Lovász Conjecture and Hom-complexes

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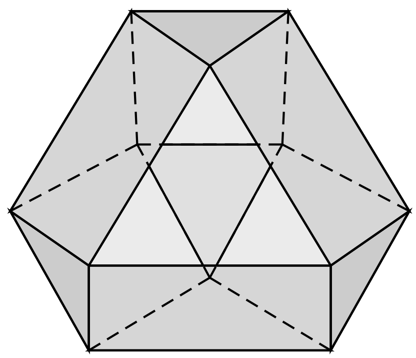


Contents

1 Introduction

The main purpose of this paper is to investigate lower bounds for the chromatic number of graphs. More precisely we will investigate the substance and application of two theorems: The first by L. Lovász and the second by E. Babson and D. N. Kozlov.

Theorem 1.1 (Lovász)**.** *Let G be a graph s.t. the connectivity of $Hom(K_2, G)$ is k for some integer $k \geq -1$. Then*

$$\chi(G) \geq conn(Hom(K_2, G)) + 3.$$

Theorem 1.2 (Babson and Kozlov)**.** *Let G be a graph s.t. the connectivity of $Hom(C_{2r+1}, G)$ is k for some integer $k \geq -1$. Then*

$$\chi(G) \geq conn(Hom(C_{2r+1}, G)) + 4.$$

The last one is also known as the Lovász conjecture. Both theorems are based on the connectivity of the Hom-complex. Connectivity is discussed in section 3.4, while we'll introduce and investigate the Hom-complex in section 4.

We will further investigate the Stiefel-Whitney classes and their application in the proof of the Lovász conjecture. The definition will be given in section 3.6, while in section 5.1 we will give an outline of the proof.

At last we will discuss the connection between theorem 1.1 and 1.2 in section 5.2 and further development possibilities in section 5.3.

2 Graphs

We begin with some basic definitions and notations for graphs. Then we proceed with a definition of the category of graphs and some more definitions. Finally we take a look at graph coloring and the chromatic number of graphs.

Definition 2.1. We define a *graph* G as a set of vertices denoted $V(G)$ and a set of edges $E(G) \subseteq V(G) \times V(G)$ s.t. if we have $(u, v) \in E(G)$ then also

$(v,u) \in E(G)$. If for any vertex $v \in V(G)$, (v,v) is an edge of G, then we'll call this a *loop*. In short our graphs are undirected and allowing loops.

In other words a graph is a collection of vertices connected by undirected edges. In our definition we allow loops, but we'll later explicitly demand of some graphs, that they are loopfree; e.g. don't have any loops.

A basic property of graphs is the existence of neighbours.

Definition 2.2. If we for two vertices $u, v \in V(G)$ have that $(u,v) \in E(G)$, then we say that u and v are *neighbours*, and we write this $u \sim v$.

One vertex can have many neighbours, and this naturally induces a notion of neighbourhood.

Definition 2.3. For a vertex $v \in V(G)$ we define the *neighbourhood* of v as the set

$$N_G(v) = \{u \in V(G) \mid (u,v) \in E(G)\} = \{u \in V(G) \mid u \sim v\}.$$

This will be written $N(v)$ when appropriate.

Traversing along successive edges of a graph, we obtain a sequence of edges. If we only allow the same edge to be used once, we get what is called a path. More precisely we have:

Definition 2.4. A *path* in G is a sequence of vertices $v_1, v_2, \ldots, v_n$ s.t. $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)$ are different edges of G. If $n \neq 1$ and $v_n = v_1$, it is also called a *cycle*.

We might add, that in the literature the definition of a path does not always require different edges. It doesn't matter which definition we use though, and we will thus call the latter a *walk*.

We continue with the next definitions of connectedness.

Definition 2.5. For a graph G two vertices $u, v \in V(G)$ are *connected* if there is a path in G from u to v.

In general we have that:

Definition 2.6. A graph G is *connected*, if for any pair of distinct vertices $u, v \in V(G)$, there is a path from u to v.

We also want to study relations between graphs, and a natural way to do so, is to define a graph homomorphism like this.

Definition 2.7. A *graph homomorphism* of G and H, is a map

$$f : V(G) \to V(H),$$

s.t. if $u \sim v$, then $f(u) \sim f(v)$.

This induces some restrictions on the maps between graphs, and it follows intuitively to introduce a collection of all homomorphisms between a pair of graphs.

Notation 2.8. For two graphs G and H, let $\mathrm{Hom}_0(G, H)$ denote the set of all graph homomorphisms from G to H. The notation is a special case of a more general definition, that we will see in section 4.

We want to introduce the category of graphs as the set of all graphs with the graph homomorphisms as the morphisms. To do this properly, we need the graph homomorphisms to be composable s.t. the composition has the identity and associativity properties. This is easily checked though and left to the reader.

Definition 2.9. We define *the category of graphs* as the set of all graphs as defined above and with the graph homomorphisms as the morphisms. We denote it **Graphs**.

Let us continue by defining two important families of graphs.

Definition 2.10 (Complete graphs)**.** Let G be a graph of n vertices s.t. for any $u, v \in V(G)$ we have $(u, v) \in E(G)$ if and only if $u \neq v$. This is the *complete graph* of n vertices, and we'll denote it K_n.

Definition 2.11 (Cyclic graphs)**.** A graph G of n vertices numerated from 1 to n and n edges given by

$$E(G) = \{(i, i+1) \mid 1 \leq i < n\} \cup \{(n, 1)\},$$

is called the *n-cyclic graph*, and we'll denote it C_n.

The next definition of folds will be used in our study of graph coloring later on.

Definition 2.12 (Fold)**.** Let G be a graph and $v \in V(G)$ a vertex of the graph. Then $G - v$ is understood to be the graph with vertex set

$$V(G - v) = V(G) \setminus \{v\},$$

and edge set

$$E(G - v) = E(G) \setminus \{(a, b) \in E(G) \mid a = v \text{ or } b = v\}.$$

If $N(v) \subseteq N(u)$ for some $u \in V(G - v)$ then we call $G - v$ a *fold* of G.

For a fold we have a graph homomorphism $f : G \to G - v$ where $f(w) = w$ when $w \neq v$ and $f(v) = u$.

At last we give a definition of complete bipartite subgraphs, that will be used later.

