathbf{Z}_p^\times/\mu_4) \cong \Sigma_3\times(\mathbf{Z}_p^\times/\mu_4)$.

Elaboration 13.6 ($G(3,3,3)$)**.** The realization of the group $G(3,3,3)$ from above and the realization of the group G_{26} from [119, p. 296–297] are both defined over their common character field $\mathbf{Q}(\omega)$ where $\omega=e^{2\pi i/3}$. Thus the relevant primes p are the ones satisfying $p\equiv 1 \pmod 3$. More precisely the representations are defined over $\mathbf{Z}[\frac{1}{3},\omega]$ and as this ring embeds in $\mathbf{Z}_p$ for all p as above, we see that $N_{\mathrm{GL}_3(\mathbf{Z}_p)}(G(3,3,3)) = \langle G_{26},\mathbf{Z}_p^\times\rangle$. It is easily checked that G_{26} is the semidirect product of $G(3,3,3)$ with the group $H\cong\mathrm{SL}_2(\mathbf{F}_3)$ generated by the elements

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, R_2 = \frac{1}{\sqrt{-3}}\begin{bmatrix} \omega & \omega^2 & \omega^2 \\ \omega^2 & \omega & \omega^2 \\ \omega^2 & \omega^2 & \omega \end{bmatrix}$$

The center of H is generated by the element

$$z = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and $G(3,3,3) \cap \langle H, \mathbf{Z}_p^\times \rangle = \langle -z, \mu_3 \rangle$. Thus

$$N_{\mathrm{GL}_3(\mathbf{Z}_p)}(G(3,3,3))/G(3,3,3) \cong H \circ_{C_2} (\mathbf{Z}_p^\times/\mu_3) \cong \mathrm{SL}_2(\mathbf{F}_3) \circ_{C_2} (\mathbf{Z}_p^\times/\mu_3)$$

where the central product is given by identifying $z \in H$ with the element in $\mathbf{Z}_p^\times/\mu_3$ represented by -1.

Elaboration 13.7 (G_5)**.** The realization of the group G_5 from [119, p. 280–281] is defined over the field $\mathbf{Q}(\zeta_{12})$, but the group has character field $\mathbf{Q}(\omega)$ and thus the relevant primes p are the one satisfying $p \equiv 1 \pmod 3$. Conjugation by the matrix

$$g = \begin{bmatrix} 2 & \sqrt{3}-1 \\ (\sqrt{3}-1)(1-i) & i-1 \end{bmatrix}$$

gives a realization defined over $\mathbf{Z}[\frac{1}{3}, \omega]$ which embeds in $\mathbf{Z}_p$ for all p as above. Its easily checked that G_{14} is generated by G_5 and the reflection

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & i \\ -i & 1 \end{bmatrix}$$

From this we get

$$N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G_5^g) = \left\langle G_5^g, \begin{bmatrix} 0 & 1 \\ -2\omega & 0 \end{bmatrix}, \mathbf{Z}_p^\times \right\rangle$$

and thus the homomorphism $\mathbf{Z} \times (\mathbf{Z}_p^\times/\mu_6) \longrightarrow N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G_5^g)/G_5^g$ which maps $(k, [\lambda])$ to the coset of $\lambda \begin{bmatrix} 0 & 1 \\ -2\omega & 0 \end{bmatrix}^k$ is surjective. The kernel is easily seen to be the infinite cyclic group generated by the element $(-2, [2])$ and we get $N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G_5^g)/G_5^g \cong \mathbf{Z} \circ_{\mathbf{Z}} (\mathbf{Z}_p^\times/\mu_6)$. It is easy to check that $[2]$ has a square root in $\mathbf{Z}_p^\times/\mu_6$ if and only if $p \equiv 1, 7, 19 \pmod{24}$ (that is unless $p \equiv 13 \pmod{24}$). In this case we get the simpler description $N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G_5^g)/G_5^g \cong C_2 \times (\mathbf{Z}_p^\times/\mu_6)$.

Elaboration 13.8 (G_7)**.** The realizations of the groups G_7 and G_{10} given in [119, p. 280–281] are both defined over their common character field $\mathbf{Q}(\zeta_{12})$. Thus the relevant primes p are the ones satisfying $p \equiv 1 \pmod{12}$. More precisely the representations are defined over $\mathbf{Z}[\frac{1}{2}, \zeta_{12}]$ and as this ring embeds in $\mathbf{Z}_p$ for all p as above, we see that $N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G_7) = \langle G_{10}, \mathbf{Z}_p^\times \rangle$. It is easily checked that $G_{10} = \langle G_7, C_4 \rangle$, where C_4 is the cyclic group generated by $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$. Since $G_7 \cap (C_4 \times \mathbf{Z}_p^\times) = C_2 \times \mu_{12}$ we conclude that $N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G_7)/G_7 \cong C_2 \times (\mathbf{Z}_p^\times/\mu_{12})$.

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