Fixed points and homotopy fixed points

(1)

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$\alpha : B \to B$

$$\begin{tikzcd} B^{\alpha} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "\Delta"] \\ B \arrow[r, "(1,\alpha)"'] & B \times B \end{tikzcd}$$

Tits

From homotopy Lie groups to homotopy Chevalley groups ... book

B^{α} = fixed points of α

$\boxed{\text{Broto-M : Chevalley p-local finite groups AGT 2007}}$

$$\begin{tikzcd} B^{h\alpha} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{map}(I,B) \arrow[d] \\ B \arrow[r, "(1,\alpha)"'] & B \times B \end{tikzcd}$$

$u \mapsto u(\dot{I})$

$B^{h\alpha}$ = homotopy fixed points of α

$x \overset{u}{\frown} \alpha(x)$

Joint with Carlos Broto

p-compact groups and their automorphisms, in particular Adams operations

BX p-compact group

Maximal torus $\quad B\check{T} \to BX \qquad \check{T}(X) \cong (\mathbb{Z}/p^{\infty})^r$

Weyl group

$$\begin{tikzcd} B\check{T}(X) \arrow[rr, "W(X)"] \arrow[dr] & & B\check{T}(X) \arrow[dl] \\ & BX & \end{tikzcd}$$

$\mathrm{Aut}(\check{T}(X)) = GL_r(\mathbb{Z}_p)$

$BX \rightsquigarrow (W(X), \check{T}(X))$

$\{\text{p-compact groups}\} \longleftrightarrow \{\text{p-adic reflection groups}\}$

$X \to (\check{T}(X), W(X))$

$$[BX, BX]^{\times} = \mathrm{Out}(X) = N_{\mathrm{Aut}(\check{T}(X))}(W(X)) / W(X)$$

Note : $\mathbb{Z}_p^{\times} / Z(W(X)) = \mathrm{Aut}(\mathbb{Z}/p^{\infty}) / Z(W) \xrightarrow{u \mapsto \psi^u} \mathrm{Out}(X)$ "Weyl group of the Weyl group"

{ irreducible p-compact groups } $\longleftrightarrow$ { irreducible p-adic refl. groups }

$\updownarrow$

Shephard–Todd ~~Clark–Ewing~~ classification list

$\boxed{\begin{array}{l} BX \text{ irreducible} \Rightarrow \\ Out(X) \text{ in } \mathbb{Z}_p^{\times}/Z(W) + \text{a little more} \end{array}}$

Any p-compact group has the form

simply connected simple

$$BX = B\left({}^{Y \times T}/(Z, \varphi) \right) \qquad BY = \prod BX_i \qquad \varphi : \check{Z}(Y) \to \check{T}$$

Shephard–Todd

~~Clark–Ewing~~ classification

$G(m,r,n)$ definition?

Family 1 $\quad W(SU(n+1)) = \Sigma_n \leq GL_n(\mathbb{Z}_p)$

Family 2 → 2a $\quad G(m,r,n) \quad (m,r,n) \neq (m,m,2) \quad r \mid m \mid p-1$ permutations + some diagonal matrices

→ 2b $\quad G(m,m,2) \quad m \geq 3$

Family 3 $\quad C_m \leq GL_1(\mathbb{Z}_p) = Aut(\mathbb{Z}/p^{\infty}) \quad m \mid p-1$

Sporadic $\quad G_i, \quad 4 \leq i \leq 37$

The Weyl groups of compact Lie are on this list

Alexander's groups are polynomial : $H^*(BX; \mathbb{F}_p) =$ polynomial algebra

Observation An irreduc p-compact group BX is BG for some compact Lie or it is polynomial $\left(H^*(BX; \mathbb{Z}_p) = P[\ldots] \right)$

Those that are not grt Lie are polynomial

The polynomials ones are constructed as very simple homotopy colimits of BG

Example Realization (Clark–Ewing p-compact groups) $p \nmid |W|$

$$BX = B(\check{T} \rtimes W) = \operatorname{hocolim} B\check{T} \circlearrowleft W$$

Family 3

$$H^*(BX) = H^*(B\check{T})^W$$

is polynomial, so this is a p.c.g. There are many of these

Sullivan spheres $m \mid p-1$

$$BX = B(\mathbb{Z}/p^\infty \rtimes C_m)$$

$$H^*(BX) = H^*(\mathbb{Z}/p^\infty)^{C_m} = \mathbb{F}_p[u^m], \quad |u| = 2, \quad |u^m| = 2m$$

$$X = \Omega BX \quad H^*(X) = H^*(S^{2m-1})$$

$$BX = BS^{2m-1} \qquad S^{2m-1} \text{ is a } p\text{-compact group}$$

Aguadé p-compact groups $p \mid |W|$

$$BX = \operatorname{hocolim}\left(B(\check{T}^S) \longrightarrow B\check{T} \circlearrowleft \right)$$

G_{12} $p=3$ DI_2 $H^*(BG_{12}) = \mathbb{F}_3[x_{12}, x_{16}]$ rank 2 Dickson algebra

G_{29} $p=5$

G_{31} $p=5$

G_{34} $p=7$

Friedlander's thm

From compact Lie → finite grps of Lie type

BG $\quad \psi^u : BG \to BG \quad u \in \mathbb{Z}_p^\times$

$\tau : BG \to BG$ exceptional isogeny of finite order ι

$\tau\psi^u : BG \to BG$ – what are the homotopy fixed points?

$$\begin{tikzcd} B({}^{\tau}G(u)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{map}(I, BG) \arrow[d] \\ BG \arrow[r] & BG \times BG \end{tikzcd}$$

BM: What happens if we replace BG by BX? (David Benson on [illegible])

p-compact grps → homotopy finite grps of Lie type

Thm A X 1-connected pcg, q prime power, $p \nmid q$, τ an automorphism of X of order prime to p. The pull back

$$\begin{tikzcd} BX(\tau\psi^q) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & BX \arrow[d] \\ BX \arrow[r, "{(1,\tau\psi^q)}"'] & BX \times BX \end{tikzcd}$$

is the classifying space of a p-local finite group.