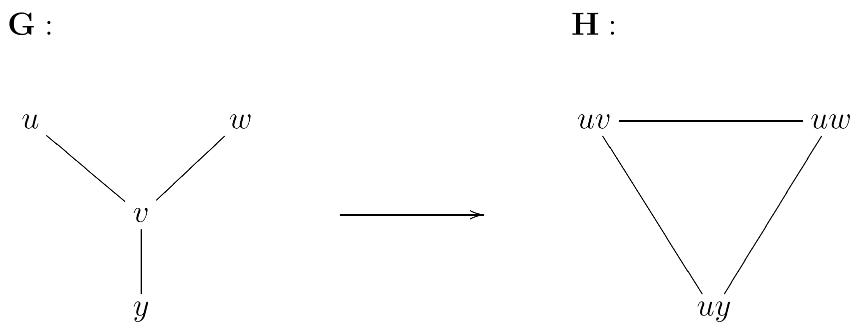
Definition 2.13. For a graph G, let $A, B \subseteq V(G)$ be non-empty sub set of vertices. Then (A, B) is a *complete bipartite subgraph* of G, if for any $x \in A$ and any $y \in B$, we have $(x, y) \in E(G)$.

Note that if G contains loops, then A and B do not have to be disjoint.

2.1 Graph coloring

We'll now go on to graph coloring. The plan is to develop tools that can be used in graph coloring, and we'll therefore state some basic definitions. When we mention graph coloring here, we'll mean vertex coloring of graphs. It is also possible to color the edges of a graph, but this problem is relatively easily converted into a vertex coloring problem.

Example 2.14. An *edge coloring* problem can be translated into a vertex coloring problem like this. Let G be the graph we want to edge color. Then we construct a derived graph H to be vertex colored like this. Each edge in $E(G)$ is added to H as a vertex. Then for each vertex v in $V(G)$ we add an edge to $E(H)$ for each pair of edges of G having v as an endpoint. That is if v is a vertex of G and (v, u), (v, w) and (v, y) are the edges of G having v as endpoint, then $((v, u), (v, w))$, $((u, u), (v, y))$ and $((v, w), (v, y))$ will be added as edges of H. This way we get all edges from G as vertices and all the constraints have been transformed into edges. Se below for a drawn example of G and H.



Definition 2.15. A *vertex coloring* of a graph G, is an assignment of a color to each vertex of G, s.t. two neighbouring vertices don't get the same color.

More precisely if we let A be a set of colors, then a vertex coloring is a map $c : V(G) \to A$ s.t. $u \sim v$ for $u, v \in V(G)$ implies $c(u) \neq c(v)$.

If G has a loop, the condition above can never be true, so a vertex coloring is never possible in this case. On the other hand if our color set A is sufficiently large, it is always possible to color any finite graph, that does not contain a loop. The interesting part though is to find the smallest number of colors s.t. there exists a coloring of a graph G.

Definition 2.16. The *chromatic number* $\chi(G)$ of a graph G is the smallest number of colors a graph can be colored by.

If G contains a loop, there is no coloring of G, and we will by convention say $\chi(G) = \infty$.

Example 2.17. If there exists a fold $G - v$ of a graph G, then G can always be colored in the same number of colors as $G - v$. This is because there is a vertex $u \in V(G - v)$ s.t. $N(v) \subseteq N(u)$, and thus v could always obtain the same color as u.

Going back to graph homomorphisms, we might ask if there is a connection between a graph homomorphism of G and H and coloring of the two graphs.

Remark 2.18. First we want to describe a coloring of a graph G in n colors as a graph homomorphism $c : G \to K_n$, each vertex of K_n representing one color. By definition a coloring of a graph is a map from the graph to a set of colors A. This corresponds to the vertices of K_n. Then the condition that two neigbouring vertices of G cannot have the same color, is equivalent to the requirement of a graph homomorphism to preserve adjacency. Hence the two descriptions must be the same.

It easily follows, that if there exists a homomorphism $f : G \to H$ and a coloring c of H in n colors, then $c \circ f$ is a coloring of G in n colors. This is because we now have graph homomorphisms

$$c \circ f : G \xrightarrow{f} H \xrightarrow{c} K_n,$$

and the composition of two homomorphisms is again a homomorphism.

We might not use all the colors in such a coloring, and thus it makes sense to look at the images of f and $c \circ f$. If the image of f is colored in less than n colors, then this is the same as $c \circ f$ not being onto. As a consequence if there exists a graph homomorphism $f : G \to H$, then $\chi(H) \geq \chi(G)$ implying, that if we also have a homomorphism $g : H \to G$, then $\chi(H) = \chi(G)$.

We'll study the relation between graph coloring and graph homomorphisms much closer later in the paper, where we'll link it to the simplicial complex $Hom(G, H)$, which is closely connected to graph homomorphisms.

3 A topological angle

We will now look into some of the topological concepts used in connection with theorems 1.1 and 1.2.

We want to define two different basic building block for our topological studies: *Simplicial complexes* and *partially ordered sets* also known as *posets*. But to make the process clear and also get a better understanding of the definitions to come, we are going to arrive at these through the *polyhedral complexes*.

Further we are going to introduce the Stiefel-Whitney classes, that are an important part of the proof of theorem 1.2.

3.1 Simplices

We begin by defining the basic building blocks – the simplices. A simplex can shortly be described as a n-dimensional versions of a triangle. E.g. the tetrahedron is a 3 dimensional simplex. More precisely we might describe a n-simplex as the convex hull of $n + 1$ affinely independent points in some Euclidean space $\mathbb{R}^m$, where $m > n$.

This yields infinitely many simplices of any dimension, and because they are all homotopic equivalent, we want to construct and work with a standard version of a n-simplex.

Definition 3.1. The standard *n-simplex* is defined as

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\}.$$

If we for any n-simplex take a k-subset of the points, we get a $k - 1$ dimensional simplex by their convex hull. These simplexes are called the *faces* of the n-simplex.

Definition 3.2. We can for any set S of n elements, construct a n-simplex based on the definition of the standard simplex above, by indexing the vertices by the elements of S. We will write this Δ^S for any set S.

The faces of Δ^S can in general be identified as all the finite subsets of S, and the notion of Δ^S will in the rest of the paper be the collection of all these simplices.

Our next step is to look at direct products of simplices. We start by looking at the product $\Delta^S \times \Delta^S$. Understanding a simplex as a CW complex with the simplices of Δ^S as cells, we consider the product to be a product of CW complexes, which is again a CW complex. For example if $|S| = 2$, $\Delta^S \times \Delta^S$ would be a square, while if $|S| = 3$, we would have a product of two solid triangles obtaining a 4 dimensional figure.

The product can be generalized by

$$\prod_{t \in T} \Delta^S,$$

where the coordinates of the product are indexed by elements of some other set T.

In general the cells of a product of CW complexes can be described as products of the cells of the CW complexes. This automatically gives us a way to describe the cells of our product above as products of simplices. The geometric structure we obtain in this way, is what we call a convex polytope – a higher dimensional version of a convex polygon.

3.2 Polyhedral complex

We are almost ready to state the definition of the polyhedral complex, but first we define the convex polytopes. We have seen a convex hull version of the definition of a simplex, and we want to continue along the same path. We'll therefore define our convex polytope like this.

Definition 3.3. A *convex polytope* is the convex hull of a finite collection of points in an Euclidean space. The points are not necessarily affinely independent.

