On Parallel Transport and Curvature

Graduate Project

Raffaele Rani
Supervisor: Jesper Michael Møller

April 6, 2009

Abstract

Given connection ∇ on a smooth vector bundle $E \to M$, with connected base space M, the set of the parallel transport maps (associated to ∇) along closed loops based at $x \in M$ form a subgroup, $\mathrm{Hol}(\nabla)$, of the general linear group on the fibre E_x, $\mathrm{GL}(E_x)$. The group $\mathrm{Hol}(\nabla)$ is known as the holonomy group of the connection and it is independent of the base point x under conjugation of elements of the general linear group. It therefore defines a global invariant for the connection. If M is simply connected, then $\mathrm{Hol}(\nabla)$ is a Lie subgroup of $\mathrm{GL}(k, \mathbb{R})$. Restricting $\mathrm{Hol}(\nabla)$ to nullhomotopic loops gives rise the restricted holonomy group, $\mathrm{Hol}^0(\nabla)$, which is exactly the identity component of $\mathrm{Hol}(\nabla)$. The Lie algebra associated to $\mathrm{Hol}^0(\nabla)$ and $\mathrm{Hol}(\nabla)$ is called the holonomy algebra, $\mathfrak{hol}(\nabla)$. The holonomy algebra is a linear subspace of $\mathrm{End}(E_x)$ and it coincides with the the subspace of $\mathrm{End(E_x)}$ generated by a special class of endomorphism obtained through the curvature tensor $R(\nabla)$ of the connection.
This important result is captured by the Ambrose-Singer holonomy theorem. In this report we investigate the dependence of parallel transport maps on the curvature, building the necessary tools to prove the Ambrose-Singer holonomy theorem.

Contents

1 Introduction

In this project we discuss some aspects of the theory of connections on vector bundles, focusing in particular on two topics, the curvature and the holonomy group of a connection. The parallel transport map associated to a given connection is an important tool to "detect" the effects of curvature. Curvature can, in fact, be understood as a measure of the extent to which parallel transport around closed loops fails to preserve the geometrical data being transported.

Given a connection ∇ on a smooth vector bundle $E \to M$ with k-dimensional connected base space M, parallel transport maps along loops based at $x \in M$ define a set of linear endomorphisms, $\mathrm{Hol}(\nabla)$, on the fibres E_x. Under composition and inverse of parallel transport maps, $\mathrm{Hol}(\nabla)$ acquires the structure of a subgroup of the general linear group $\mathrm{GL}(E_x)$ and it is known as the holonomy group of ∇. In addition, if M is simply connected then $\mathrm{Hol}(\nabla)$ is a Lie subgroup of $\mathrm{GL}(E_x)$ ($\cong \mathrm{GL}(k, \mathbb{R})$). It is interesting to observe that $\mathrm{Hol}(\nabla)$ is independent of the base point x under conjugation of elements of the general linear group and thus, it is a global invariant for the connection, whereas curvature is a local invariant.

If we restrict the definition of holonomy group to nullhomotopic loops, then $\mathrm{Hol}(\nabla)$ group is said to be restricted. The restricted holonomy group, $\mathrm{Hol}^0(\nabla)$ is a Lie subgroup of $\mathrm{GL}(E_x)$ and it is the identity connected component of $\mathrm{Hol}(\nabla)$ with an associated Lie (holonomy) algebra $\mathfrak{hol}(\nabla)$.

The curvature and the holonomy group of a connection are strictly related. One can construct a class of endomorphisms on a fibre E_x at x by acting the curvature tensor $R(\nabla)$ on a parallelely transported section and of E and then transporting the resulting section back to the starting point. It turns out that the linear subspace of $End(E_x)$ spanned my these maps coincide exactly with $\mathfrak{hol}_p(\nabla)$. This result is encapsulated by the Ambrose-Singer holonomy theorem.

The Ambrose-Singer holonomy theorem along with its "almost converse" result that the curvature $R(\nabla)_p$ of a connection ∇ at p lies in $\mathfrak{hol}(\nabla) \otimes \wedge^2 T_p^* M$, for any $p \in M$, make it evident that the holonomy algebra both constrains the curvature and is determined by it.

Moreover, the Ambrose-Singer holonomy theorem together with the Bianchi identities lie at the basis of the methods employed by Berger to prove his classification of Riemannian holonomy groups [1].

The goal of this paper is two-fold: we both aim at introducing the machinery required to prove the Ambrose-Singer holonomy theory and at investigating the details of the dependence of parallel transport on the curvature.
Chapter 2 sets the notation and defines parallel transport in terms of the pullback bundle construction.
In chapter 3 we define a smooth homotopies and use their properties to explore some basic properties of parallely transport maps and curvature and how such maps are connected to curvature.
In chapter 4 the theory of holonomy groups is introduced and some important examples are given for Riemannian symmetric spaces. Finally we discuss the relation between holonomy and curvature and work out a proof of the Ambrose-Singer holonomy theorem.

2 Preliminaries

In this chapter, we provide a brief account of some constructions in the framework of differential geometry. We shall assume that the reader is already familiar with the basic ideas of the theory of differential manifolds. We will mainly focus on establishing the notation and for sake of brevity of this review chapter many details and proofs are left to the references.
Since this chapter is designed around the notion of vector bundle, section 2.1 covers vector, tensor and form fields as they are constructed as sections of tensor and exterior products of vector bundles.
After having defined connections on a generic vector bundle in section 2.2, the focus is soon shifted to the construction of pullback bundles and the connections induced on them. This particular type of induced connection will then be used to define the parallel transport map: the main theme of this note.
We conclude this introductory chapter by shortly discussing the notion of curvature and torsion of a connection in section 2.3.

2.1 Basic Constructions in Differential Geometry

We recall that a vector bundle E with fibre space F (typical fibre) over a manifold M is a

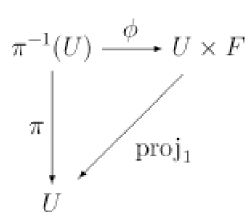

Figure 2.1: Vector bundle.

fibre bundle whose fibres are (real or complex) vector spaces. That is, E is a manifold equipped with a continuous surjective map $\pi : E \to M$, where each $E_x = \pi^{-1}(x)$ and the typical fibre F

are vector spaces. In addition, given an open neighbourhood U of each point $x \in M$, the local homeomorphisms (*trivializations*), $\phi : \pi^{-1}(U) \to U \times F$, can be chosen so that their restriction to each fibre is a linear isomorphism, $\phi : E_x \to \{x\} \times F \cong F$ [2].
If E and M are endowed with smooth structures, π is a smooth map and the local trivializations can be chosen to be diffeomorphisms, then we say that the vector bundle E is *smooth*. In what follows we will always refer to smooth vector bundles unless we specify otherwise. Depending on what we wish to emphasize, we may sometimes leave out some or all of the elements of the definition and simply use E, $E \to M$ or $\pi : E \to M$ to denote a vector bundle.

Figure 2.2: An everyday example of fibre bundle: the intuition behind the term "fibre bundle" is evident. This hairbrush is like a fibre bundle in which the base space M is a cylinder and the bristles are the fibres E_x. The map $\pi : E_x \to M$ would take all the points on any bristle to the point on the cylinder where the bristle attaches.

Now, let M be a smooth n-dimensional manifold, with tangent bundle TM and cotangent bundle T^*M. Then TM and T^*M are vector bundles over M [3].

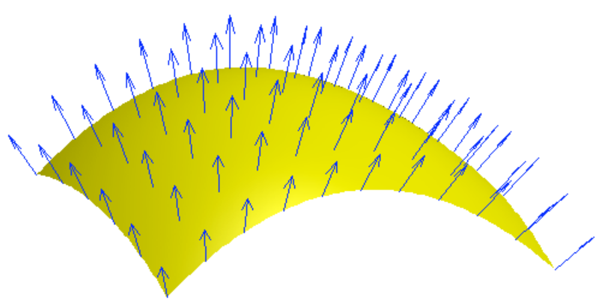

Figure 2.3: A map associating to each point on a surface a vector normal to it can be viewed a section.

A *section* of a vector bundle is a section of its map π, that is a continuos map $\sigma : M \to E$ such that $\pi \circ \sigma = Id_M$. The map that associates to each point on a surface a vector normal to it can be viewed as an example of section of the tangent bundle.

If E is a vector bundle over M, we use the notation $C^\infty(E)$ for the vector space (under pointwise addition and scalar multiplication) of smooth (global) sections of E [3]. The elements of $C^\infty(TM)$ and $C^\infty(T^*M)$ are called *vector fields* and 1-*form fields* respectively. The symbol $C^\infty(M)$ will be used to denote the vector space of smooth functions (maps from M to $\mathbb{R}^n$) on M.

By taking tensor products of the vector bundles TM and T^*M we obtain the bundles of tensors on M. A $\binom{k}{l}$-*tensor field* T is the smooth section of a bundle $\bigotimes^k T^*M \otimes \bigotimes^l TM$ for some $k, l \in \mathbb{N}$ [1].

Let U be an open set in M, and $(x^1, \ldots, x^n)$ coordinates on U (i.e. the component functions of a chart $\phi : U \to \mathbb{R}^n$). Then, at each point $x \in U$, $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ form a basis for T_xU, the tangent space at x. We will often abbreviate the operators $\frac{\partial}{\partial x^i}$ to ∂_i. Hence, any smooth vector field v on U can be uniquely written as $v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}$ for some smooth component functions $v^1, \ldots, v^n : U \to \mathbb{R}$.

Similarly, at each $x \in U$, $dx^1, \ldots, dx^n$ are a basis for T_x^*U, the cotangent space at x. Thus we can uniquely write any 1-form α on U as $\alpha = \sum_{i=1}^n \alpha_i dx^i$ for some smooth component functions $\alpha_1, \ldots, \alpha_n : U \to \mathbb{R}$.

In the same way, a tensor T in $C^\infty(\bigotimes^k T^*M \otimes \bigotimes^l TM)$ yields the following coordinate representation

$$T = \sum_{\substack{1 \le a_i \le n,\ 1 \le i \le l \\ 1 \le b_j \le n,\ 1 \le j \le k}} T_{b_1 \ldots b_k}{}^{a_1 \ldots a_l} dx^{b_1} \otimes \cdots \otimes dx^{b_k} \otimes \frac{\partial}{\partial x^{a_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{a_l}}.$$

To avoid the cumbersome proliferation of summation symbols every time we write a tensor equation in component form, it is useful to adopt the Einstein summation convention. According to this rule, summation is implied whenever one index is repeated in a lower and upper position. With this convention the component expressions for a vector, a 1-form and a $\binom{k}{l}$ -tensor are, respectively, recast as the more compact form

$$v = v^i \frac{\partial}{\partial x^i}, \qquad \alpha = \alpha_i dx^i, \qquad T = T_{b_1 \ldots b_k}{}^{a_1 \ldots a_l} dx^{b_1} \otimes \cdots \otimes dx^{b_k} \otimes \frac{\partial}{\partial x^{a_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{a_l}}.$$

As a generalization of 1-forms, we define *p-forms* to be *totally antisymmetric* (*alternating*) tensors of type $\binom{p}{0}$. i.e. ω is a p-form if for any exchance of any two arguments a_i and a_j,

$$\omega(a_1 \ldots a_i \ldots a_j \ldots a_p) = -\omega_(a_1 \ldots a_j \ldots a_i \ldots a_p)$$

.

