Equivariant Euler characteristic

1. Definition

$\Delta \circlearrowleft G$

invariant of this situation

$$\chi_r(\Delta, G) = \frac{1}{|G|} \sum_{X \in Hom(\mathbb{Z}^r, G)} \chi(C_\Delta(X))$$

$$\tilde{\chi}_r(\quad) = \quad \tilde{\chi}(\quad)$$

p prime

$$\chi_r^p(\Delta, G) = \frac{1}{|G|} \sum_{X \in Hom(\mathbb{Z} \times \mathbb{Z}_p^{r-1})} \chi(C_\Delta(X))$$

$$\tilde{\chi}_r^p(\Delta, G) = \ ? \ \tilde{\chi}$$

~~whe does it come from~~ Where does it come from?

2. WHAT IS IT?

<u>Example</u> Trivial action

$$\chi_r(\Delta, G) = \chi(\Delta)\, |Hom(\mathbb{Z}^r, G)| / |G|$$

$$= \chi(\Delta)\, |Hom(\mathbb{Z}^{r-1}, G)/G|$$

$$\chi_r(*, G) = |Hom(\mathbb{Z}^{r-1}, G)/G| \qquad \tilde{\chi}_r(*, G) = 0$$

$$\tilde{\chi}_r(\Delta, G) = \chi_r(\Delta, G) - \chi_r(*, G) \qquad \chi_2(*, G) = k(G)$$

<u>Free action:</u> $\chi_r(\Delta, G) = \frac{1}{|G|}\chi(\Delta) = \chi(\Delta/G)$

$$\sum \tilde{\chi}_r(S^n, \mathbb{Z}_2)\, x^r = (-1)^{n-1} \frac{x}{(1-x)(1-2x)}$$

/64/ 2/p

Rewriting:

$$\chi_{r+1}(\Delta,G) \overset{A}{=} \sum_{gG\in G/G} \chi_r\left(C_\Delta(g), C_G(g)\right)$$

ordinary Eu weighted by

$$\overset{B}{=} \sum_{\sigma G\in \Delta/G} (-1)^{d(\sigma)} \left| Hom(\mathbb{Z}^r, C_G(\sigma))/C_G(\sigma)\right|$$

$$\overset{C}{=} \frac{1}{|\cdot|}\sum_{[A]} |G:N_G(A)|\,\varphi_{\mathbb{Z}}(A)\,\chi(C_\Delta(A))$$

$r=1,2$:

Burnside

$$\chi_1(\Delta,G) \overset{def}{=} \frac{1}{|G|}\sum \chi(C_\Delta(g)) = \chi(\Delta/G)$$

$$\chi_2(\Delta,G) = \begin{cases} \overset{A}{=} \sum_{gG\in G/G} \chi_1\left(C_\Delta(g)/C_G(g)\right) \\ \overset{B}{=} \sum_{\sigma G} (-1)^{d(\sigma)} k(C_G(\sigma)) \end{cases}$$ /7.1/

A combined with

Thm $$K_G^*(|\Delta|) = \left(\bigoplus_{g\in G} K^*(C_{|\Delta|}(g))\otimes\mathbb{C}\right)^G$$

shows

$$\chi_2(\Delta,G) = \sum_{G/G} \chi\left(H^*(C_\Delta(g))^{C_G(g)}\right)$$

$$= \dim K_G^0(|\Delta|)\otimes\mathbb{C} - \dim K_G^1(|\Delta|)\otimes\mathbb{C} = \chi_{K_G}(|\Delta|)$$

(3)/10

6min/page

No similar interpretation for $r>2$. Except

$\underline{r>2}$

Thm Hopkins-Kuhn-Ravenel, Tamanoi

$$\chi_r(\Delta, G) = \chi_{K(r)}(|\Delta|_{hG})$$

3. Equivariant ECs and grp conjectures

From now on: $\Delta = \mathcal{S}_G^{s+*}$ = nontrivial s-subgrps of G

$\Delta = \Delta(\mathcal{S}_G^{s+*})$ simplices of nontrivial s-subgrps of G

$\Delta(\mathcal{S}_G^{s+*}) \ni \sigma = (P_0 < P_1 < \dots < P_d)$ a d-simplex

Brown Thm $|G|_s \mid \tilde{\chi}(\mathcal{S}_G^{s+*}) \Leftarrow$ (Frobenius Thm: $|G|_s \mid k(G_d)$) stronger