This implies directly, that every simplex is a convex polytope. But we also get other objects of various dimension. For example the convex polygons are

2-dimensional convex polytopes, while a dice shaped cube is a 3-dimensional convex polytope.

Unlike with the simplices, we are not able to decide the dimension of the convex polytope based on the number of points required to define it. The following remark is useful to understand convex polytopes.

Remark 3.4. For any collection of points defining a convex polytope, there exists a uniquely determined minimal set of points defining the same convex polytope.

Using this minimal representation of a convex polytope, we can define a face of a convex polytope.

Definition 3.5. Let S be the minimal set of points describing a convex polytope. For any subset of S spanning a n-dimensional hyperplane L s.t. L does not divide the convex polytope (e.g. all points are either on L or on the same side of L), a *face* of a convex polytope is the convex hull of all points in S on L.

Thus for the cube mentioned above, if we take three corners on the same side, then all four corners of the side are included in the plane, and the convex hull of the four points is a face of the convex polytope.

There are obvious difficulties in deciding the faces of an arbitrary convex polytope, because different subsets of the points defining a convex polytope, define the same face. A complex polytope is thus a much less regular structure than the simplex.

We'll organize collections of convex polytopes in a polyhedral complex like this.

Definition 3.6. A *polyhedral complex* is a set P of convex polytopes s.t.

- Any face of a convex polytope in P is also in P.
- Any intersection of two convex polytopes $\sigma_1, \sigma_2 \in P$ is a face of both σ_1 and σ_2.

In this way the polyhedral complex can be understood as a collection of convex polytopes glued together along their faces. The products of simplices seen above, are convex polytopes. Further a polyhedral complex is also a CW complex.

3.3 Simplicial complex

Although the construction of polyhedral complexes from direct products of simplicial complexes is simple and intuitive, polyhedral complexes do not offer the flexibility we want. We will therefore continue with the definition of a simplicial complex.

Definition 3.7. A *simplicial complex* S with vertex set V, is a set of subsets of V that is stable under subset. That is if $\sigma_1 \in S$ and $\sigma_2 \subset \sigma_1$, then $\sigma_2 \in S$.

Clearly the subsets can be understood as simlices as defined in section 3.1.

The purpose of defining the simplicial complexes is that we are then able to use some general tools, that are not available when working with the more complex polyhedral complexes. Fortunately we can relatively easily obtain a simplicial complex from a polyhedral complex by doing barycentric subdivision.

3.3.1 Barycentric subdivision

To be able to define a barycentric subdivision of a convex polytope, we first have to consider the barycenter.

Definition 3.8. The *barycenter* of a convex polytope of dimension n is the intersection of all $(n-1)$-dimensional hyper planes dividing the polytope into two part of equal size (by some defined measure).

For example the barycenter of a line segment would usually be the point dividing the segment into two segment of equal length, while the barycenter of a polygon would be the intersection of lines dividing the polygon into two parts with the same area.

But how we find the barycenter of a convex polytope does not matter in our case, as long as it is uniquely defined and inside the convex hull of the points.

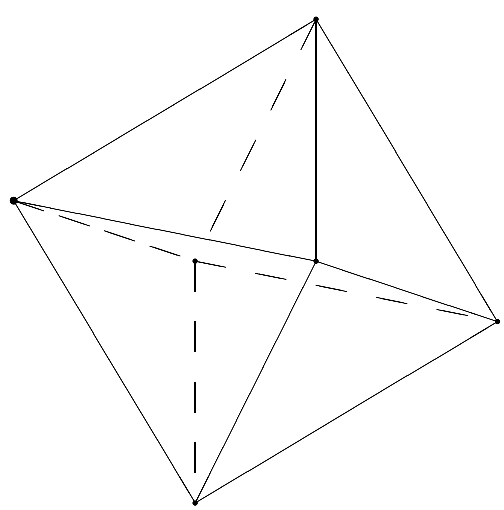
We are now able to define barycentric subdivision of a polytope P using the barycenters of P and all of its faces.

Definition 3.9. Let P be a convex polytope of dimension n. Then *barycentric subdivision* is a division of P into n-dimensional simplices in the following way. For each possible sequence $F_0, F_1, \ldots, F_n$ of faces of P meeting the requirement that F_i is a face of F_{i+1} for all $0 \leq i < n$, we have a simplex with vertices $v_0, v_1, \ldots, v_n$, where v_i is the barycenter of F_i.

We can see the barycentric subdivision as a slicing of the convex polytope into simplices s.t. simplices are covering the convex polytope, and more importantly the intersection of two simplices is again a simplex.

Example 3.10. Let us look at the octahedron and do barycentric subdivision on it. The octahedron is a polytope consisting of 6 vertices, 12 edges, 8 triangles and the whole octahedron. Given a vertex F_0 we have 4 possible choices of edge and 2 choices of face. Thus there are 48 possibilities of sequences F_0, F_1, F_2, F_3 giving 48 simplices.

Octahedron:



3.4 Connectivity

With connectivity we mean connectivity in the sense of simplicial complexes (or more general CW complexes) as seen in definition 3.14. But first we need to build up to it.

Definition 3.11. A simplicial complex H is said to be *connected*, if any two points of H are connected by some path.

In this case we mean a topological path, but it is easily seen, that a simplicial complex is connected if and only if the 1-skeleton of the simplex is a connected graph. This is true because a simplicial complex is a CW-complex, and the only cells having disconnected boundary are the 1-cells.

Definition 3.12 (k-connected)**.** For an integer $k \geq -1$, a simplicial complex H is said to be *k-connected*, if we for any $-1 \leq i \leq k$ have, that any continuous map $S^i \to K$ is homotopy equivalent to a constant map.

Example 3.13. For example a polygon would be homotopy equivalent to S^1, and it is well known that not all maps $S^1 \to S^1$ are homotopy equivalent to a constant map. Thus a polygon is not 1-connected, but it is 0-connected.

Any shape that is homotopy equivalent to a sphere is 1-connected, because any circle can be deformation retracted to a point on the sphere. But it is not 2-connected.

Definition 3.14. The *connectivity* of a simplicial complex H is the largest integer k s.t. H is k-connected. We will write this $conn(H)$.

Remark 3.15. If our simplicial complex H consists of only a single point (or if it deformation retracts to a point), then any map $S^n \to H$ is homotopy equivalent to a constant map. In this case we will consider the connectivity to be infinite.

Example 3.16. A disconnected simplicial complex is (-1)-connected, but not 0-connected. Hence it has connectivity -1.

Remark 3.17. With regard to the computational complexity of deciding a lower bound for the chromatic number of a graph, theorem 1.2 does not give much. Even for small values of k it is very hard to decide if a graph is k-connected (see for example [Koz] 1.1.3 for a discussion of this). Later in section 3.6 we will define an other value, the Stiefel-Whitney classes, connected to the connectivity, which we hope to be easier to compute.