[1] When mentioning the order (type) of a tensor we follow the most commonly used notation in Mathematics: the "upper" (covariant order) indicates the number of vectors that the multilinear map takes as arguments. Similarly the "lower" (contravariant) order tells us about the number "covectors" (1-forms). Notice, however, that some authors, especially in the Theoretical Physics literature, tend to exchange the role of these two numbers. This results, for instance, in p-forms being called $\binom{0}{p}$ antisymmetric tensors.

The set of all the p-forms is closed under addition and scalar multiplication and therefore it constitutes a subspace, denoted by $\Lambda^p T^*M$, of $\bigotimes^p T^*M$, i.e. the space of all $\binom{p}{0}$-tensors. Because of the (anti-) symmetry if its elements, $\Lambda^p T^*M$ has dimension $\binom{p}{n}$ (when $p \leq n$, 0 otherwise, by anti-symmetry of the forms) with $n = \dim T_x M$. Again, in a local coordinates at x, a p-form ω can be written as

$$\omega = \sum_{\substack{1 \leq a_1 < \cdots < a_p \leq n \\ 1 \leq a_i \leq n}} \omega_{a_1 \dots a_p} dx^1 \otimes \cdots \otimes dx^p.$$

Notice, however, that the restrictions on the summation are not evident (notationally) in the Einstein summation convention, but they are indeed accounted for in practice [2] .
One can associate a p-form to any tensor of type $\binom{p}{0}$ through the *alternating map*, Alt : $\bigotimes^p T^*M \to \Lambda^p T^*M$. The map Alt is defined as

$$\operatorname{Alt} T(V_1, \dots, V_p) = \frac{1}{p!} \operatorname{sgn} \sigma \sum_{\sigma \in S^p} T(V_{\sigma(1)}, \dots, V_{\sigma(p)}),$$

where S^p is the group of permutations of p objects and $sgn\sigma$ gives $+1$ if the permutation σ is obtained by an even number of transpositions and -1 if this number is odd.
There is an important natural operation defined on differential forms called the *exterior product*. Let α be a l-form and β be a p-form, then The exterior product associates to a l-form α and a p-form β a $l+p$-form, $\alpha \wedge \beta$, obtained by "antisymmetrizing" the tensor product $\alpha \otimes \beta$ through the alternating map:

$$\alpha \wedge \beta := \operatorname{Alt}(\alpha \otimes \beta).$$

The exterior product is bilinear, associative and anticomutative ($\alpha \wedge \beta = (-1)^{lp} \beta \wedge \alpha$).
The p^{th} exterior power of the cotangent bundle, $\bigwedge^p T^*M$ is a real vector bundle over M with fibres of dimension $\binom{n}{p}$ (for $n \geq p$, 0 otherwise, n) and the space of its smooth section is denoted by $C^\infty(\bigwedge^p T^*M)$.
We observe that $\bigwedge^p T^*M$ is a subbundle of $\bigotimes^p T^*M$ [2].
For every smooth manifold there are unique linear maps $d : C^\infty(\bigwedge^p T^*M) \to C^\infty(\bigwedge^{p+1} T^*M)$, defined for each integer $p \geq 0$, that extend the concept of differential of a function (a 0-form) [3]. The *exterior derivative* d of a function is a 1-form which we identify with its *differential*, df.
In addition, d satisfies

$$\alpha \wedge \beta = (d\alpha) \wedge \beta + (-1)^l \alpha \wedge (d\beta), \quad \forall\, \alpha \in C^\infty(\wedge^l T^*M),\ \beta \in C^\infty(\wedge^p T^*M)$$

and

$$d^2 = d \circ d = 0.$$

[2] The very same observation holds for any tensor that possesses symmetries in its arguments. For example, the subspace of totally symmetric tensors of type $\binom{p}{0}$ has dimesion $\binom{n+p-1}{p}$, for any p and n in $\mathbb{N}$ [4].

Differential forms in the kernel of d are said to be *closed forms* and a p-form that can be written as the exterior derivative of a $p-1$-form is called *exact*. As $d^2 = 0$, every exact form is also closed [3].
Consider any two smooth vector fields, v and w. The *Lie bracket*, $[v, w]$, is another smooth vector field on M, defined as the commutator of v and w, i.e. for a function f on M we have

$$[v, w]f = v(wf) - w(vf).$$

The Lie bracket is bilinear, antisymmetric and satisfies the Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

for any three vector fields x, y, z on M.

2.2 Connections on Vector Bundles

The definition of vector bundle makes precise the idea "attaching vectors [3] to the points" of a manifold. To compare and manipulate objects defined on different fibres, we need, however, to provide a method to set up such operations in a consistent fashion (fibres are otherwise linear spaces with no natural relation between them).
There are a number of different angles from which to approach the notion of connection, a device to "connect" or identify fibres over infinitesimally nearby points, each definition being useful for a certain framework of questions and applications. In analogy with the notion of directional derivative, we will define a connection as a derivation on sections of a vector bundle. Our discussion here will however be limited to using connections (covariant derivatives) to define a *parallel transport* map. Whereas a connection is a means to compare sections at nearby points [5], the parallel transport map allows to (parallely) move a vector along a curve to an arbitrary point on a manifold.
To generalize the notion of directional differentiation for sections of a vector bundle $E \to M$ along a vector field X a map

$$\nabla : C^\infty(TM) \times C^\infty(E) \to C^\infty(E)$$

which we write as $(X, \sigma) \mapsto \nabla_X \sigma$, must satisfy the following properties:

a) It must be $C^\infty(M)$-linear in X (we expect that after rescaling the "direction" vector, the derivative along X should only rescale by the same factor):

$$\forall\, f, g \in C^\infty(M), \quad \nabla_{(fX_1+gX_2)}\sigma = f\nabla_{X_1}\sigma + g\nabla_{X_2}\sigma.$$

[3] In this context we will use the word "vector" to mean an element of a fibre (vector space) of a vector bundle. The term "vector field" will be reserved to denote a section of the tangent bundle, while we will simply use "section" when referring to an arbitrary vector bundle.

b) It must be $\mathbb{R}$- (or $\mathbb{C}$-) linear in σ

$$\forall\, a, b \in \mathbb{R}\ (or\ \mathbb{C}), \quad \nabla_X(a\sigma_1 + b\sigma_2) = a\nabla_X\sigma_1 + b\nabla_X\sigma_2.$$

c) It must satisfy a Leibniz-like rule

$$\nabla_X f\sigma = (X \cdot f)\sigma + f\nabla_X\sigma \quad \forall\, f \in C^\infty(M),\ \sigma \in C^\infty(E).$$

Since ∇ is to be a derivation, it has to satisfy some kind of product rule. The only product defined in an abstract vector bundle is the multiplication of a section with a smooth function. Here we have assumed that the ∇ behaves like an ordinary directional derivative when acting on a smooth function, thus $X \cdot f = df(X)$ denotes the action of X on f (which in local coordinates equals $X^a\partial_a f$, the *Lie derivative* of f by X).

Condition **a)** on ∇ defines a map

$$\nabla\sigma : C^\infty(TM) \to C^\infty(E), \quad X \mapsto \nabla_X\sigma$$

for any section $\sigma \in C^\infty(E)$. This is a $C^\infty(M)$-linear bundle map between the tangent bundle and an arbitrary vector bundle over the same base manifold M. By naming $\mathrm{Hom}(TM, E)$ the space of $C^\infty(M)$-linear bundle maps from TM to E we have the natural isomorphim (see [2, 6] for details)

$$\nabla\sigma \in \mathrm{Hom}(TM, E) \cong C^\infty(T^*M \otimes E).$$

Summarizing, we can recast the observations above as the following definition:

Definition 2.2.1. *Let $E \to M$ a smooth vector bundle over a smooth manifold M. A (linear) connection or covariant derivative ∇ is a linear map $\nabla : C^\infty(E) \to C^\infty(E \otimes T^*M)$ satisfying the Leibniz rule*

$$\nabla f\sigma = f\nabla\sigma + \sigma \otimes df, \tag{2.1}$$

whenever $\sigma \in C^\infty(E)$ and f is a smooth function on M.

We notice that connections do exist on any vector bundle and if $m \in M$, then the value of the section $\nabla_X\sigma$ at m depends only on the value of X and σ at m [5, 7–9].
Now let $x \in M$ and U be an open neighbourhood of x such that a chart for M and a local trivialization of E are both defined on U. Assuming that M has dimension l, we obtain the local coordinate vector fields $\partial_1, \ldots, \partial_l$. The subset $E\,|_U = \pi^{-1}(U)$ of E is again a vector bundle [3] with the restriction of π as its projection map. For fibres of dimension n, a basis of $\mathbb{R}^n$ yields a basis $\mu_1, \ldots, \mu_n$ of (smooth) sections of $E\,|_U$ (a *local frame*) through the identification

$$E\,|_U \cong U \times \mathbb{R}^n.$$

For a connection ∇ on E, we define ln^2 smooth functions (on $E\,|_U$) by

$$\nabla_{\partial_i}\mu_j =: \Gamma^k_{ij}\mu_k.$$

The "section component" functions Γ^k_{ij} are called *Christoffel symbols*.
More generally, we can write a similar expression for the covariant derivative of a section of $E_{|U}$, $\sigma = \sigma^k\mu_k$ along a generic vector $V = V^i\partial_i$

$$\begin{aligned}\nabla_V\sigma &= \nabla_{V^i\partial_i}\sigma^k\mu_k \\ &= V^i(\partial_i\sigma^k + \Gamma^k_{ij}\sigma^j)\mu_k.\end{aligned} \tag{2.2}$$

Consider now two manifolds B and M, a vector bundle E over B and a smooth map $f : M \to B$. Recall that we can construct a bundle over M by "pulling back" the fibres of E [9, 10]. The *pullback bundle* f^*E is defined as

$$f^*E = \{(x, e) \in M \times E \mid f(x) = \pi(e)\} \subset M \times E.$$

The projection onto the second factor, $F(x, e) = e$, gives the commutative diagram below.

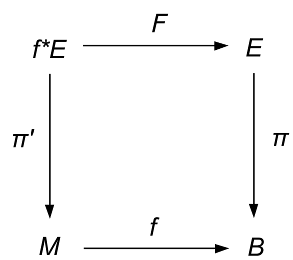

Figure 2.4: Pullback of a vector bundle.