Quillens Thm $O_s(G) \neq 1 \Rightarrow \mathcal{S}_G^{s+*} \simeq * \Rightarrow \tilde{\chi}_r(\mathcal{S}_G^{s+*}) = 0\ \forall r$

actually a new formulation

Quillen's Conj: $\forall r: \tilde{\chi}_r(\mathcal{S}_G^{s+*}) = 0 \Rightarrow O_s(G) \neq 1$ New formulation of Quillen conj

Webb's Thm $\tilde{\chi}_1(\mathcal{S}_G^{s+*}, G) = 0$ (Symonds's thm)

$KRC_p(G)$: $\tilde{\chi}_2(\mathcal{S}_G^{s+*}) = \chi_2(\mathcal{S}_G^{s+*}) - k(G)$

compute $\tilde{\chi}_2$

$$\tilde{\chi}_2(\mathcal{S}_G^{s+*}) \overset{B}{=} \sum_{\sigma \in \Delta(\mathcal{S}_G^{s+*})/G} (-1)^{d(\sigma)} k(C_G(\sigma)) - k(G)$$

An grp theoretic homology approximation thm

$$\operatorname{hocolim}_{\sigma \in \Delta(\mathcal{S}_G^{s+*})/G} BC_G(\sigma) \simeq BG$$

$KRC_p(G)$

$$\boxed{-\tilde{\chi}_2(\mathcal{S}_G^{s+*}) = z_s(G)}$$

$$z_s(G) = \#\{\chi \in \mathrm{Irr}(G, \mathbb{C}) \mid \chi(1)_s = |G|_s\}$$

Can replace $\mathcal{S}_G^{s+*}$ by $\mathcal{S}_G^{s+*\,\mathrm{rad}}$, $\mathcal{S}_G^{s+*+\mathrm{rad}}$

④ 24 min

The KRC_p is equivalent to

$AWC_p(G)$: (1986)

$$\sum_{P_G \in \mathcal{S}_G^{st*}/G} z_s\left(N_G(P)/P\right) = k_{s'}(G) - z_s(G)$$

$AWC_p(G)$ known for: G p-solvable, Σ_n, finite grp of Lie type in char s, $GL_n(\mathbb{F}_q)$ for all q. Reduction thms reduces to checking a long list of conditions for all simple groups

OBS $AWC_p(G) \Leftrightarrow AWC_s(G/O_sG)$

$KRC_p(G)$ known for: G p-solvable

$O_p(G) \neq 1$: $z_s(G) = 0$ and $\mathcal{S}_G^{st*} \simeq_G *$.

($AWC_p(G)$ is not known when $O_pG \neq 1$)

FALSE Proof of $AWC_p(G)$

$O_s(G \times C_s) \neq 1 \Rightarrow KRC_s(G \times C_s) \Leftrightarrow AWC_s(G \times C_s) \Rightarrow AWC_s(G)$

The point is that KRC and AWC are equiv in this sense:

$$\forall S \leq G : KRC_s(N_G(S)/S) \Leftrightarrow \forall S \leq G : AWC_s(N_G(S)/S)$$

not

$$\cancel{KRC_s(G) \Leftrightarrow AWC_s(G)}$$

Hard to justify what follows (?) ~~Exam~~ Computations of ~~Equivariant~~ Eues

~~Vl~~ Analogues of $\tilde{\chi}_r(\mathcal{S}^{st*}_G, G)$ - what can it typically be?

(Partitions of n) $\circlearrowleft \Sigma_n$

If we become very good at calculating Equiv Eues we can solve the KR ρ

$$GL^+_n(\mathbb{F}_q) = Aut(\mathbb{F}_q^n)$$

$$GL^-_n(\mathbb{F}_q) = Aut(\mathbb{F}_{q^2}^n, \langle\ ,\ \rangle) \quad \langle\ ,\ \rangle = \langle\ ,\ \rangle^q$$

$$Sp_{2n}(\mathbb{F}_q) = Aut(\mathbb{F}_q^{2n}, \langle\ ,\ \rangle) \quad \langle\ ,\ \rangle = -\langle\ ,\ \rangle$$

$\cancel{EG_r(X) = \sum \tilde{\chi}_r(G_n}$ $\quad 5 = char(\mathbb{F}_q) \quad \tilde{\chi}_r(G_n(q))$

$$= \tilde{\chi}_r(\mathcal{S}^{st*}_{G_n(q)}, G_n(q))$$

Will describe $\tilde{\chi}_r(G_n(q))$ and $\tilde{\chi}_r^{\ell}(G_n(q))$ ~~either~~ as power series expanded after n or after r.