3.5 Partially ordered set

We also want to describe the product of simplices as a *poset* (short for partially ordered set).

Definition 3.18. A *partially ordered set* is a set P with a binary relation $\leq$ over P s.t.

- $a \leq a$ for all $a \in P$ (transitivity)
- if $a \leq b$ and $b \leq a$ for $a, b \in P$, then $a = b$ (asymmetry)
- if $a \leq b$ and $b \leq c$ for $a, b, c \in P$, then $a \leq c$ (transitivity)

holds. We will normally call a partially ordered set a *poset*.

The first thing we want to show, is that the products of simplices as described in section 3.1, can also be described as posets with the cells as elements. If we take $(s_1, s_2, \dots, s_n)$ and $(\tau_1, \tau_2, \dots, \tau_n)$ as two different cells of a product $\prod \Delta^S$ of n simplices, then we define an order of the two as

$$(s_1, s_2, \dots, \sigma_n) \leq (\tau_1, \tau_2, \dots, \tau_n)$$

if and only if $\sigma_i \subseteq \tau_i$ for all $1 \leq i \leq n$.

This way we get that two cells are orderly related if one is contained in the other. Intuitively a cell is contained in another cell if it is a face of it.

3.6 Stiefel-Whitney classes

To get a better understanding of the Lovász conjecture, we have to take a closer look at Stiefel-Whitney classes. This requires us to start with some basic definitions.

Definition 3.19. Let G be a group, and X a set. Then a *group action* of G on X is a function $G \times X \to X$ that satisfies the two following conditions.

- $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and all $x \in X$.
- $e \cdot x = x$ for all $x \in X$, where e is the identity element in G.

Group actions have something to do with symmetries of the set X. For example if X is the graph K_3, then group actions on K_3 would be functions sending vertices to vertices and edges to edges thus rotating around various axes.

If we have a group action G, that does not fix any element of X, we call it a free group action. Formally we define it like this.

Definition 3.20. Let G be a group and X a set. A *free group action* of G on X is a group action with the extra condition, that

$$g \cdot x = x \quad \Rightarrow \quad g = e$$

for all $g \in G$, all $x \in X$ and e the identity element in G.

For a given group G, any set X admitting a free G action, is called a G-space.

We'll in the rest of this section consider X to be a CW-complex. Specially our arguments will apply to simplicial complexes.

Before we are ready to proceed with the definition of Stiefel-Whitney classes, we need to state some facts from the general theory of principal G-bundles. Firstly if X is a G-space, then there exists a G-equivariant map $\omega : X \to \mathbf{E}G$. $\mathbf{E}G$ is in this case a contractible space on which G acts freely. Further the induced map $\omega/G : X/G \to \mathbf{E}G/G = \mathbf{B}G$ is uniquely determined upto homotopy (see [Hus] for more details).

Now let us get to our concrete case. Let $G = \mathbb{Z}_2$ and $\mathbf{E}G = S^\infty$ using the antipodal map as the $\mathbb{Z}_2$ action. We then have a $\mathbb{Z}_2$-equivariant map $\omega : X \to S^\infty$, and an induced map $\omega/\mathbb{Z}_2 : X/\mathbb{Z}_2 \to S^\infty/\mathbb{Z}_2 = \mathbb{RP}^\infty$.

Now we want to look at the cohomology groups of $X/\mathbb{Z}_2$ and $\mathbb{RP}^\infty$ and their $\mathbb{Z}_2$-algebras. Firstly $H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2[z]$, where z is the non trivial class of $H^1(\mathbb{RP}^\infty; \mathbb{Z}_2)$. Thus there is one trivial and one nontrivial class in each cohomology group.

Secondly the map $\omega/\mathbb{Z}_2$ above induces a $\mathbb{Z}_2$-algebra homomorphism $\omega^* : H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) \to H^*(X/\mathbb{Z}_2; \mathbb{Z}_2)$. The image of this homomorphism applied to z is an element of $H^1(X/\mathbb{Z}_2; \mathbb{Z}_2)$. This is what we'll define as the *first Stiefel Whitney class* of X, and we denote it $\varpi_1(X)$.

From the homomorphism property of ω^* we get, that $\omega^*(z^k) = \varpi_1^k(X)$, and as we'll see later, we are mainly interested in finding the largest k s.t. $\varpi_1^k(X) \neq 0$.

Example 3.21. Let us look at how to calculatethe Stiefel-Whitney class of a simplicial complex X, where X is a $\mathbb{Z}_2$-space.

It is clear, that the free $\mathbb{Z}_2$ action ψ on X pairs simplices of the same dimension in X together in orbits of two. We want to use this in our construction of the $\mathbb{Z}_2$ map $\varphi : X \to S^\infty$. To define φ, the first thing we do, is to divide all the points (or 0-simplices) of X into two sets A and B s.t. if $x \in A$ then $\psi(x) \in B$. Then we let $\{a, b\}$ be the two points of the standard description of S^∞, and we map the points in A to a and the points in B to b.

Now the 1-simplices of X can be divided into three sets based on their endpoints. Two sets for both endpoints being in A or B and one for *multicolored* edges, where the endpoints are in different sets. It's obviously the multicolored edges, that are the interesting ones. As mentioned the 1-simplices can be paired in orbits of two by ψ, and it follows that the multicolored edges can be paired. To define f on the 1-simplices, we now map edges with both endpoints in A resp. B to a resp. b. Further we map the multicolored edges to the standard 1-simplices of S^∞ - let us call them e_1 and e_2 - s.t. if the edge x is mapped to e_1 then $\psi(x)$ is mapped to e_2.

Our next step is to understand the Stiefel-Whitney class obtained by this construction. The space $X/\mathbb{Z}_2$ obtained by identifying x with $\psi(x)$ for all simplices of X, has one simplex for each orbit in X. Thus the cochain complex $C^*(X/\mathbb{Z}_2;\mathbb{Z}_2)$ has one generator for each orbit of X. Let us denote the generator corresponding to any simplex $\sigma \in X$ by τ_σ. Thus $\tau_\sigma = \tau_{\psi(\sigma)}$. For S^∞ the orbit corresponding to the 1-simplices e_1 and e_2 is the generator of $H^1(\mathbb{RP}^\infty;\mathbb{Z}_2)$. Call it z. Now the image of the morphism $(\varphi/\mathbb{Z}_2)^*$, induced by φ when applied to z gives us the identity on non-multicolored 1-simplex orbits, and the generator on multicolored 1-simplex orbits. Thus the image, which by definition is the first Stiefel-Whitney class, is the sum of the generators of the multicolored 1-simplex orbits.