The restriction of F to each fibre $(f^*E)_p$ over $p \in M$ is, then, an isomorphism onto a fibre $E_{f(p)}$ and the projection map π' is given by the projection on the first factor, i.e. $\pi'(x, e) = x$. We say that F is a *bundle morphism* covering f.
In this scheme, composition defines a pulback operation on the sections of E: if σ is a section of E over B, then the *pullback section* $f^*\sigma = \sigma \circ f$ is a section of f^*E over M.
Moreover, given a local trivialization of E, (U, ψ) there is a corresponding local trivialization of f^*E constructed as $(f^{-1}U, \phi)$, with $\phi(x, e) = (x, \mathrm{proj}_2(\psi(e)))$ [9] (the symbol proj_2 denotes projection on the second coordinate of $U \times \mathbb{R}^n$).
Connections on vector bundles naturally induce connections on derived bundles, such as direct sum, multilinear, dual or exterior product bundles. For instance, a connection ∇ on TM extends naturally to connections all the bundles of tensors $\bigotimes^k T^*M \otimes \bigotimes^l TM$ for $k, l \in \mathbb{N}$. All of these induced connections on tensor bundles are conventionally also written as ∇. Through ∇ we can thus differentiate any tensor field on M.
Pullback bundles are no exception and we can define a *pullback connection*, i.e. a map

$$f^*\nabla : C^\infty(TM) \times C^\infty(f^*E) \to C^\infty(f^*E).$$

Let $E \to M$ be a vector bundle with fibres of dimension m. It is necessary to first remark that a frame $(\mu_1, \dots, \mu_n)$ on E (the same reasoning holds for local frames covering M) induces a set of pullback sections $(f^*\mu_1, \dots, f^*\mu_n)$ on the fibres of f^*E.
Since, the fibres of E and f^*E are identified pointwise by a bundle morphism, this "pulled back frame" $(f^*\mu_1, \dots, f^*\mu_n)$ is actually a local frame on f^*E. As any section of f^*E can be written as a linear combination of such pullback sections, by linearity of the connection it suffices to define $f^*\nabla$ on sections of the form $f^*\sigma$ [11].
Let ∇ be a connection on E then any tangent vector $X \in TM$ and any section σ of E we can define a connection on f^*E as

$$(f^*\nabla)_X f^*\sigma := f^*(\nabla_{f_* X}\sigma),$$

with f_*X being the pushforward of X through f (a more detailed discussion on pullback connections can be found in [10, 12]).
We can apply the above construction to pull back a vector bundle through a curve on a manifold. Let M be a manifold, $E \to M$ a vector bundle over M, and ∇ a connection on E. Let $\tilde{\gamma} : [0, 1] \to M$ be a smooth curve on M parametrized by the unit interval. Smoothness of $\tilde{\gamma}$ means by definition that there exists a smooth curve γ defined on some open interval containing $[0, 1]$ that extends $\tilde{\gamma}$. We can thus work with γ and restrict back to the original interval when we need to, obtaining in this way a smooth map between manifolds [4].
The pullback bundle $\gamma^*(E)$ over $[0, 1]$ is then a vector bundle with fibre $E_{\gamma(t)}$ over $t \in [0, 1]$, where E_x is the fibre of E over $x \in M$ (the fibres of E and γ^*E are identified pointwise). Let σ be a smooth section of $\gamma^*(E)$ over $[0, 1]$, so that $\sigma(t) \in E_{\gamma(t)}$ for each $t \in [0, 1]$, we say that the section $\sigma(t)$ is a *section along the curve* γ.
The connection ∇ is pulled back under γ to give a connection on $\gamma^*(E)$ over $[0, 1]$.

Definition 2.2.2. *Let $E \to M$ be a vector bundle, $\gamma : I \to M$ a smooth curve; we say that the section σ of $\gamma^*(E)$ is parallel along γ if its covariant derivative under the pullback connection vanishes,*

$$\gamma^*(\nabla_{\gamma_*\partial_t}\sigma(t)) = \nabla_{\dot{\gamma}(t)}\sigma(t) = 0 \qquad \forall\, t \in [0, 1], \tag{2.3}$$

where $\dot{\gamma}(t)$ is $\frac{d}{dt}\gamma(t)$ regarded as a vector of $T_{\gamma(t)}M$.

Again the (pointwise) identification of the fibres allows us to write the element $\gamma^*(\nabla_{\gamma_*\partial_t}\sigma(t))$ in the fibre E_t of $\gamma^*(E)$ as $\nabla_{\dot{\gamma}(t)}\sigma(t)$ in the fibre $E_{\gamma(t)}$ of E, in equation (2.3).
In local coordinates, we can write (2.3) in terms of Christoffel symbols through (2.2) to give

$$\partial_t\sigma^j(t)\mu_j(t) + \dot{\gamma}(t)\sigma^j(t)\Gamma^j_{ik}(t)\mu_j(t) = 0,$$

thus it defines a system of first order ordinary equations for the components of $\sigma(t)$

[4] This definition of smoothness is equivalent to the component functions $\tilde{\gamma}^i$ having one-sided derivatives of all orders at the endpoints (in any local coordinate systems). The values on $[0, 1]$ of any continuous function of the derivatives of $\tilde{\gamma}^i$ are independent of the extension we choose for $\tilde{\gamma}$ [8].

$$\partial_t \sigma^j(t) + \dot{\gamma}(t)\sigma^j(t)\Gamma^j_{ik}(t) = 0.$$

Therefore, for each possible initial value $e \in E_{\gamma(0)}$, by the Cauchy-Lipschitz theorem [13] there exists a unique, smooth solution σ on all of $[0,1]$ with $\sigma(0) = e$. We shall use this fact to define parallel transport.

Definition 2.2.3. *Let M be a manifold, E a vector bundle over M, and ∇ a connection on E. Suppose that $\gamma : [0,1] \to M$ is smooth, with $\gamma(0) = x$ and $\gamma(1) = y$, $x, y \in M$. Then for each $e \in E_x$, there exists a unique smooth section σ of $\gamma^*(E)$ satisfying $\nabla_{\dot{\gamma}(t)}\sigma(t) = 0$ for $t \in [0,1]$ and $\sigma(0) = e$. Define $P_\gamma(e) = \sigma(1)$. Then $P_\gamma : E_x \to E_y$ is a well-defined linear map, called the parallel transport map.*

This definition easily generalizes to the case when γ is continuous and piecewise smooth, by requiring σ to be continuous and differentiable whenever γ is differentiable.
As we have seen above, if x_0 and x_1 are point in M, the fibres E_{x_0} and E_{x_1} (over x_0 and x_1 respectively) can be identified by choosing a curve γ with $\gamma(0) = 1$ and $\gamma(1) = x_1$ and parallely transporting each section $\sigma_0 \in E_{x_0}$ to E_{x_1} along γ. This identification depends only on the choice of γ. Moreover, from parallel transport we can recover the notion of covariant derivative [8, 14]. In fact, for a parallel transport map P_γ, the following relation holds for any $\sigma \in C^\infty(E)$

$$\nabla_{\dot{\gamma}(0)}\sigma := \lim_{t\to 0} \frac{P_\gamma \sigma(t) - \sigma(0)}{t}.$$

A connection is therefore an infinitesimal version of parallel transport, in this sense it "connects" fibres at nearby points.

2.3 Curvature and Torsion

Associated to a connection there are two important tensor fields, that play a very important role in Geometry and in many of its applications to Theoretical Physics, namely *curvature* and *torsion*.
The approach we take to define curvature makes use of vector fields and the Lie bracket of vector fields.
Let ∇ be a connection on a vector bundle $E \to M$, then there exists a unique 2-form $R(\nabla)$ "with values in the end bundle $\mathrm{End}(E) = E \otimes E^*$", that is $R(\nabla) \in \mathrm{End}(E) \otimes \Lambda^2 T^*M$, such that it defines a map $R(\nabla) : C^\infty(TM) \times C^\infty(TM) \times C^\infty(E) \to C^\infty(E)$ given by:

$$R(\nabla)(X,Y)\sigma = \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma - \nabla_{[X,Y]}\sigma. \tag{2.4}$$

By definition $R(\nabla)$ is multilinear in X, Y and σ and antisymmetric in X and Y, i.e. $R(\nabla)(X,Y)\sigma = -R(\nabla)(X,Y)\sigma$.
One way to understand the curvature of a connection is the following. Define $V_i = \partial_i$ for $i = 1, \ldots n$ (provided that n is the dimension of the fibres of E). Then V_i is a vector field over some open subset U of M and $[V_i, V_j] = 0$. For $\sigma \in C^\infty(E)$ we may interpret $\nabla_{V_i}\sigma$ as a kind

of "partial derivative" $\frac{\partial \sigma}{\partial x^i}$ of the section σ (see [6], for a detailed discussion on how to make the identification partial / covariant derivative precise.
Then equation (2.4) implies that

$$R(\nabla)(V_i, V_j)\sigma = \frac{\partial^2 \sigma}{\partial x^i \partial x^j} - \frac{\partial^2 \sigma}{\partial x^j \partial x^i}.$$

Thus the curvature of ∇ measures how much these "partial derivatives" of σ fail to commute. Now, partial derivatives of functions do commute, $\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i} = 0$, if $f \in C^\infty(M)$. However this does not hold in general for arbitrary sections of E. Behind this "noncummatitivity" lurks the fact that the "background space" is *curved*. In chapter 3 we will see how curvature captures the dependence of parallel transport on the chosen path.
Now let ∇ be a connection on the tangent bundle TM of M, rather than on a general vector bundle. There is a unique map

$$T(\nabla) : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM),$$

associated to ∇ called the *torsion* of the connection.
The torsion is defined by

$$T(\nabla)(V, W) = \nabla_V W - \nabla_W V - [V, W]. \tag{2.5}$$

From this definition we recover that the torsion is tensorial and anti-symmetric in V and W. A connection is called *torsionless* if the corresponding torsion vanishes. If the curvature vanishes, then the connection is called *flat*. A connection that is both torsionless and flat is *locally Euclidean*, meaning that there exist local coordinates for which all of the Christoffel symbols vanish [5, 6, 14].
For a torsion-free connection ∇, the curvature $R(\nabla)$ and its derivative $\nabla R(\nabla)$ have certain extra symmetries, known as the *Bianchi identities* [1].

Proposition 2.3.1. *Let M be a manifold and ∇ a torsion-free connection on its tangent bundle TM. Then the curvature $R(\nabla)$ of ∇ satisfies the following tensor equations, known as the first and second Bianchi identities respectively*

$$R(\nabla)(X, Y)\sigma + R(\nabla)(\sigma, X)Y + R(\nabla)(Y, \sigma)X = 0,$$

for $\sigma \in C^\infty(TM)$ and

$$\nabla_X R(\nabla)(Y, Z) + \nabla_Z R(\nabla)(X, Y) + \nabla_Y R(\nabla)(Z, X) = 0.$$

3

Parallel Transport and Curvature

We now set up some of the theory that will allow us to define the holonomy group of a connection and eventually will come into play in the Ambrose-Singer holonomy theorem [15]. This collection of basic results are however interesting on their own as they clarify the relation between curvature and parallel transport.

3.1 Parallel Frames

We begin by taking a closer look at some properties of the parallel transport map that lead to the definition of *parallel frames along a curve*.
Before we continue, however, it is important to remark that the definitions and results presented below for smooth curves and smooth sections along smooth curves extend naturally to piecewise smooth curves and sections. In fact, suppose that $\gamma : I \to M$ is piecewise smooth curve and σ is a piecewise smooth section along γ. Then, all we need to do is choose and work with a subdivision $\cdots < t_{i-1} < t_i < t_{i+1} < \dots$ of I such that both γ and σ are smooth on the subintervals $[t_{i-1}, t_i]$, so that the definition of parallel transport applies as we defined it for piecewise smooth curves in section 2.2.
Moreover, the composition and inverse of piecewise smooth paths are piecewise smooth paths. Let x, y and $z \in M$ and let $\alpha : [0,1] \to M$ and $\beta : [0,1] \to M$ be piecewise smooth paths in M with $\alpha(0) = x$, $\alpha(1) = y = \beta(0)$ and $\beta(1) = z$. We define the paths α^{-1} and $\beta\alpha$ by

$$\alpha^{-1} = \alpha(1-t), \qquad \beta\alpha(t) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} .$$

Then α^{-1} and $\beta\alpha(t)$ are piecewise smooth in M with $\alpha^{-1}(0) = y$, $\alpha^{-1}(1) = x$, $\alpha\beta(0) = x$ and $\alpha\beta(1) = z$. This is one of the reasons to consider piecewise smooth curves, which along with the following results, encapsulates the defining properties of holonomy groups (as we will see in the next chapter).