$G_n = GL_n^+(\mathbb{F}_q)$

$$FGL^+_{r+1}(q,x) = \sum_{n\geq 0} \tilde{\chi}_r(GL_n^+(\mathbb{F}_q))\, x^n \quad (r \text{ fixed}, n \text{ varies})$$

Thm
$$= \prod_{0\leq j\leq r} (1-q^j x)^{(-1)^{r-j}\binom{r}{j}}$$

start with $j=r$

$$FGL_{r+1}(q,x) = \frac{(1-q^r x)^{\binom{r}{r}}\,(1-q^{r-2}x)^{\binom{r}{r-2}}}{(1-q^{r-1}x)^{\binom{r}{r-1}}}$$

$= 1+$

$$FGL_2(q,x) = \frac{1-qx}{1-x} = (1-qx)(1+x+x^2+\cdots)$$
$$= 1+(1-q)x+(1-q)x^2+\cdots$$

$-Z_s(GL_n(\mathbb{F}_q)) = 1-q$ so $KRC_s(GL_n^+(\mathbb{F}_q))$ holds.

$FGL^+_4(q,x) = 1-(q-1)^3(x+(3q^2+1)x^2+$
$(6q^4+q^3-3q^2+1)x^3+\cdots$

Coefficients are irreducible polynomials - significance?

Alternatives:

Thm
$$FGL^+_{r+1}(q,x) = \exp\left(-\sum (q^n-1)^{r}\frac{x^n}{n}\right)$$
$$= \prod (1-x^n)^{a^+_{r+1}(q,n)}$$

$$a^+_{r+1}(q,n) = \frac{1}{n}\sum_{d|n}\mu(n/d)(q^d-1)^r$$

For $r=1$: $a^+_2(q,n) = \frac{1}{n}\sum_{d|n}\mu(n/d)(q^d-1) = IM_n(q)$

$$GGL_n^+(q,x) = \sum_{r\geq 0} \tilde{\chi}_{r+1}(GL_n^+(\mathbb{F}_q))\, x^r \quad (n \text{ fixed}, r \text{ varies})$$

Thm

$$GGL_n^+(q,x) = \frac{1}{n!}\sum_{\lambda \vdash n} \frac{(-1)^{r(\lambda)}\, T(\lambda)}{1 - x\prod_{i\in\lambda}(q^i - 1)}$$

$\lambda \vdash n$ partition of n $\quad \lambda = (2,1,1) \vdash 4$

$r(\lambda)$ = number of parts in λ $\quad r(\lambda) = 3$

$T(\lambda)$ = length of conjugacy class of λ in Σ_n $\quad T(\lambda) = 6$

Example

$$2!\, GGL_2^+(q,x) = \frac{-1}{1-(q^2-1)x} + \frac{1}{1-(q-1)^2 x} \quad (GL_2^+(\mathbb{F}_q))$$

$$3!\, GGL_3^+(q,x) = \frac{-1}{1-(q^3-1)x} + \frac{3}{1-(q^2-1)(q-1)x} + \frac{-1}{1-(q-1)^3 x} \quad (GL_3^+(\mathbb{F}_q))$$

$$= (q-1)x + 3q^2(q-1)^2 x + (q-1)^3(6q^4 - q^3 + 3q^2 + 1)$$

Idea of proof

$$\varphi_G(A) \neq 1 \Rightarrow C_{\varphi_G^{St*}}(A) \simeq * \Rightarrow \tilde{\chi}(C_{\varphi_G^{St*}}(A)) = 0$$

$$\tilde{\chi}_{r+1}(GL_n^+(\mathbb{F}_q)) = \sum \tilde{\chi}_r\big(\underbrace{C_{\Delta(\varphi_q)}(g), C_{GL_n^+(q)}(g)}_{\text{this is a product simplicial complex}}\big)$$

described by Rainer char polynomials $\longrightarrow g \in [GL_n^+(\mathbb{F}_q)]$

$(q, |g|) = 1$

p-primary version

Thm $$FGL^+_{r+1}(p,q,x) = \exp\left(-\sum (q^n-1)_p^r \frac{x^n}{n}\right)$$
$$= \prod (1-x^n)^{a^+_{r+1}(p,q,n)}$$

$$a^+_{r+1}(p,q,n) = \frac{1}{n}\sum \mu(n/d)\,(q^d-1)_p^r$$

theory of $FGL^+_\alpha(p,q,x)$? Is there an $KRC_S(G,p)$? $Z_S(G)_p$?

$$GGL^+_n(p,q,x) = \frac{1}{n!}\sum_{\lambda\vdash n} \frac{(-1)^{r(\lambda)}T(\lambda)}{1-x\prod_{i\in\lambda}(q^i-1)_p}$$

Combinatorics

$$\sum_{\lambda\vdash n}\prod_{d\in B(\lambda)}\binom{a_{r+1}(q,d)}{-E(\lambda,d)} = \tilde{\chi}_{r+1}(GL_n(\mathbb{F}_q))$$

For $r=1$:
$$\sum_{\lambda\vdash n}\prod_{d\in B(\lambda)}\binom{IM_d(q)}{-E(\lambda,d)} = 1-q \quad \text{known}$$

not known for higher r

and not known for p-version

$GL_n^-(\mathbb{F}_q)$

is similar but throw in $\pm$ signs here and the

$$FGL^-_{2r+1}(q,x) = Z(E_1^r, -x)^{-1}$$

Z = Hasse-Weil zeta funct.