The Stiefel-Whitney class is clearly in the kernel of δ because for any 2-simplex in $\sigma \in X$, either all corners are in the same set A or B, or two are in one and one is in the other. If all are in the same, then $\delta\varpi_1(\sigma)$ is trivial, while in the other case exactly two of the edges have endpoints in different set and are therefore evaluated non-trivially. Thus we have $\delta\varpi_1(\sigma) = 1 + 1 + 0 = 0$.

It is more difficult to decide if the Stiefel-Whitney class is in the image of δ. If X is disconnected, it might be, but if X is connected, then for any vertex $x \in X$ there is a path from x to $\psi(x)$. And because x and $\psi(x)$ are in different sets A and B, an odd number of the edges on the path are evaluated non-trivially by $\varpi_1(X)$. And this gives an obstruction, because this would demand x and $\psi(x)$ to be both equal and different at the same time.

At last let us look at how to find the powers $\varpi_1^k(X)$. We see it now as a product in the algebra $H^*(X/\mathbb{Z}_2;\mathbb{Z}_2)$. For any n-simplex $\sigma \in X/\mathbb{Z}_2$ denoted $\sigma = (v_1, v_2, \ldots, v_n)$ we have

$$\varpi_1^k(\sigma) = \prod_{i=1}^{n-1} \varpi_1(v_i, v_{i+1}).$$

Clearly all the edges (v_i, v_{i+1}) have to be evaluated non-trivially by ϖ_1, if

$\varpi_1^k(\sigma)$ is to be evaluated non-trivially. So $\varpi_1^k(\sigma)$ can only be non-trivial if the ordered set of vertices are alternately in A and B. In addition to this, we have to consider if $\varpi_1^k(\sigma)$ is in the image of δ and thus vanishes on the boundary.

We'll need the next property of the Stiefel-Whitney class later.

Remark 3.22. Assume that Y is an other $\mathbb{Z}_2$-space s.t. there is a $\mathbb{Z}_2$ equivariant map $\varphi : X \to Y$; e.g. a map that is commutable with the $\mathbb{Z}_2$ actions. Then there is a $\mathbb{Z}_2$ equivariant map $v : Y \to S^\infty$ inducing the same structure as ω did with X replaced by Y. We also have $v \circ \varphi : X \to S^\infty$ as an other equivariant map on the same spaces as ω. In addition the induced quotient map $\omega/\mathbb{Z}_2$ is uniquely determined upto homotopy, implying that $\omega^* = ((v \circ \varphi)/\mathbb{Z}_2)^*$. Now $v \circ \varphi$ as a commutable diagram induces the following equivalence

$$((v \circ \varphi)/\mathbb{Z}_2)^* = (v/\mathbb{Z}_2)^* \circ (\varphi/\mathbb{Z}_2)^*,$$

giving at last, that $\varpi_1(X) = (\varphi/\mathbb{Z}_2)^*(\varpi_1(Y))$. We will refer to this property as the Stiefel Whitney classes being *functorial*. (see also [Koz] section 3.1.1.)

4 Hom-complex

In the introduction we have expressed the Lovász conjecture with the Hom-complex. And now we will explain this complex as well as consider how to calculate it.

In the literature there are two different but equivalent definitions, that are dominant. One is based on maps of T into the subsets of vertices of G, while the other takes a simplicial complex approach. We begin with the latter.

4.1 A simplicial complex approach

For any graph G let $\Delta^{V(G)}$ be the simplex having $V(G)$ as vertices. Then we can identify the simplices of $\Delta^{V(G)}$ with the finite subsets of $V(G)$ as in definition 3.2. Now for any graph T let

$$C(T,G) := \prod_{x\in V(T)} \Delta^{V(G)},$$

be the direct product of these simplices indexed by the vertices of T with the additional requirement, that for any product of cells in $C(T,G)$, only finitely many of them may have positive dimension. This gives a polynomial complex, which again can be transformed into a simplicial complex through barycentric subdivision.

Example 4.1. The cells of $C(T,G)$ can be described as

$$\sigma = \prod_{x\in V(T)} \sigma_x,$$

where σ_x is a simplex of $\Delta^{V(G)}$ indexed by vertex x of T.

We can also determine the dimension of σ as the sum

$$\dim \sigma = \sum_{x\in V(T)} \dim \sigma_x,$$

where, as mentioned above, only finitely many cells have positive dimension. Thus the sum is finite.

Example 4.2. Let us see some examples with K_2 and K_3. $C(K_3,K_2)$ is a direct product of 3 2-simplices (segments), and thus it is a dice shaped cube.

The $C(K_2,K_3)$ complex is a direct product of two 3-simplices (triangles) giving a 4 dimensional object in some way resembling a torus (the product of the borders are homotopy equivalent to $S^1 \times S^1$, but then we have to fill out both the inside and the outside of the torus in some way).

Now we can define $Hom(T,G)$ as a subcomplex of $C(T,G)$ like this.

Definition 4.3 (Hom-complex)**.** We define $Hom(T,G)$ as the subcomplex of $C(T,G)$ where each cell $\sigma = \prod_{x\in V(G)} \sigma_x$ of $C(T,G)$ is in $Hom(T,G)$ if and only if, we for any edge $(u,v) \in E(T)$ have that (σ_u,σ_v) is a bipartite subgraph of G.

From this definition we see directly, that the 0-cells of $Hom(T, G)$ are exactly the graph homomorphisms from T to G. Thus we see, that our denotation of all graph homomorphisms from T to G by $Hom_0(T, G)$, complies with this general definition. We also see that the Hom-complexes have a great deal to do with graph homomorphisms, and it inspires our first statement about Hom-complexes.

Proposition 4.4. *The Hom-complexes are fully described by their 0-cells, thus making the Hom-complexes only depending on the graph homomorphisms from T to G.*

The statement is part of proposition 2.2.2 of [Koz], where a proof can also be found.

To further our understanding, we return to our examples with K_2 and K_3.

Example 4.5. Recall that $Hom(K_3, K_2)$ is a subcomplex of $C(K_3, K_2)$. We also know that there are no graph homomorphisms from K_3 to K_2, and by proposition 4.4 we see, that $Hom(K_3, K_2)$ must be empty.

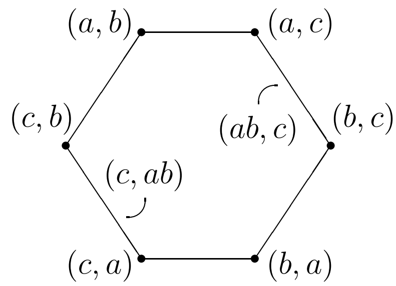
Swapping the graphs on the other hand, $Hom(K_2, K_3)$ is a subcomplex of $C(K_2, K_3)$ – the direct product of two triangles. Because K_3 has no loops, we also require that the cells of $Hom(K_2, K_3)$ are products of two disjoint subsets of $V(K_3)$. In this way we get 6 points and 6 segments

$$(a, b), (a, c), (b, c), (b, a), (c, a), (c, b)$$

and

$$(a, bc), (ab, c), (b, ca), (bc, a), (c, ab), (ca, b),$$

where we have labeled the vertices of K_3 by a, b and c. This gives us the hexagon we see below.