Lemma 3.1.1. *Given a piecewise smooth path α and its inverse α^{-1}, defined as above, then P_α and $P_{\alpha^{-1}}$ are inverse maps.*

Proof. Suppose we have $e_x \in E_x$ and $P_\alpha(e_x) = e_y \in E_y$. Then there is a unique parallel section σ of $E_{\alpha(t)}$ with $\sigma(0) = e_x$ and $\sigma(1) = e_y$. Now if we define $\tilde{\sigma}(t) = \sigma(1-t)$, then $\tilde{\sigma}$ is a parallel section of $(\alpha^{-1})^*(E)$. We have that $\tilde{\sigma}(0) = e_y$ and $\tilde{\sigma}(1) = e_x$, which implies $P_{\alpha^{-1}}(e_y) = e_x$. Thus, if $P_\alpha(e_x) = e_y$, then $P_{\alpha^{-1}}(e_y) = e_x$ and so $P_{\alpha^{-1}}$ is the inverse map of P_α [1]. ■

In particular, if γ is any piecewise smooth curve in M, then P_γ is invertible and

$$(P_\gamma)^{-1} = P_{\gamma^{-1}}. \tag{3.1}$$

Lemma 3.1.2. *Given two piecewiese smooth paths $\alpha : [0,1] \to M$ and $\beta : [0,1] \to M$ be piecewise smooth paths in M with $\alpha(0) = x$, $\alpha(1) = y = \beta(0)$ and $\beta(1) = z$ and their composition $\beta\alpha$ defined as above, then*

$$P_\beta \circ P_\alpha = P_{\beta\alpha}. \tag{3.2}$$

Proof. The proof of this result parallels that of lemma 3.1.1. If we have $e_x \in E_x$, $P_\alpha(e_x) = e_y \in E_y$ and $P_\beta(e_y) = e_z \in E_z$. Then, there exists a unique parallel section γ_α of $E_{\alpha(t)}$ with $\sigma_\alpha(0) = e_x$ and $\sigma_\alpha(1) = e_y$. Likewise, Then, there exists a unique parallel section σ_β of $E_{\beta(t)}$ with $\sigma_\beta(0) = e_y$ and $\sigma_\beta(1) = e_z$.
If we define

$$\tilde{\sigma}(t) = \begin{cases} \sigma_\alpha(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \sigma_\beta(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}, \tag{3.3}$$

then $\tilde{\gamma}$ is a parallel section of $(\beta\alpha)^*(E)$.
So we have that $\tilde{\gamma}(0) = e_x$ and $\tilde{\gamma}(1) = e_z$, which implies $P_{\beta\alpha}(e_x) = e_z$. Thus, if $P_\beta \circ P_\alpha(e_x) = P_\beta(P_\alpha(e_x)) = e_z$, then $P_{\beta\alpha}(e_x) = e_z$ and so

$$P_\beta \circ P_\alpha = P_{\beta\alpha}.$$

■

In what follows we let M be a smooth manifold, $E \to M$ a (smooth) vector bundle over M and $\gamma : I \to M$ a smooth curve. This notation will be used repeatedly.

Lemma 3.1.3. *Let $t_0 \in I$ and $\sigma_1, \dots, \sigma_k$ be parallel sections along γ. Suppose that $\sigma_1(t_0), \dots, \sigma_k(t_0)$ form a basis of $E_{\gamma(t_0)}$, then $\sigma_1(t), \dots, \sigma_k(t)$ is a basis for $E_{\gamma(t)}$ for all $t \in I$.*

Proof. Let P_γ be the parallel transport map along γ from $p = \gamma(t_0)$ to $q = \gamma(t)$. By definition we have $\sigma_i(t) = P_\gamma \sigma_i(t_0)$. Since P_γ is a linear bijective map (lemma 3.1.1 shows the existence of the inverse map), it maps basis elements to basis elements. So for all $t \in I$, $\sigma_i(t)$ $i = 1, \dots, k$ is a basis for $E_{\gamma(t)}$. ■

Definition 3.1.4. *Let $\Phi = (\sigma_1, \dots, \sigma_n)$ be a n-tuple of parallel section along a curve γ. We say that Φ is a frame of the vector bundle E along γ if $\sigma_1(t), \dots, \sigma_n(t)$ is a basis of the fibre $E_{\gamma(t)}$ for all $t \in I$.*

We call $C^\infty_\gamma(E)$ the set of smooth sections along the curve γ (i.e. those sections $\sigma(t)$ in $E_{\gamma(t)}$ for all the $t \in I$).
Now let $\Phi = (\sigma_1, \dots, \sigma_k)$ be a parallel frame along γ and let σ be a smooth section in $C^\infty_\gamma(E)$. Then there is a smooth map $p : I \to \mathbb{R}^k$ (or $\mathbb{C}^k$), such that

$$\sigma = \sum_{i=1}^{k} p^i \sigma_i.$$

The map p is called the *principal part of σ with respect to* Φ.
Since the sections σ_i are parallel along γ, by the Leibniz rule for the connection, we have

$$\nabla_{\partial_t} \sigma = \sum_{i=1}^{k} (\partial_t p^i) \sigma_i. \tag{3.4}$$

By selecting a parallel frame we can thus reduce covariant derivatives of a section to partial derivatives of its principal part. We can restate this result as the following lemma:

Lemma 3.1.5. *Let σ in $C^\infty_\gamma(E)$, $t_0 \in I$ and P_γ be the parallel transport map along γ from $\gamma(t)$ to $q = \gamma(t_0)$. Then*

$$P_\gamma \nabla_{\partial_t} \sigma = \partial_t P_\gamma \sigma. \tag{3.5}$$

Proof. Choose a parallel frame $\Phi = (\sigma_1, \dots, \sigma_k)$ of E along γ and let p be the principal part of the section σ with respect to Φ. By the definition of parallel frame we have

$$P_\gamma \sigma = \sum_{i=1}^{k} p^i(t_0) \sigma_i(t_0)$$

and

$$P_\gamma \nabla_{\partial_t} \sigma = \sum_{i=1}^{k} (\partial_t p^i)(t_0) \sigma_i(t_0).$$

In analogy with (3.4) we may write $\nabla_{\partial_t} P_\gamma \sigma := \partial_t P_\gamma \sigma$, hence we can define the action of ∂_t on a parallely transported section $P_\gamma \sigma$ as the sum of the partial derivative of the principal part multiplied by the the basis sections at t_0, i.e.

$$\partial_t P_\gamma \sigma := \sum_{i=1}^{k} (\partial_t p^i)(t_0) \sigma(t_0).$$

It follows that

$$P_\gamma \nabla_{\partial_t} \sigma = \partial_t P_\gamma \sigma.$$

■

3.2 Parallel Transport and Curvature

We start with generalizing the concept of parallel transport along a curve to parallel transport along a family of piecewise smooth curves, namely along a *piecewise smooth homotopy.*

Definition 3.2.1. *Let let I be an interval and M be a smooth manifold. Define continuous map*

$$H : I \times [a, b] \to M, \quad H(s, t) = h_s(t)$$

where $h_s : [a, b] \to M$ is a family of piecewise smooth maps, and $H(s, t)$ is smooth in s. The map H is called a piecewise smooth homotopy on M.

We say that a piecewise smooth homotopy is proper when $H(s, a) = p$ and $H(s, b) = q$ for some points $p, q \in M$ and all $s \in I$ (all the curves in the family begin and end at the same points).
The map H traces a "net" of curves on M. For each s in I we have a piecewise smooth curve h_s from $h_s(a)$ to $h_s(b)$. In addition, we can smoothly "move" from one curve h_s to another by fixing t in $[a, b]$ and letting s run through I. Thus, for a fixed t, H selects a smooth curve $I \to M$ on the net. Any two points in the image of H can therefore be joined by a (on the whole) piecewise smooth curve. In this sense we can apply the pullback bundle construction to define a section "along the homotopy" and set up a parallel transport map. We denote a section along H by $\sigma(s, t)$ (the "coordinates" (s, t) tell us on which curve h_s and at what point of the curve the section is). We will only write σ when it is clear from the context that we mean one of such sections.
We will focus exclusively on parallel transport along the curves h_s and use $P_{s,t}$ to denote parallel transport through the map P_{h_s} from the point $h_s(t)$ to $h_s(b)$ (in a similar fashion we write $P^{-1}_{s,b}$ for the inverse map from $h_s(b)$ to $h_s(t)$).
Consider a vector bundle $E \to M$ endowed with a connection ∇ with curvature R. Through piecewise smooth homotopies we can gain some important insight into the relation between parallel transport and curvature. In order to investigate this relation we construct a map $R_{s,t}$: $C^\infty(E) \to C^\infty(E)$, composing parallel transport with the curvature map (defined by (2.4)). First we transport the section γ from $h_s(b)$ to $h_s(t)$ and then we apply the curvature map on γ and the "tangent vectors" $\partial_t H(s, t)$ and $\partial_s H(s, t)$. Finally we transport $R(\partial_t H(s, t), \partial_s H(s, t))\sigma$ back to $h_s(b)$. In symbols we have

$$R_{s,t} = P_{s,t} \circ R(\partial_t H(s, t), \partial_s H(s, t)) \circ P^{-1}_{s,t}. \tag{3.6}$$

Lemma 3.2.2. *Let σ be a piecewise smooth section along a piecewise smooth homotopy H (defined as above), such that $\nabla_{\partial_t}\sigma(s, t) = 0$ and $\nabla_{\partial_s}\sigma(s, a) = 0$ for all $s \in I$.*
Then

$$\nabla_{\partial_s}\sigma(s, b) = \left(\int_a^b R_{s,t} dt \right) \sigma(s, b). \tag{3.7}$$

To prove the lemma 3.2.2 we will need the following result:

Lemma 3.2.3. *Let $E \to M$ be a smooth vector bundle, $U \subset \mathbb{R}^2$ be open and $f = f(x,y)$ be a smooth map from U to M. If σ is a smooth section of E on $F(U)$* [1]*, then*

$$\nabla_{\partial_x}\nabla_{\partial_y}\sigma - \nabla_{\partial_x}\nabla_{\partial_y}\sigma = R(\partial_x f, \partial_y f)\sigma. \tag{3.8}$$

Proof of lemma 3.2.3. Using the definition of curvature given in (2.4) we have

$$R(\partial_x f, \partial_y f)\sigma = \nabla_{\partial_x}\nabla_{\partial_y}\sigma - \nabla_{\partial_y}\nabla_{\partial_x}\sigma - \nabla_{[\partial_x,\partial_y]}\sigma,$$

but the Lie bracket of the coordinate vector fields ∂_x and ∂_y in $\mathbb{R}^2$ vanishes, making the term $\nabla_{[\partial_x,\partial_y]}\sigma$ disappear. We are thus left with equation (3.8).

■

Equation (3.8) actually follows from the more general property

$$R(f_*X, f_*Y)\sigma = \nabla_X\nabla_Y\sigma - \nabla_Y\nabla_X\sigma - \nabla_{[X,Y]}\sigma,$$

for a smooth map $f : N \to M$ between smooth manifolds and $X, Y \in C^\infty(TN)$. We refer to [10] for a proof of this statement.