There is a consequence of

The proof splits into 2 cases

Action of G on BX Split the proof in two cases

An action of G on BX is a fibration

Proof splits into 2 cases

(5)

$$BX \to BX_{hG} \to BG$$

The fixed points of this action BX^{hG} is the space of sections.

$\boxed{\text{Can change the fibration by } Out(BG) \text{ - this is the advantage of p-compact compared to algebraic}}$

Homotopy fixed points for actions prime to p

<u>Thm B</u> $\rho : G \to Out(X)$ outer action

(1) ρ lifts to an action of G on BX

(2) BX^{hG} is a pcg and

this almost determines

$$H^*(BX^{hG}; \mathbb{Q}_p) = S[\underbrace{QH^*(BX, \mathbb{Q}_p)_G}_{\text{a small vector space}}] \quad X^{hG}$$

(3) $X \simeq X^{hG} \times X/X^{hG}$ and X/X^{hG} is an H-space (some new H-space)

(4) If $H^*(BX; \mathbb{F}_p)$ is polyn so is $H^*(BX^{hG}; \mathbb{F}_p)$

$F_4 \simeq X_{12} \times F_4/D_{12}$ at $p=3$

$\boxed{\text{Here we step in the world of pcgs}}$

<u>Thm C</u> $q \equiv 1 \bmod p$, $q \neq 1$

$BX(q)$ is the classifying space of a p-local finite group.

$\boxed{\text{Here we move to p-local finite groups}}$

Thm E X 1-connected, τ has order prime to p — describe in Thm B

Why we can split (6) the proof of Thm A in two parts

(1) If $q \equiv 1 \bmod p$ then $BX(\tau q) = BX^{h\langle\tau\rangle}(q)$

(2) If q, q' have the same multiplicative order mod p and $\nu_p(1-q^r) = \nu_p(1-q'^r)$

then $BX(q) = BX(q')$.

Thm D $q \equiv 1 \bmod p$, $q \neq 1$. Exotic p-local finite groups

- $BX_{29}(q)$, $BX_{34}(q)$ at $p = 5, 7$
- $BX(m,r,n)(q)$ for $n \geq p$, $r > 2$

Example The p-local finite groups $BX_{12}(q)$ and $BX_{31}(q)$ are not exotic

$p=3$ $\quad BX_{12}(q) = B({}^2F_4(2^{3^\ell}))^\wedge_3 \qquad \ell = \nu_3(q^2-1)$

$p=5$ $\quad BX_{31}(q) = B(E_8(2^{2^{m+1}}))$ if $\nu_5(q^4-1) = \nu_5(1+2^{4m+2})$

$BX_{12} = \operatorname{hocolim} B({}^2F_4(2^{3^n}))^\wedge_3$

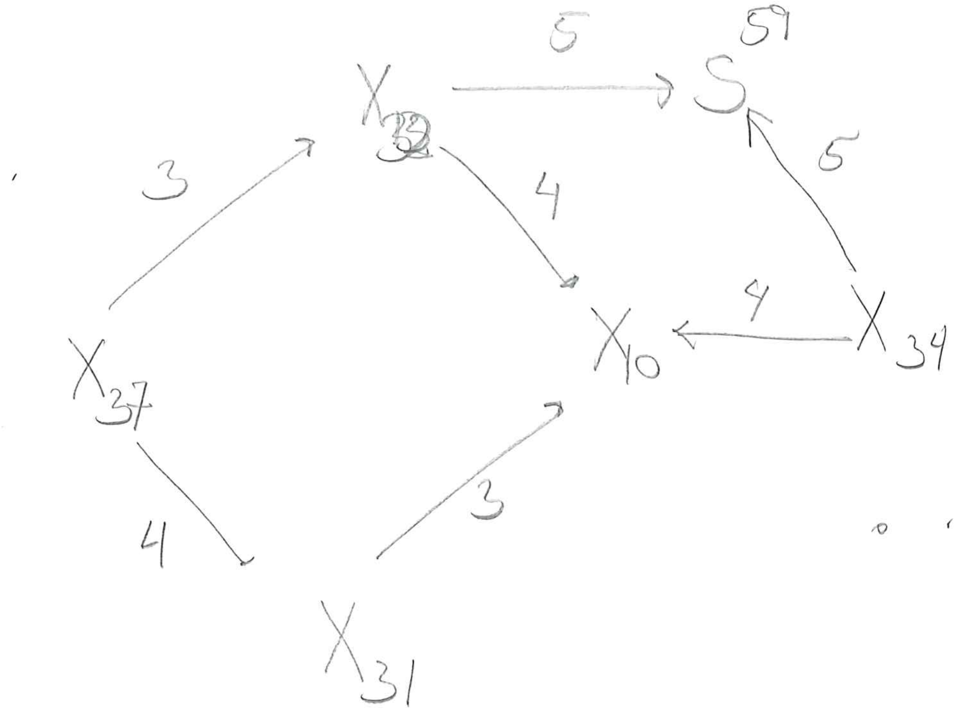
$BX_{31} = \operatorname{hocolim} B(E_8(2^{5^n}))^\wedge_5$

$BX(m,1,n)(q) = BGL_{mn}(q)$

Only want to rank

Φ_{eq} $\quad 1-q^n$

cyclotomic pol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