The hexagon is homotopy equivalent to S^1 and thus its connectivity is 0, and by theorem 1.1, $\chi(K_3) \geq 3$. Nothing non-trivial but nice to see the theorem in action.

4.2 An other definition

Sometimes an other equivalent definition of Hom-complexes is more useful than the one seen above. We will therefore state it here.

Definition 4.6. For graphs T and G we define $Hom(T,G)$ to be the poset with elements given by maps $\eta : T \to 2^G \setminus \{\emptyset\}$, s.t. if $(x,y) \in E(T)$ then $(\tilde{x},\tilde{y}) \in E(G)$ for every $\tilde{x} \in \eta(x)$ and $\tilde{y} \in \eta(y)$.

The inclusion of the poset is given by $\eta \leq \eta'$ if $\eta(x) \subseteq \eta'(x)$ for all $x \in V(G)$.

Comparing the two definitions it is clear, that the vertices of σ_x in the product of cells in 4.3 correspond to the subset, which x gets mapped to in 4.6. Likewise the conditions on the edges of T are the same. Therefore the two definitions only differ by their notation.

4.3 Properties of Hom

To get a better understanding of the Hom-complexes, we want to investigate some properties of them.

First we want to connect them to the neighbourhood complexes, that Lovász used in the original version of theorem 1.1.

Definition 4.7. Let G be a graph. Then we let the *neighbourhood complex* of G be the simplicial complex $\mathcal{N}(G)$ defined in the following way: The vertices of $\mathcal{N}(G)$ are all non-isolated vertices of G, and the simplices of $\mathcal{N}(G)$ are all subsets of $V(G)$ that have any common neighbour.

Now the following theorem holds for any graph G.

Theorem 4.8. *There is a homotopy equivalence*

$$\mathcal{N}(G) \cong Hom(K_2, G),$$

for any graph G.

Theorem 1.1 was originally formulated by Lovász ([Lov]) using $\mathcal{N}(G)$, but it was later shown (see for example [BaKo] prop. 4.2) that $\mathcal{N}(G)$ and $Hom(K_2, G)$ are homotopy equivalent, and thus a reformulation of the theorem to the form stated here was possible. Indeed the two complexes are of the same homotopy type according to [Koz2].

The advantage of using the Hom-complexes instead of the neighbourhood complexes is, that $Hom(K_2, G)$ is only a special case of a general family of complexes. The general case opens up for more similar theorems, and in fact the proof of theorem 1.2 by Babson and Kozlov ([BaKo2]) underlines this.

It has further been conjectured, that the following statement would be true.

Conjecture 4.9. *For any graphs G and H the statement*

$$Hom(G, H) \leq \chi(H) - \chi(G) - 1$$

holds.

This statement has though been disproved by a counter example by Hoory and Linial in [HoLi].

We have been looking at folds above, and it is relevant to ask how the Hom-complexes are affected by folds. This would enable us to classify whole families of Hom-complexes. It turns out, that the following statement is true.

Theorem 4.10. *Let G and H be two arbitrary graphs and let $G-v$ be a fold of G. Let further $i : G-v \to G$ be the inclusion map, and $f : G \to G-v$ be the map $f(w) = w$ for all $w \neq v$ and $f(v) = u$. Then there are induced homotopy equivalences*

$$f_H : Hom(G-v, H) \to Hom(G, H),$$

$$i_H : Hom(G, H) \to Hom(G-v, H),$$

$$f_{H'} : Hom(H, G) \to Hom(H, G-v),$$

$$i_{H'} : Hom(H, G-v) \to Hom(H, G).$$

An argument for the first two equivalences can be found in [BaKo] as proposition 5.1, while the last two are proved in [Cuk2] lemma 3.1.

The Hom-complexes do also have functorial properties in the sense, that the following two propositions hold.

Proposition 4.11. *$Hom(T, -)$ is a covariant functor from* **Graphs** *to the simplicial complexes.*

Proposition 4.12. *$Hom(-, G)$ is a contravariant functor from* **Graphs** *to the simplicial complexes.*

An argument for these propositions can be found in [Koz] section 2.4.3.

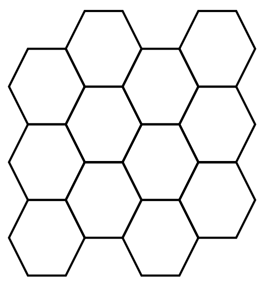
4.4 Hom-complex examples

We have previously seen, that $Hom(K_2, K_3)$ is a hexagon, and now we want to look at some other similar examples. We start by looking at complete graphs.

Example 4.13. $Hom(K_3, K_4)$ – Let us start with maps from K_3 into K_4. Obviously the vertices of K_3 have to be mapped to disjoint sets of vertices of K_4. Thus when we have a graph homomorphism sending the vertices of K_3 to vertices of K_4, it is only possible to exchange one of them at a time to obtain a new graph homomorphism. In this way we get that each graph homomorphism is included in 3 multihomomorphisms as defined in definition 4.6.

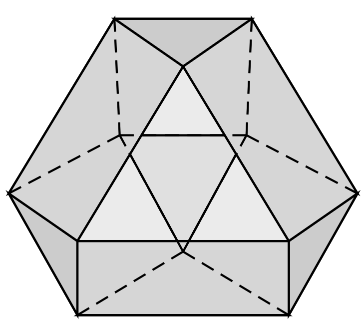
Further if we fix one of the vertices of K_3 to one in K_4, then we get a hexagon structure in the same way as for $Hom(K_2, K_3)$ in example 4.5.

It turns out, that the whole $Hom(K_3, K_4)$ complex is a grid of hexagon as shown in the figure below, where the same 12 hexagons keep repeating periodically both horizontally and vertically.



Example 4.14. $Hom(K_2, K_4)$ – Now we take an example, that should give some higher dimensional objects. Let us look at multihomomorphisms from K_2 into K_4. Then if we fix one end of K_2, we have 3 options for the other end resulting in cells that are triangles. Further if the first end of K_2 is mapped to one of two cells of K_4, then we have the possibility of one of the other two for the second end of K_2. Thus we get a square cell.

These are the maximal cells, and combining all the cells, we get 8 triangles and 6 squares. The resulting complex is the polyhedron we see below.



If we look at these examples, we see that $Hom(K_2, K_3)$ is homotopy equivalent to S^1, $Hom(K_3, K_4)$ is homotopy equivalent to a wedge of 12 circles, while $Hom(K_2, K_4)$ is homotopy equivalent to S^2. Babson and Kozlov have shown in [BaKo] that in general $Hom(K_m, K_n)$ is homotopy equivalent to a wedge of $(n-m)$-dimensional spheres. In fact from their work we might extract the following theorem for the case $m = 2$.