Proof of lemma 3.2.2. By (3.8) and the assumptions we made on σ we have

$$\nabla_{\partial_s}\nabla_{\partial_t}\sigma = \nabla_{\partial_t}\nabla_{\partial_s}\sigma + R(\partial_t H, \partial_s H)\sigma = R(\partial_t H, \partial_s H)\sigma.$$

By lemma 3.1.5 we can write

$$\begin{aligned}
\partial_t P_{s,t}\nabla_{\partial_s}\sigma(s,t) &= P_{s,t}(\nabla_{\partial_t}\nabla_{\partial_s}\sigma(s,t)) \\
&= P_{s,t}(R(\partial_t H(s,t), \partial_s H(s,t))\sigma(s,t)) \\
&= P_{s,t}(R(\partial_t H(s,t), \partial_s H(s,t))P_{s,b}^{-1}(\sigma(s,b))) \\
&= R_{s,t}\sigma(s,b).
\end{aligned} \tag{3.9}$$

Before continuing we observe that if we define $\mathrm{End}(E_{h_s(b)})$ to be the space of all linear maps from $E_{h_s(b)}$ to itself, then $R_{s,t} \in \mathrm{End}(E_{h_s(b)})$ ($R_{s,t}$ is a composition of linear maps that begins and ends at $E_{h_s(b)}$). For each $s \in I$, $R_{s,t}$ traces out a curve in $\mathrm{End}(E_{h_s(b)})$. Now, under scalar multiplication and composition of maps, $\mathrm{End}(E_{h_s(b)})$ is given the structure of a vector space, therefore we can integrate along the curves given by $R_{s,t}$.
For a fixed s, the integral $\int_a^b R_{s,t}dt$ is again a linear map from $E_{h_s(b)}$ to $E_{h_s(b)}$, therefore it belongs to $\mathrm{End}(E_{h_s(b)})$.
By definition of $P_{s,t}$, $P_{s,b} = Id_{E_{h_s(b)}}$. As we assumed that $\nabla_{\partial_s}\sigma(\cdot, a) = 0$ we can write

$$\nabla_{\partial_s}\sigma(s,b) = P_{s,b}\nabla_{\partial_s}\sigma(s,b) - P_{s,a}\nabla_{\partial_s}\sigma(s,a). \tag{3.10}$$

[1] Being an open subset of $\mathbb{R}^2$, U is itself a manifold. Therefore the pullback bundle construction we presented in section 2.2 applies to this case too and our definition of section "along a curve" extends to the map f.

By using (3.9), we get

$$\begin{aligned}P_{s,b}(\nabla_{\partial_s}\sigma(s,b)) - P_{s,a}(\nabla_{\partial_s}\sigma(s,a)) &= \int_a^b \partial_t P_{s,t}(\nabla_{\partial_s}\sigma(s,t))dt \\ &= \int_a^b R_{s,t}\sigma(s,b)dt \\ &=: \left(\int_a^b R_{s,t}dt\right)\sigma(s,b).\end{aligned}$$

The last step defines the action of $\int_a^b R_{s,t}dt$ on the section $\sigma(s,b) \in E_{h_s(b)}$. ■

An interesting application of lemma 3.2.2 occurs in the case of H being a proper piecewise smooth homotopy. Note that now we are dealing with a "net" of curves on M and we can parallel transport and take covariant derivative in both the t and s direction. The principal part of a section also depends on both t and s in this case. So we have an analogous version of the result of lemma 3.2.2 for the action of ∂_s:

$$P_{s,t}\nabla_{\partial_s}\sigma = \partial_s P_{s,t}\sigma. \tag{3.11}$$

Under the assumptions of lemma 3.2.2, $\sigma(\cdot, a)$ is constant and (3.10) can be restated as

$$\nabla_{\partial_s}\sigma(s,b) = P_{s,b}\nabla_{\partial_s}\sigma(s,b) = \partial_s P_{s,b}\sigma(s,b) = \partial_s\sigma(s,b).$$

Then (3.7) yields

$$\partial_s\sigma(s,b) = \left(\int_a^b R_{s,t}dt\right)\sigma(s,b), \tag{3.12}$$

which is a first order linear ordinary differential equation for $\sigma(s,b)$.
If $P_{s,a}$ denotes parallel translation from $h_s(a)$ to $h_s(b)$ (in the case of a proper piecewise smooth homotopy all the curves h_s originate at $h_s(a)$ and end at $h_s(b)$), then by our assumptions on σ, $\sigma(s,b) = P_{s,a}(\sigma(s,a))$, which allows for (3.12) to be restated as

$$\partial_s P_{s,a}(\sigma(s,a)) = \left(\int_a^b R_{s,t}dt\right)(P_{s,a}(\sigma(s,a))). \tag{3.13}$$

Therefore (since $\sigma(\cdot, a)$ is constant) we have

$$\partial_s P_{s,a} = \left(\int_a^b R_{s,t}dt\right)P_{s,a}. \tag{3.14}$$

In this sense, curvature tells us about the dependence of parallel transport on the chosen path. An intuitive example is provided by parallel transporting a tangent vector along the geodesics (great circles) of a sphere.

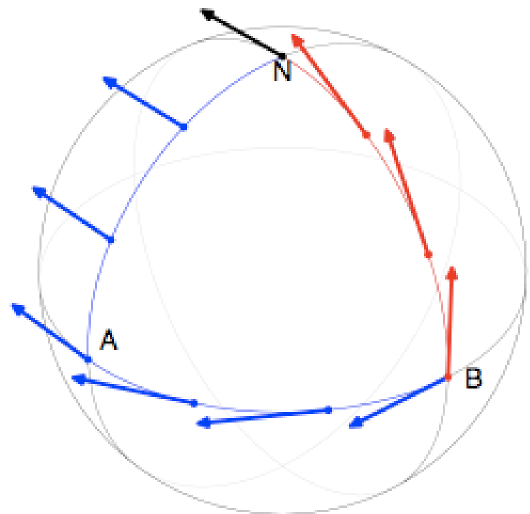

Figure 3.1: Parallel transport on the sphere.

Start with a vector at N and first transport it to B along the meridian connecting N to B and then along the path N-A-B (see figure 3.2). The results of the two operations differ! The two vectors transported to B are twisted by an angle.
Now let $p \in M$, $A, B \in T_pM$ and $f : U \to M$ be a smooth map with

$$f(0) = p, \qquad \partial_x f|_0 = A, \qquad \partial_y f|_0 = B,$$

where $U \subset \mathbb{R}^2$ is an (open) neighbourhood of 0.
Consider the homotopy $H(s,t) : [0,1] \times [0,1] \to M$ of piecewise smooth curves $h_s(t)$ defined by

$$h_s(t) = \begin{cases} f(4st, 0) & \text{if } 0 \le t \le \frac{1}{4} \\ f(s, s(4t-1)) & \text{if } \frac{1}{4} \le t \le \frac{1}{2} \\ f(s(3-4t), s) & \text{if } \frac{1}{2} \le t \le \frac{3}{4} \\ f(0, 4s(1-t)) & \text{if } \frac{3}{4} \le t \le 1 \end{cases} \tag{3.15}$$

The domains of the maps $h_s(t)$ of the family defined through f are "square loops" of side s. For, fix s, $(4st, 0)$ spans the segment of x-axis from 0 to s as t changes from 0 to $\frac{1}{4}$. Likewise $(s, s(4t-1))$ is the vertical segment from $(s, 0)$ to (s, s) at t goes from $\frac{1}{4}$ to $\frac{1}{2}$ and $(s(3-4t), s)$ connects (s, s) to $(0, s)$ for t ranging over $[\frac{1}{2}, \frac{3}{4}]$. The loop is completed by $(0, 4s(1-t))$ with t spanning $[\frac{3}{4}, 1]$. Thus the parameter s scales the loops ranging from 0 to 1, see figure 3.2. Thus when $s \to 0$, the loop shrinks to one point, namely $p \in M$.
By making use of this special piecewise smooth homotopy the next theorem expresses the infinitesimal dependence of parallel translation on the curvature.

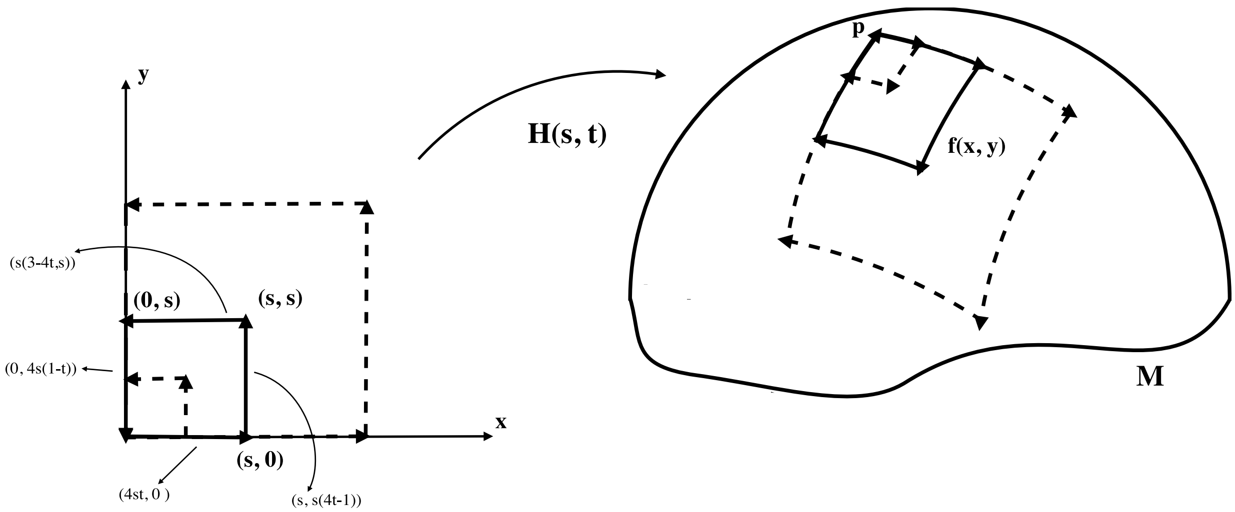

Figure 3.2: The "shrinking" homotopy.