E = super singular elliptic curve

$$FGL_2^-(q,x) = 1 + (q+1)x + \cdots$$

$$Z_s(GL_n^-(\mathbb{F}_q)) = q+1$$

Sketch of proof: s-regular classes described by their self-dual characteristic polys

$Sp_{2n}(\mathbb{F}_q)$

<u>Thm</u> $$FSp_{r+1}(q,x) = \prod_{\substack{0 \leq j \leq r \\ j \not\equiv r \bmod 2}} (1-q^j x)^{-\binom{r}{j}} \qquad (n \text{ varies})$$

$$FSp_2(q,x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots \quad \tilde{\chi}_2(Sp_{2n}(\mathbb{F}_q)) = 1 = Z_s(Sp_{2n}(\mathbb{F}_q))$$

Alternative: $$FSp_{r+1}(q,x) = \frac{\exp\left(\sum (q^n+1)^r \frac{x^n}{2n}\right)}{\exp\left(\sum (q^n-1)^r \frac{x^n}{2n}\right)} = \exp\left(\sum \left((q^n+1)^r - (q^n-1)^r\right) \frac{x^n}{2n}\right)$$

$$= \prod (1-x^n)^{c_{r+1}(q,n)}$$

$$c_{r+1}(q,n) = \sum \mu(n/d)\left((q^d-1)^r - (q^d+1)^r\right)$$

For

$$CSp_n(q,x) = \sum \pm \hat{\chi}_{r+1}(Sp_{2n}(\mathbb{F}_q))\, x^r$$

<u>Thm</u> $$CSp_n(q,x) = \frac{1}{n!} \sum_{\lambda \vdash n} \frac{T(\lambda)}{2^{r(\lambda)}} \sum_{\varepsilon \in \{-1,+1\}^{\lambda}} \frac{\prod \varepsilon}{1 - x \prod_{i \in \lambda} (q^i + \varepsilon(i))}$$

<u>Sketch of proof</u> s-regular classes of $Sp_{2n}(\mathbb{F}_q)$ are described by their self-reciprocal characteristic polynomials

combinatorics

$$T_{a^+_{r+1}(q)} (1-x)^{\varepsilon} = FGL^+_{r+1}(q,x)^{\varepsilon}$$

what is coeff of x^n in $FGL^{\pm}_{r+1}(q,x)$

$$T_{a^-_{r+1}(q)} (1-x)^{\varepsilon} = FGL^-_{r+1}(q,x)^{\varepsilon}$$

$$T_{c^-_{r+1}(q)} (1-x)^{\varepsilon} = FSp^-_{r+1}(q,x)$$

$$\sum_{\lambda \vdash n} \prod_{d \in B(\lambda)} \binom{+a^{\pm}_{r+1}(q,d)}{-E(\lambda,d)} = \chi_{r+1}(GL^{\pm}_n(\mathbb{F}_q))$$

$$\sum_{n_0+n_1+\dots+n_r=n} \prod_{0 \le j \le r+1} (-1)^{n_j} \binom{(-1)^{r-j}\binom{r}{j}}{n_j} q^{jn_j} = \tilde{\chi}_{r+1}(GL_n(\mathbb{F}_q))$$

known

$\underline{r=2}$ $$\sum_{\lambda \vdash n} \prod_{d \in B(\lambda)} \binom{IM_d(q)}{-E(\lambda,d)} = 1-q \qquad a^+_2(q,d) = \sum \mu(n/d)(q^d-1)$$

not known

$$\sum_{\lambda \vdash n} \prod_{d \in B(\lambda)} \binom{a^-_2(q,d)}{-E(\lambda,d)} = \cancel{1}+q \qquad a^-_2(q,d) = \sum \mu(n/d)(-1)^d(q^d-(-1)^d)$$

not known

$$\sum_{\lambda \vdash n} \prod_{d \in B(\lambda)} \binom{c_2(q,d)}{-E(\lambda,d)} = 1$$

Also p-primary versions
Also ± 1 versions