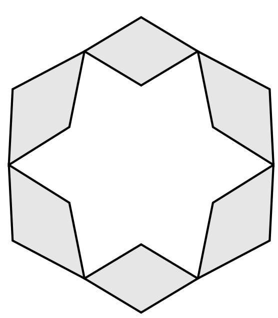
Theorem 4.15. *For an integer n, $Hom(K_2, K_n)$ is homotopy equivalent to S^{n-2}.*

This follows from proposition 4.5 and 4.6 in [BaKo].

Let us proceed with complexes of cycles.

Example 4.16. $Hom(C_4, C_3)$ – Look at graph homomorphisms from C_4 into C_3. Let us label the vertices of C_4 by A, B, C and D successively and the vertices of C_3 1, 2 and 3. Then the notation 1212 means that we map A to 1, B to 2 and so fort.

Let us continue to find the polyhedral complex, where the maximal cells will be equivalent to this: A maps to 1, B maps to $\{2, 3\}$, C maps to 1 and D maps to $\{2, 3\}$. Thus these cells are squares, and exploring all the combinations, we find that there are 6 such cells, that are connected to each others on the corners. E.g. the cell described above is connected to one mapping B and D to 2 and A and C to $\{1, 3\}$ at the vertex 1212. This gives us the complex below.



Clearly the $Hom(C_4, C_3)$ is homotopy equivalent to S^1. In fact in [Cuk2] it is shown, that for integers m and n, in the complex $Hom(C_m, C_n)$ every connected component is homotopy equivalent to S^1 or homeomorphic to a point.

These are just some basic and simple examples of Hom-complexes. They are not really complicated, but it is clear, that when the graphs become larger, everything rapidly becomes much more complex. It is therefore interesting to develop tools to better understand the Hom-complexes as done in for example [Cuk], [Cuk2] and [BaKo].

5 Perspective and prospects

5.1 An outline of the proof of Lovász conjecture

We'll now give a rough outline of the proof that Babson and Kozlov in [BaKo2] gave of Lovász conjecture. The proof is based on using the Stiefel-Whitney classes of Hom-complexes to construct a lower bound for graph coloring.

The first thing we want to establish, is that the Hom-complexes stated in the theorem are $\mathbb{Z}_2$-spaces.

Theorem 5.1. *Let G be a graph with a $\mathbb{Z}_2$ action $\varphi : G \to G$, s.t. φ flips an edge. That is for some edge $(a, b) \in E(G), a \neq b$, we have $\varphi(a) = b$. Then for any graph H, we get an induced $\mathbb{Z}_2$ action $\varphi_H : Hom(G, H) \to Hom(G, H)$. Further if H has no loops, then φ_H is a free $\mathbb{Z}_2$ action.*

Proof. In theorem 4.12 it is stated, that $Hom(-, H)$ is a functor, and the induced $\mathbb{Z}_2$ action follows from this.

Let us now suppose, that H is loop free, and that φ_H is not free. Then there would be a cell $\eta : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$, that is fixed by φ_H.

We have that φ_H maps η to $\eta \circ f$ implying that $\eta = \eta \circ f$, because η is fixed. It follows that $\eta(a) = \eta \circ f(a) = \eta(b)$. We have assumed, that η is a cell of $Hom(G, H)$, and therefore there is a vertex v of H included in

$\eta(a) = \eta(b)$. By definition of $Hom(G, H)$ this implies, that $(v, v) \in E(H)$ contradicting, that H is loop free. $\square$

To prove theorem 1.2 we don't have to look at the case, when G has a loop, because these graphs do not have any legal coloring. Thus if we can find a graph G fulfilling the assumptions in theorem 5.1, then all relevant Hom-complexes will be $\mathbb{Z}_2$-spaces.

It is further easily seen, that for $G = C_{2r+1}$ and $G = K_n$, there are well known $\mathbb{Z}_2$ maps flipping an edge. Thus every Hom-complex we are going to look at, has a Stiefel-Whitney class.

An other consequence of theorem 5.1 is the following corollary.

Corollary 5.2. *For any graph G having a $\mathbb{Z}_2$ action flipping an edge, we have that $Hom(G, -)$ is a covariant functor from loop free graphs to $\mathbb{Z}_2$-spaces.*

Our next step is to show that nonvanishing of powers of Stiefel-Whitney classes are obstruction to graph coloring.

Theorem 5.3. *Let G be a loop free graph, and T a graph with a $\mathbb{Z}_2$ action that flips an edge as described in theorem 5.1. If for some integers $k \geq 0$, $m \geq 1$ we have that $\varpi_1^k(Hom(T, G)) \neq 0$ and $\varpi_1^k(Hom(T, K_m)) = 0$, then $\chi(G) \geq m + 1$.*

Proof. Assume in addition to the assumptions of the theorem, that G is m colorable. Then there is a graph homomorphism $c : G \to K_m$. Now we may apply corollary 5.2 to get an induced $\mathbb{Z}_2$-map $c^T : Hom(T, G) \to Hom(T, K_m)$. Further the Stiefel-Whitney classes are functorial according to remark 3.22, so we get an induced morphism $(c^T/\mathbb{Z}_2)^* : \varpi_1^k(Hom(T, K_m)) \to \varpi_1^k(Hom(T, G))$. Now $\varpi_1^k(Hom(T, K_m)) = 0$ implies $\varpi_1^k(Hom(T, G)) = 0$, and we have a contradiction. $\square$

This is connected to the formulation of theorem 1.2 by the following lemma.

Lemma 5.4. *If a $\mathbb{Z}_2$ space X is k-connected, then there exists a $\mathbb{Z}_2$ map $\varphi : S^{k+1} \to X$. In particular $\varpi_1^{k+1}(X) \neq 0$.*

Proof. Let us first describe S^i inductively in the following way. Let first S^0 be two points. Then for $i > 0$, let S^i be the join of S^{i-1} and $\{a, b\}$. In this way S^{k+1} will be a join of $k+2$ copies of S^0.

We will now define φ in the following way. First we define φ on the fist copy of S^0 by mapping one point to an arbitrary point x of X, and then map the other point to the free $\mathbb{Z}_2$ action on x. Then for $1 < i \leq k+1$ we will map the join of the first $i+1$ copies of S^0 by extending the map of the first i copies to the join $S^{i-1} * \{a, b\}$. This can be done because X is k-connected, and thus any map $f : S^{i-1} \to X$ is homotopically equivalent to a point. Thus if we map a to this point and b to the $\mathbb{Z}_2$ action on a, then the extension follows intuitively from the homotopy.