Theorem 3.2.4. *Let M be a smooth manifold, $p \in M$, H the piecewise smooth homotopy defined above. Let P_s be the parallel transport map along the curve h_s defined from $h_s(0)$ to $h_s(1)$. Then*

$$\partial_s P_s|_{s=0} = 0$$

and

$$\partial_s \partial_s P_s|_{s=0} = 2R(A, B)|_{s=0}.$$

Proof. By the definition of H and the antisymmetry of the curvature tensor in its two "vector arguments" we obtain

$$R(\partial_t H, \partial_s H) = \begin{cases} 0 & \text{if } t \leq \frac{1}{4} \text{ or } t \geq 1 \\ 4sR(\partial_y f, \partial_x f) & \text{if } \frac{1}{4} \leq t \leq \frac{3}{4} \end{cases} \tag{3.16}$$

As usual, if we define $R_{s,t}$ as in (3.6) with $a = 0$ and $b = 1$, then by (3.12) we get

$$\partial_s P_s = \left(\int_{\frac{1}{4}}^{\frac{3}{4}} R_{s,t} dt \right) P_s. \tag{3.17}$$

It follows that, since $R_{s,t} = 0$ at 0, $\partial_s P_s|_{s=0}$ also vanishes.
Now for the second derivative we have

$$\partial_s(\partial_s P_s)|_0 = \lim_{s \to 0} \frac{\partial_s P_s(s)|_s - \partial_s P_s|_0}{s} = \lim_{s \to 0} \frac{\partial_s P_s|_s}{s} \tag{3.18}$$

From (3.16) we have

$$\frac{R_{s,t}}{4s} = P_s R(\partial_y f, \partial_x f) P_s^{-1}.$$

Now substituting (3.17) in the expression for the limit gives us

$$\partial_s(\partial_s P_s)|_0 = \lim_{s \to 0} \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} R_{s,t} dt|_s}{s}$$

Since parallel translation depends continuously on the path ((3.14)),

$$\frac{1}{4s} R_{s,t} = P_{s,t} R(\partial_y f, \partial_x f) P_{s,t}^{-1} \to R(A, B)$$

uniformly in t (t is in the interval $[0, 1]$) for $s \to 0$. We thus get

$$\partial_s(\partial_s P_s)|_0 = \int_{\frac{1}{4}}^{\frac{3}{4}} 4R(A, B) dt|_0 = \frac{1}{2} 4R(A, B)|_0 = 2R(A, B)|_0.$$

■

4

Holonomy Groups and the Ambrose-Singer Holonomy Theorem

Parallel transport along closed curves (loops) defines a group of endomorphisms on the fibers of a vector bundle. The holonomy group of a generic connection is therefore a subgroup of the general linear group of each fiber and, being independent of the base point of the loop, it defines a global invariant for the connection. In this chapter we introduce the notion of holonomy group and explain some of its basic properties.
As curvature provides a measure of the extent to which parallel transport along loops fails to preserve the geometrical data being transported, it is no surprise that holonomy groups turn out to be deeply related to curvature. In particular, it turns out that the Lie algebra of an holonomy group both constrains curvature and is determined by it.
Using the results we obtained in chapter 3 we investigate this relation as it is captured by the Ambrose-Singer theorem.
For sake of completeness, we have collected some useful definitions and properties about Lie groups that will frequently be used throughout the chapter.
We remind the reader that all the manifolds we will consider in this chapter are assumed to be connected.

4.1 Lie Groups and Lie Algebras: Some Elementary Facts

A *Lie group* G is a group endowed with the structure of a smooth manifold (or, more generally, of the disjoint union of finitely many smooth manifolds) for which the *multiplication* ($G \times G \to G$, $(g,h) \mapsto g\dot{h}$) and *inverse* ($G \to G$, $g \mapsto g^{-1}$) maps are smooth.

G1) A *Lie subgroup*, H of a Lie group G is a subgroup of G endowed with a smooth manifold structure, making it into a Lie group and an immersed submanifold of G through a smooth inclusion map.

G2) A closed subgroup of a Lie group is always a Lie subgroup.

G3) If G is a connected Lie group, then the component containing the identity is a Lie subgroup of G.

G4) Every arcwise connected [1] subgroup of a Lie group is a Lie subgroup [16].

Given two Lie groups G and H, the map $\Phi : G \to H$ is called a *Lie group homomorphism* if it is a continuous group homomorphism. If, in addition, Φ is bijective with continuous inverse, Φ is called a *Lie group isomorphism*.
A *representation* of a Lie group G on a vector space V over a field $\mathbb{K}$, $\dim(V) \geq 1$, is a Lie group homomorphism from G to the general linear group over V:

$$\Pi : G \to \mathrm{GL}(V).$$

The space V is called the *representation space* and it dimension is known as the *dimension of the representation*.
Let Π be a finite dimensional representation of a Lie group G on a space V. A subspace W of V is called *invariant* if $\Pi(g)w \in W$ for all $w \in W$ and $g \in G$. We say that an invariant subspace W is *nontrivial* if $W \neq \{0\}$ and $W \neq V$. A representation with no nontrivial invariant subspaces is called *irreducible*. If the Lie group homomorphism is one-to-one the representation is called *faithful*.
Given a Lie group there exists a special class of vector fields characterized by invariance under group action. Define the *left translation* $L_a : G \to G$, by $L_a g = ag$ with $a, g \in G$ (similarly we can define a *right translation*). The map L_a is a diffeomorphism and it induces a map $L_{a*} : T_g G \to T_{ag} G$ between the tangent spaces at g and $ag \in G$. A vector field X on G is said to be a *left-invariant* vector field if $L_{a*} X|_g = X|_{ag}$.
Fix a point $p \in G$, a vector $V \in T_p G$ defines a unique left invariant vector field X_V over G by $X_V|_g = L_{g*} V$, $g \in G$. In fact, we have $X_V|_{ag} = L_{ag*} V = (L_a L_g)_* V = L_{a*} L_{g*} V = L_{a*} X_V|_g$.
Conversely, a left-invariant vector field X defines a unique vector $V = X|_p \in T_p G$. We denote the set of left invariant vector fields by $\mathfrak{g}$. The map $T_p G \to \mathfrak{g}$, that associates X_V to V is an isomorphism, and therefore $\mathfrak{g}$ is a vector space isomorphic to $T_p G$. Since $\mathfrak{g}$ is a subset of $C^\infty(G)$, the Lie bracket defined in section 2.1 is also defined on $\mathfrak{g}$. In addition, $\mathfrak{g}$ is also closed under the Lie bracket: if we consider two points g and $ag = L_a g$ in G and apply L_{a*} to the Lie bracket of $X, Y \in \mathfrak{g}$, we have

$$L_{a*}[X, Y]|_g = [L_{a*} X|_g, L_{a*} Y|_g] = [X, Y]|_{ag}.$$

Thus, $[X, Y] \in \mathfrak{g}$.

[1] A note on connectedness.
We will use the term *connected* (or *path-connected*) for a topological space X to mean that any two of its points can be joined by a continuous path. We say that X is *arcwise connected* if any two distinct points can be joined by an *arc*, that is a path which is a homeomorphism (a continuous map with continuous inverse between $[0, 1]$ and its image in M). The set $X = \{a, b\}$ with the trivial topology is connected with paths $\gamma : [0, 1] \to X$, defined by $\gamma(t) = a$ for all $t \neq 1$ and $\gamma(1) = b$. It is not arwise connected since γ is not even an injective map.
Finally, being simply connected will mean that every loop based at a point can be continuosly/smoothly contracted to the constant loop.

We call $\mathfrak{g}$ (the set of the left-invariant vector fields endowed with the Lie bracket $[\ ,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$) the *Lie algebra* associated to the Lie group G.

A1) A vector subspace of a Lie algebra which is also closed under the Lie bracket is a Lie subalgebra.

A2) If H is a Lie subgroup of a Lie group G, then the Lie algebra $\mathfrak{h}$ of H is a Lie subalgebra of $\mathfrak{g}$, the Lie algebra of G.

A3) Let $\mathfrak{g}$ be the Lie algebra of a Lie group G, then each Lie subalgebra of $\mathfrak{g}$ is the Lie algebra of exactly one Lie subgroup of G.

A4) The Lie algebra of a connected Lie group is isomorphic to the Lie algebra of its connected component cointaining the identity.

A linear map $\phi : \mathfrak{g} \to \mathfrak{h}$, between the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ is called a *Lie algebra homomorphism* if it preserves the Lie brackets, i.e. $\phi([X, Y]) = [\phi(X), \phi(Y)]$. If, in addition, ϕ is also bijective we have a *Lie algebra isomorphism*.
Every Lie group homomorphism gives rise to a Lie algebra homomorphism [17]. To see this let us introduce the notions of *one-parameter subgroup* and *exponential map*. Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. A Lie group homomorphism $\phi : \mathbb{R} \to G$ is called a one-parameter subgroup of G. Let X be a vector of $\mathfrak{g}$, it can be shown that there exists a unique one-parameter subgroup such that $\lambda_X : \mathbb{R} \to G$ such that $\lambda'(0) = X$ [18].
Then, the exponential map in G, $\exp : \mathfrak{g} \to G$, is defined by

$$\exp : X \mapsto \lambda_X(1).$$

Any Lie group homomorphism between two Lie groups gives rise through the exponential map to a Lie algebra homomorphism between the correponding Lie algebras [17, 19]. Let $\Phi : G \to H$ be a Lie group homomorphism, then there exists a unique Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ such that

$$\Phi(\exp(X)) = \exp(\phi(X)).$$

As we did for Lie groups we can now define a *representation* of a Lie algebra $\mathfrak{g}$ on a vector space V over a field $\mathbb{K}$, $\dim(V) \geq 1$, as a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra associated to the general linear group over V:

$$\pi : \mathfrak{g} \to \mathfrak{gl}(V).$$

The terms *invariant*, *nontrivial* and *irreducible*, *faithful* are defined for representations of Lie algebras exactly in the same way in which we introduced them for Lie groups.
We conclude this section with an example of Lie group and corresponding Lie algebra representation. Consider a Lie group G with Lie algebra $\mathfrak{g}$. For each g in G we define a linear map *adjoint map* (also known as the *the adjoint action* of G on $\mathfrak{g}$) $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ to be $\mathrm{Ad}_g(a) = gag^{-1}$,

for all $a \in \mathfrak{g}$. The map Ad_g is invertible with inverse $\mathrm{Ad}_{g^{-1}}$. Since each element of G defines such a map we have the map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$.
It can be shown that the map Ad is a group homomorphism [17]. Moreover, being a group homomorphism into the general linear group of $\mathfrak{g}$, *Ad* is a representation of G, which is referred to as the *adjoint representation*.
Similarly for any X in the Lie algebra $\mathfrak{g}$ there exists a linear, generated by the the adjoint map of G, $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$. This map has the property that $\mathrm{ad}_X(Y) = [X, Y]$ with $Y \in \mathfrak{g}$ [17, 19]. The maps ad_X give rise to the Lie algebra homomorphism $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by $\mathrm{ad}(X) = \mathrm{ad}_X$. The map ad is therefore a representation of $\mathfrak{g}$, named the adjoint representation.
We refer to [19] for an extensive treatment on Lie groups.

4.2 Loops and Holonomy Groups

After our brief aside on Lie groups, we now define a special class of piecewise smooth paths. Notice that all the manifolds we will consider from now on are assumed to be connected (even if we will not state this property explicitly).

Definition 4.2.1. *Let $E \to M$ be a vector bundle and ∇ a connection on E. Fix a point $p \in M$. A piecewise smooth path $\gamma : [0, 1] \to M$ is said to be a loop based at p if $\gamma(0) = \gamma(1) = p$.*

We notice that if γ is a loop based at p, then the parallel transport map $P_\gamma : E_p \to E_p$ is an invertible linear map. It therefore belongs to $\mathrm{GL}(E_p)$, the group of linear invertible transformations on E_p. This leads to the definition of our main object: the *holonomy group* of a connection:

Definition 4.2.2. *We define the holonomy group $\mathrm{Hol}_p(\nabla)$ of the connection ∇ to be the group $\mathrm{Hol}_p(\nabla) := \{P_\gamma |\ \gamma \text{ is a loop based at } p\} \subset \mathrm{GL}(E_p)$.*

If α and β are loops based at p, so are α^{-1} and $\beta\alpha$ (their inverse and composition). By lemma 3.1.1 and 3.1.2 we have that $P_\alpha^{-1} = P_{\alpha^{-1}}$ and $P_\beta \circ P_\alpha = P_{\beta\alpha}$. Thus, if P_α and P_β belong to $\mathrm{Hol}_p(\nabla)$, so do P_α^{-1} and $P_{\beta\alpha}$. This means that the set $\mathrm{Hol}_p(\nabla)$ is closed under the operations of composition and taking the inverse and therefore is a subgroup of $\mathrm{GL}(E_p)$, which justifies calling $\mathrm{Hol}_p(\nabla)$ a group.
Since M is connected, we can find a piecewise smooth path $\gamma : [0, 1] \to M$ joining any two points $p = \gamma(0)$ and $q = \gamma(1)$ in M.
We have the parallel transport map $P_\gamma : E_p \to E_q$. If now α is a loop based at p, then the composition $\gamma\alpha\gamma^{-1}$ gives a loop based at q and lemma 3.1.2 yields $P_{\gamma\alpha\gamma^{-1}} = P_\gamma \circ P_\alpha \circ P_\gamma^{-1}$. Thus, $P_\gamma \circ P_\alpha \circ P_\gamma^{-1} \in \mathrm{Hol}_p(\nabla)$ if $P_\alpha \in \mathrm{Hol}_p(\nabla)$. So, for all the elements of $\mathrm{Hol}_p(\nabla)$ it holds that

$$P_\gamma \,\mathrm{Hol}_p(\nabla) P_\gamma^{-1} = \mathrm{Hol}_q(\nabla). \tag{4.1}$$

From which it follows that the holonomy groups are independent of the base point. In fact, suppose that E has fibre $\mathbb{R}^k$, then the identification $E_p \cong \mathbb{R}^k$ induces a group isomorphism

$GL(E_p) \cong GL(k,\mathbb{R})$. From this viewpoint we may consider $\mathrm{Hol}_p(\nabla)$ as a subgroup H of $GL(k,\mathbb{R})$. If we choose a different local trivialization $E_p \cong \mathbb{R}^k$, then there exists some $a \in GL(k,\mathbb{R})$ such that aHa^{-1} is a subgroup of $GL(k,\mathbb{R})$. The holonomy group of a connection is therefore a subgroup of the general linear group, defined up to a conjugation. In addition, through (4.1) we see that there is no distinction between different base points: $\mathrm{Hol}_p(\nabla)$ and $\mathrm{Hol}_q(\nabla)$ do yield the same subgroup of $GL(k,\mathbb{R})$, up to conjugation. We have thus proven:

Proposition 4.2.3. *Let $E \to M$ be a vector bundle with fibre $\mathbb{R}^k$, and ∇ a connection on E. For each $p \in M$, the holonomy group $\mathrm{Hol}_p(\nabla)$ may be regarded as a subgroup of $GL(k,\mathbb{R})$ defined up to conjugation within $GL(k,\mathbb{R})$. In this sense it is independent of the base point p.*

We simply write the holonomy groups of the connection ∇ as $\mathrm{Hol}(\nabla) \subset GL(k,\mathbb{R})$ by implicitly supposing two subgroups of $GL(k,\mathbb{R})$ to be equivalent if they are conjugate in $GL(k,\mathbb{R})$.
In the same fashion, proposition 4.2.3 extends to vector bundles with complex fibers.
In the case of vector bundles over simply-connected manifolds we have that $\mathrm{Hol}(\nabla)$ is a connected Lie group.

Proposition 4.2.4. *Let simply-connected manifold M, E a vector bundle over M with fibre $\mathbb{R}^k$ and ∇ a connection on E. Then $\mathrm{Hol}(\nabla)$ is a connected Lie subgroup of $GL(k,\mathbb{R})$.*

Proof. Choose a base point p in M and consider a loop γ based at p. Since M is simply connected, the loop γ can be continuously contracted to a constant loop c through the (continuous) homotopy

$$H : [0,1] \times [0,1] \to M,$$

with $H(0,t) = \gamma$ and $H(1,t) = c$. In addition, $H(s,t) = \gamma_s(t) : [0,1] \to M$ satisfies $\gamma_s(0) = \gamma_s(1) = p$ for all $s \in [0,1]$.
Notice the usual requirement for the definition simply connectedness is that the above homotopy depend continuously on s and t. However, as shown in [5], we can also suppose that H depends on both s and t in a piecewise smooth way. Thus, we have that $s \to P_{\gamma_s}$ is a piecewise smooth map from $[0,1]$ to $\mathrm{Hol}_p(\nabla)$. Now, as γ_0 is the constant loop at p, it follows that $P_{\gamma_0} = Id$ and $P_{\gamma_1} = P_\gamma$.
Therefore, each $P_\gamma \in \mathrm{Hol}_p(\nabla)$ can be joined to P_{γ_0} by a piecewise smooth path within $\mathrm{Hol}_p(\nabla)$.
By **G4)** of section 4.1, every arcwise connected subgroup of a Lie group is a connected subgroup.
So $\mathrm{Hol}_p(\nabla)$ is a connected Lie subgroup of $GL(k,\mathbb{R})$.

■

When M is not simply-connected it is convenient to define a *restricted* version of the holonomy group.

Definition 4.2.5. *Let $\gamma : [0,1] \to M$ be a loop based at $p \in M$, we say that γ is null-homotopic if there is a piecewise smooth homotopy*

$$H : [0,1] \times [0,1] \to M,$$

such that $H(0,t) = \gamma(t)$ and $H(1,t) = c$, where c is the constant loop based at p.

Definition 4.2.6. *Let M be a (smooth) manifold, $p \in M$, E a vector bundle over M with $\mathbb{R}^k$ and ∇ a connection on E. We define the restricted holonomy group* $\mathrm{Hol}_p^0(\nabla)$ *to be*

$$\mathrm{Hol}_p^0(\nabla) := \{P_\gamma | \ \gamma \text{ is a null-homotopic loop based at } p\}.$$

Then $\mathrm{Hol}_p^0(\nabla)$ is a subgroup of $\mathrm{GL}(E_p)$. Again, we may consider $\mathrm{Hol}_p^0(\nabla)$ as a subgroup of $\mathrm{GL}(k,\mathbb{R})$ defined up to conjugation. As before, it is then independent of the base point and we write it as $\mathrm{Hol}^0(\nabla) \subseteq \mathrm{GL}(k,\mathbb{R})$.
For the reader interested in more algebraic topological aspects of the theory we have collected some properties of $\mathrm{Hol}_p^0(\nabla)$ in the following proposition.

Proposition 4.2.7. *Let M be a manifold, E a vector bundle over M with fibre $\mathbb{R}^k$ and ∇ a connection on E. Then* $\mathrm{Hol}_p^0(\nabla)$ *is connected Lie subgroup of* $\mathrm{GL}(k,\mathbb{R})$. *In particular, it is the connected component of* $\mathrm{Hol}(\nabla)$ *that contains the identity and is a normal subgroup of* $\mathrm{Hol}(\nabla)$. *There is a natural, group homomorphism between the fundamental group of M and* $\mathrm{Hol}(\nabla)/\,\mathrm{Hol}^0(\nabla)$. *Thus, if M is simply connected, then* $\mathrm{Hol}(\nabla) = \mathrm{Hol}^0(\nabla)$.

Proof. The very same argument used to prove proposition 4.2.4 yields that the restricted holonomy group $\mathrm{Hol}^0(\nabla)$ is a connected Lie subgroup of $\mathrm{GL}(k,\mathbb{R})$.
Now to show that it is a normal subgroup, we fix $p \in M$ and let α and β be loops based a p with β being null-homotopic. Then, also the composition $\alpha\beta\alpha^{-1}$ is null-homotopic. Thus, if $P_\alpha \in \mathrm{Hol}_p(\nabla)$ and $P_\beta \in \mathrm{Hol}_p^0(\nabla)$, we have that $P_{\alpha\beta\alpha^{-1}} = P_\alpha P_\beta P_\alpha^{-1}$ is in $\mathrm{Hol}_p^0(\nabla)$. Therefore $\mathrm{Hol}_p^0(\nabla)$ is a normal subgroup of $\mathrm{Hol}_p^0(\nabla)$.
The group homomorphism $\phi : \pi_1(M) \to \mathrm{Hol}_p(\nabla)/\,\mathrm{Hol}_p^0(\nabla)$ is given by the $\phi([\gamma]) = P_\gamma \cdot \mathrm{Hol}_p^0(\nabla)$, which associates the homotopy class $[\gamma]$ of $\pi_1(M)$ with the coset $P_\gamma \cdot \mathrm{Hol}_p^0(\nabla)$.
Since $\pi_1(M)$ is countable, the quotient group $\mathrm{Hol}_p(\nabla)/\,\mathrm{Hol}_p^0(\nabla)$ is also countable. Therefore, $\mathrm{Hol}_p^0(\nabla)$ is the connected component containing the identity. ∎

Now, given a vector bundle $E \to M$ with fiber $\mathbb{R}^k$ and a connection ∇ on E we can define the *holonomy algebra* of the restricted holonomy group.

Definition 4.2.8. *We define the holonomy algebra,* $\mathfrak{hol}^0(\nabla)$ *to be the Lie algebra of the restricted holonomy group* $\mathrm{Hol}^0(\nabla)$.

Similarly, for $\mathrm{Hol}_p^0(\nabla)$ (which is a Lie subgroup of $\mathrm{GL}(k,\mathbb{R})$) we define $\mathfrak{hol}_p(\nabla)$ to be the Lie algebra of $\mathrm{Hol}_p^0(\nabla)$ for any $p \in M$. The space $\mathfrak{hol}_p(\nabla)$ is a Lie subalgebra of $\mathrm{End}(E_p)$.
We conclude this section with an observation. First we notice that since $\mathrm{Hol}^0(\nabla)$ is the identity component of $\mathrm{Hol}(\nabla)$, by **A5)** their Lie algebras coincide. Also, by **A3)**, the lie algebra $\mathfrak{hol}(\nabla)$ is a Lie subalgebra of $\mathfrak{gl}(k,\mathbb{R})$ (defined up to the adjoint action of $\mathrm{GL}(k,\mathbb{R})$).
Moreover, although $\mathrm{Hol}^0(\nabla)$ is a Lie subgroup of $\mathrm{GL}(k,\mathbb{R})$, it may not necessarily be a closed subgroup, and subsequently it may not a submanifold of $\mathrm{GL}(k,\mathbb{R})$. For example, consider the inclusion of $\mathbb{R}$ in $T^2 = \mathbb{R}^2/\mathbb{Z}$ given by $t \mapsto (t + \mathbb{Z}, \sqrt{2}t + \mathbb{Z})$ for $t \in \mathbb{R}$. This is a non-closed Lie subgroup of a Lie group. Even if, the restricted holonomy group, $\mathrm{Hol}^0(\nabla)$, is closed, the "full" group, $\mathrm{Hol}(\nabla)$, may or may not be closed in $\mathrm{GL}(k,\mathbb{R})$.

4.3 Examples: Riemannian Holonomy Groups

A wealth of examples of holonomy groups is found for connection on Riemannian manifolds. Let (M, g) be a Riemannian manifold of dimension n, endowed with a Levi-Civita connection ∇. Then $\nabla g = 0$ and g is a constant tensor in the sense of definition A.0.3. By proposition A.0.4 if $p \in M$, then the action of $\mathrm{Hol}_p(\nabla)$ on T_pM preserves the metric $g|_p$ on T_pM. The group $\mathrm{Hol}_p(\nabla)$ lies therefore in the subgroup of $GL(T_pM)$ of transformations under which $g|_p$ is invariant. In other words, $\mathrm{Hol}_p(\nabla)$ is isomorphic to a subgroup of the orthogonal group $O(n)$. Some well-known examples of Riemannian holonomy groups are [19]

- the holonomy group of $\mathbb{R}^n$ is the trivial group $\{Id\}$,
- the unit spheres $\mathcal{S}^n$ with the round metric have holonomy group $\mathrm{SO}(n)$,
- the hyperbolic space $\mathcal{H}^n$ with the hyperbolic metric has holonomy group $\mathrm{SO}(n)$,
- the projective spaces $\mathbb{CP}^n$ with the Fubini-Study metric has the unitary group $U(n)$ as holonomy group.