Applying remark 3.22 to our map, we get that

$$\varphi^*(\varpi_1^{k+1}(X)) = \varpi_1^{k+1}(S^{k+1}) \neq 0,$$

and this clearly shows that $\varpi_1^{k+1}(X) \neq 0$.

□

The next step in the proof is to answer to which graphs T and integers n the following equation holds.

$$\varpi_1^{n-\chi(T)+1}(Hom(T, K_n)) = 0, \text{ for all } n \geq \chi(T) - 1. \tag{1}$$

It turns out, that it holds for all n when $T = K_m, m \geq 2$ or $T = C_{2r+1}, r \geq 1$. This is not easily proved though specially in the case when $T = C_{2r+1}$ and n is even. A proof can be found in [BaKo2].

Finally there is one more big hurdle before we are able to give the proof of Lovász conjecture (theorem 1.2). And that is theorem 5.5, which will not be proved here.

Consider one of the two maps mapping K_2 to the $\mathbb{Z}_2$ invariant edge of C_{2r+1}, e.g. the edge being flipped by the $\mathbb{Z}_2$-action used above. Let us write it $\iota : K_2 \to C_{2r+1}$. Because $Hom(-, H)$ is a contravariant functor, ι induces a map of $\mathbb{Z}_2$ spaces $\iota_{K_n} : Hom(C_{2r+1}, K_n) \to Hom(K_2, K_n)$. And this map induces a $\mathbb{Z}_2$ algebra homomorphism

$$\iota_{K_n}^* : H^*(Hom(K_2, K_n); \mathbb{Z}) \to H^*(Hom(C_{2r+1}, K_n); \mathbb{Z}).$$

Now the theorem states this.

Theorem 5.5. *Assume n is even. Then $2 \cdot \iota_{K_n}^*$ is a 0-map.*

This is stated as theorem 2.6 in [BaKo2] and proved in section 4.

At last we give the proof of Lovász conjecture (theorem 1.2).

Proof of Lovász conjecture. The Hom-complex is (-1)-connected if and only if it is nonempty, and because there clearly are no graph homomorphisms from odd cycles to bipartite graphs, and a two-colorable graph is bipartite, the case $k = -1$ is trivially true. We may thus assume that $k \geq 0$.

Now let k be even. Then $Hom(C_{2r+1}, G)$ is k-connected, and by lemma 5.4, $\varpi_1^{k+1}(Hom(C_{2r+1}, G) \neq 0$. Odd cycles are 3 colorable, and as we have remarked above, equation (1) holds when T is an odd cycle. By setting $n = k + 3$ in the equation we get that $\varpi_1^{k+1}(Hom(C_{2r+1}, K_{k+3})) = 0$ for all positive even k. At last odd cycles clearly have the required $\mathbb{Z}_2$ action to fulfill the requirements of theorem 5.3, and we get $\chi(G) \geq k + 4$ for all even k.

Let us continue with k odd. Suppose also indirectly that $\chi(G) \leq k + 3$. By condition of the conjecture, $Hom(C_{2r+1}, G)$ is k-connected, and by lemma 5.4 we have a map $f : S^{k+1} \to Hom(C_{2r+1}, G)$. We also have a $k+3$ coloring map $c : G \to K_{k+3}$ because $\chi(G) \leq k + 3$. In corollary 5.2 we see, that $Hom(C_{2r+1}, -)$ is a covariant functor from loop free graphs to $\mathbb{Z}_2$-spaces, and thus the coloring map induces a $\mathbb{Z}_2$ map between the $\mathbb{Z}_2$-spaces

$$c^{C_{r2+1}} : Hom(C_{2r+1}, G) \to Hom(C_{2r+1}, K_{k+3}).$$

Further the map ι from theorem 5.5 gives us a $\mathbb{Z}_2$-space map

$$\iota_{K_{k+3}} : Hom(C_{2r+1}, K_{k+3}) \to Hom(K_2, K_{k+3}).$$

The last space $Hom(K_2, K_{k+3})$ is homotopy equivalent to S^{k+1} by theorem 4.15.

Combining these maps we get a map

$$h = f \circ c^{C_{2r+1}} \circ \iota_{K_{k+3}} : S^{k+3} \to S^{k+3}.$$

Moving to cohomology this gives an induced homomorphism on the $k+1$ dimensional groups. Formally we may write this map $h^* : \mathbb{Z} \to \mathbb{Z}$. By theorem 5.5, $2 \cdot \iota^*_{K_{k+3}}$ is a 0-map, implying that $2 \cdot h^* = 0$, and thus also $h^* = 0$.

Now we have our counter example, because in [Hat] proposition 2B.6 it is stated, that such a map has to have odd degree, and thus cannot be a 0-map. And this concludes the proof. □

5.2 Connection to Lovász theorem

Lovász conjecture (theorem 1.2) gives a lower bound for the chromatic number of a graph G based on the Hom-complexes $Hom(C_{2r+1}, G)$. But how is this comparable to the original theorem 1.1 proved by Lovász?

In [Sch] C. Schultz points out, that Babson and Kozlov partially proved the stronger statement, that if k is the height of the Stiefel-Whitney class $\varpi_1(Hom(C_{2r+1}, G))$ (e.g. the largest k s.t. $\varpi_1^k(X) \neq 0$), then $\chi(G) \geq k+3$.

He also points out, that the usual proof of theorem 1.1 yields an even stronger boundary

$$\chi(G) \geq ind_{\mathbb{Z}_2} Hom(K_2, G) + 2,$$

where $ind_{\mathbb{Z}_2}(X)$ is the minimum k s.t. there is a $\mathbb{Z}_2$-map $X \to S^k$.

Finally he proceeds to show, that

$$h(\varpi_1(Hom(C_{2r+1}, G)) + 1 \leq h(\varpi_1(Hom(K_2, G))) \leq ind_{\mathbb{Z}_2} Hom(K_2, G),$$

where h is the mentioned height function of the Stiefel-Whitney classes. And this shows indeed, that the stronger version of Lovász conjecture can never yield a stronger boundary, than the one we get from the usual proof of theorem 1.1.

5.3 Conclusion and further development

As we have seen in section 5.2, the proof of Lovász conjecture does not give the best boundary for chromatic numbers. And a natural question is therefore to ask how we might obtain the best boundary. In [HoLi] a counter example shows that conjecture 4.9 is not true. But we might still find other methods to improve.

Several attempts to better the understanding of the Hom-complexes have been made. In [Doc] A. Dochtermann tries to do this with the introduction of a notion of x-homotopy of graph maps. In [Cso] Csorba and Lutz try to connect the Hom-complexes to manifolds, specially PL manifolds. Some of this might give some result in the future.

An other interesting question to ask, is about computability. Hom-complexes might be easily computed, but certainly their connectivity is not. In this matter the Stiefel-Whitney classes come in handy, because they are probably much more computable.

Ultimately one might always ask oneself if it is the results that are important, or it is the methods developed in the quest for the results.

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