The manifolds in the list above are examples of *irreducible symmetric spaces* (see [19]). From the theory of symmetric spaces one has that the action of the holonomy group of such class of spaces induces a representation on the tangent spaces, T_xM, of M. This representation is faithful and it is known as *holonomy representation*. Through these properties, the holonomy group of a Riemannian symmetric space is easily found. A large number of holonomy groups in fact occur in this way: every compact, connected, simple Lie group is the holonomy group of an irreducible Riemannian symmetric space, with the adjoint representation as holonomy representation [1]. For a through introduction to the theory of symmetric space we refer the reader to [19].

4.4 The Ambrose-Singer Holonomy Theorem

Proposition 4.2.3, stating the independence of $\mathrm{Hol}(\nabla)$ on any base point (up to conjugation), shows that the holonomy group is a *global* invariant of a connection (compare with curvature that, instead, is a local invariant as it varies from point to point on the manifold).
For a given connection on a vector bundle, the holonomy group (or the Lie algebra) is closely related to its curvature. More precisely, the holonomy algebra both constrains the curvature and is determined by it, as the results below illustrate.

Proposition 4.4.1. *Let M be a manifold, E a vector bundle over M and ∇ a connection on E. Then for each $p \in M$ the curvature $R(\nabla)|_p$ of ∇ at p lies in $\mathfrak{hol}_p(\nabla) \otimes \bigwedge^2 T_p^*M$.*

In other words, the holonomy group of a connection places a linear restriction upon its curvature. The result above follows from a similar one obtained for principal bundles. As the theory of principle bundles is beyond the scope of this note, we refer to [1] for a proof and a review of the relations between principal and vector bundles.

The relations between holonomy algebra and curvature is strengthened by the Ambrose-Singer holonomy theorem [15] that we state and prove below.
With the notation of chapter 3 we set

$$R_\gamma(u, v) = P_\gamma^{-1} \circ R(P_\gamma u, P_\gamma v) \circ P_\gamma, \tag{4.2}$$

where $\gamma : [0, 1] \to M$ is a piecewise smooth curve in manifold M with $\gamma(0) = p$, $\gamma(1) = q$, $p, q \in M$ and $u, v \in T_pM$. The map $R_\gamma(u, v)$ is an endomorphism of E_p.

Theorem 4.4.2. *[Ambrose-Singer Holonomy Theorem] Let M be a manifold, E a vector bundle over M and ∇ a connection in E. Fix $p \in M$ so that $\mathfrak{hol}_p\nabla$ is a Lie subalgebra of* $\mathrm{End}(E_p)$*. Then, $\mathfrak{hol}_p\nabla$ is the subalgebra generated by the endomorphisms $R_\gamma(u, v)$ for all piecewise smooth curves γ and vector u and v in T_pM.*

Proof. Let $\gamma : [0, 1] \to M$ be a piecewise smooth curve with $\gamma(0) = p$, $\gamma(1) = q$, $p, q \in M$ $u, v \in T_pM$. Let $U \subset \mathbb{R}^2$ be an open neighbourhood of 0 and $f : U \to M$ be a smooth map with $f(0) = q = \gamma(1)$, $\partial_x f(0) = P_\gamma u$ and $\partial_y f(0) = P_\gamma v$.
Define loops $h_s(t)$, $0 \le s \le 1$ based at q as in (3.15). Then the piecewise smooth curves

$$\gamma_s = \gamma h_s \gamma^{-1}$$

are null-homotopic loops at p (as h_s contracts to q, γ_s contracts to p).
For $0 \le \tau \le 1$, let P_τ be the parallel transport map along the curve γ_s with $s = \tau^2$. Then, by (3.14) and theorem 3.2.4, $P_\tau : [0, 1] \to \mathrm{GL}(E_p)$ is a continuously differentiable curve contained in $H = \mathrm{Hol}_p^0(\nabla)$ and

$$\partial_\tau P_\tau|_{\tau=0} = P^{-1} \circ R(P_\gamma v, P\gamma u) P_\gamma = R_\gamma(v, u) \in T_pH \cong \mathfrak{hol}_p(\nabla).$$

Thus the endomorphisms $R_\gamma(v, u)$ contain the generators of $\mathfrak{hol}_p(\nabla)$. It follows thats $\mathfrak{hol}_p(\nabla)$ is the subspace of $\mathrm{End}(E_p)$ with elements $R_\gamma(v, u)$. ∎

The Ambrose-Singer holonomy theorem shows that $\mathfrak{hol}_p(\nabla)$ is the vector subspace of $\mathrm{End}(E_p)$ spanned by the endomorphisms $R_\gamma(u, v)$. Thus, $R(\nabla)$ determines $\mathfrak{hol}(\nabla)$ and, hence $\mathrm{Hol}^0(\nabla)$. As a simple example we may consider a flat connection, $R(\nabla) = 0$. Then $\mathfrak{hol}(\nabla) = 0$, from which it follows that $\mathrm{Hol}^0(\nabla) = Id$.

A

Appendix: Constant Tensors

Let ∇ be a connection on the tangent bundle TM of a manifold M. As we mentioned in 2.2, ∇ extends to connections on all tensor bundles $\bigotimes^k T^*M \otimes \bigotimes^l TM$ for $k, l \in \mathbb{N}$ (remember that we still use ∇ to denote this extended connection).

Definition A.0.3. *A tensor field T is called constant if*

$$\nabla T = 0.$$

The constant tensors on M are determined entirely by the holonomy group of the connection.

Theorem A.0.4. *Let M be a manifold, ∇ be a connection on TM. Fix point p in M and let $H = \mathrm{Hol}_p(\nabla)$. Then H acts naturally on the tensor powers $\bigotimes^k T^*M \otimes \bigotimes^l TM$. Suppose $S \in C^\infty\big(\bigotimes^k T^*M \otimes \bigotimes^l TM\big)$ is a constant tensor field, then $S|_p$ is fixed by the action of H on $\bigotimes^k T^*M \otimes \bigotimes^l TM$. Conversely, if $S|_p \bigotimes^k T^*M \otimes \bigotimes^l TM$ is fixed by H, it extends to a unique constant tensor field $S \in C^\infty\big(\bigotimes^k T^*M \otimes \bigotimes^l TM\big)$.*

Proof. The main idea in the proof is that constant tensors are invariant under parallel transport along paths within the manifold M. Let us see how this works in detail.
Let γ be a loop based at p and $P_\gamma \in \mathrm{GL}(E_p)$ the parallel transport map using the extension of ∇ to $E = \bigotimes^k T^*M \otimes \bigotimes^l TM$. Then $P_\gamma \in \mathrm{Hol}_p(\nabla) = H$, and there is some element $h \in H$ such that $P_\gamma = h$. Moreover, for every $h \in H$ we have $P_\gamma = h$ for some loop γ in M based at p. Now, since $\nabla S = 0$, the pullback $\gamma^*(S)$ is a parallel section of the pullback bundle $\gamma^*(E)$ over $[0,1]$. Therefore $P_\gamma(S|_{\gamma(0)}) = S|_{\gamma(1)}$, but since $\gamma(0) = \gamma(1) = p$, we have $P_\gamma(S|_p) = S|_p$. Thus, $h(S|_p) = S|_p$ for all $h \in H$ and $S|_p$ is fixed by the action of H on E_p.
For the converse, suppose that $S|_p \in E_p$ is fixed by H. We will define a tensor field $S \in C^\infty(E)$ which satisfies the given properties. Let q be any point of M. Since M is connected, there is a piecewise smooth joining p and q. Let $\alpha : [0,1] \to M$ and $\beta : [0,1] \to M$ with $\alpha(0) = \beta(0) = p$ and $\alpha(1) = \beta(1) = q$ be two such paths. Let $P_\alpha, P_\beta : E_p \to E_q$ be the parallel transport maps, so that $P_{\alpha^{-1}\beta} = P_\alpha^{-1} P_\beta$. Now $\alpha^{-1}\beta$ is a loop based at p, thus $P_{\alpha^{-1}\beta} = h$ for some $h \in H$. By

assumption we have $h(S|_p) = (P_\alpha^{-1} P_\beta)S|_p = S|_p$, giving $P_\beta S|_p = P_\alpha S|_p$, which shows that the element $P_\alpha S|_p$ of E_q only depends on depends on q and not on the path α.
Define a section S of E by $S_q = P_\alpha S|_p$ where α is any piecewise smooth path from p to q, then S is well-defined. If γ is any path in M, then the pullback section $\gamma^*(S)$ is parallel. Thus, S is differentiable with $\nabla S = 0$ and clearly $S \in C^\infty(E)$. ■

Bibliography

[1] D. D. Joyce. *Riemannian Holonomy Groups and Calibrated Geometries*. Oxford Graduate Texts in Mathematics. Oxford University Press, 2007.

[2] I. Madsen and J. Tornehave. *From Calculus to Cohomology*. Cambridge University Press, 1999.

[3] J. M. Lee. *Introduction to Smooth Manifolds*. Springer, 2006.

[4] C. O. Aguilar. *The Dimension of the Symmetric k-Tensors*. Website. www.mast.queensu.ca/~cesar/math_notes/dim_symmetric_tensors.pdf.

[5] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry, Volume 1*. Wiley Classics Library. John Wiley & Sons, INC., 1996.

[6] L. I. Nicolaescu. *Lectures on the Geometry of Manifolds*. World Scientific, 1996.

[7] S. Ramanan. *Global Calculus*. Graduate Studies in Mathematics. American Mathematical Society, 2005.

[8] J. M. Lee. *Riemannian Geometry, an Introduction to Curvature*. Springer, 1997.

[9] A. Kovalev. *Vector Bundles*. Website, 2004. http://www.dpmms.cam.ac.uk/~agk22/vb.pdf.

[10] W. Ballmann. *Vector Bundles and Connections*. Website, 2002. http://www.math.uni-bonn.de/people/hwbllmnn/archiv/concurvb.ps.

[11] C. H. Taubes. *Metrics, Connections and Gluing Theorems*. Regional Conference Series in Mathematics. Amer. Math. Soc., 1996.

[12] F. Zheng. *Complex Differential Geometry.* Studies in Advanced Mathematics. American Mathematical Society, 2000.

[13] S. Smale M. W. Hirsch. *Differential Equations, Dynamical Systems, and Linear Algebra.* Pure and Applied Mathematics. Academic Press, 1974.

[14] J. Jost. *Riemannian Geometry and Geometric Analysis.* Universitext. Springer, 2005.

[15] I. M. Singer W. Ambrose. A Theorem on Holonomy. *Transactions Amer. Math Soc.*, **75** :428–443, 1953.

[16] H. Yamabe. On an Arcwise Connected Subgroup of a Lie Group. *Osaka Journal of Mathematics*, **2** :13–14, 1950.

[17] B. C. Hall. *Lie Groups, Lie Algebras, and Representations.* Springer, 2004.

[18] M. M. Alexandrino. *Introduction to Lie Groups, Adjoint Action and its Generalizations.* Preprint, 2009. http://arxiv.org/abs/0901.2374v1.

[19] S. Helgason. *Differential Geometry, Lie Groups and Symmetric Spaces.* Pure and Applied Mathematics. Academic Press, 1978.