Chromatic numbers of manifolds. ~~Origin of~~ this conference

$\boxed{\text{Origin of conference}}$

Joint work with Frank Lutz

K simplicial complex finite $s = 1, 2, 3, \ldots$

Tonimage Complexity
$\lceil \;\rceil$

Def An s-coloring in r colors is a map
$$\mathrm{col} : V(K) \longrightarrow \{1, 2, \ldots, r\}$$

$\mathrm{col} : K \to \Delta[r]$ with fiberpoint fibers of dim $< s$.

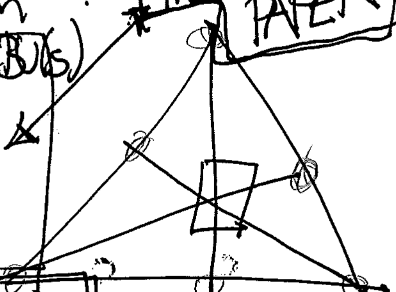
so that ~~no~~ K does not contain any monochrome s-simplices.

The s-chromatic number $\chi_s(K) = r$ is the minimum number ~~so that~~ means that there is an s-coloring in r colors but no s-coloring in $(r-1)$-~~co~~ colors.

- $\chi_s(K)$ depends only on the s-skeleton K^s of K
- $\chi_1(K)$ is the usual chromatic number of the graph K^1 Memory
- ~~χ_1 $|K|$~~ $|V(K)| \geq \chi_1(K) \geq \chi_2(K) \geq \cdots \geq \chi_d(K) \geq \chi_{d+1}(K) = 0$, $d = \dim(K)$
- $K_1 \subseteq K_2 \Rightarrow \chi_s(K_1) \leq \chi_s(K_2)$
- Barycentric subdivision : THROW AWAY PAPER Pachner moves ~~Problem~~ are not well-behaved wrt coloring

K has an s-coloring in r-colors

$$BU(s) \times \cdots \times BU(s)$$
$$\downarrow$$
$$DJ(K) \longrightarrow \underbrace{BU(1) \times \cdots \times BU(1)}_{\mathrm{BU}(K)} {}^{\#} \longrightarrow BU$$



M Δ-able mfld $\boxed{\text{MANIFOLDS}}$

Def $\chi_s(M) = \sup\{\chi_s(K) \mid |K| = M\}$

$g \to \chi_1(M_g)$ $g \to \chi_1(N_g)$
$g \to \chi_2(M_g)$ $g \to \chi_2(N_g)$

Exercise Determine the functions $g \to \chi_1(M_g), \chi_2(M_g)$

Exercise

Determine the functions $g \to \chi_1(M_g)$ $g \to \chi_2(M_g)$

$g \to \chi_1(N_g)$ $g \to \chi_2(N_g)$

Partial answer:

$\chi_1(S^2) = 4$ $\quad \chi_1(S_g) = H(g)$ $\quad g \geq 1$ ✓

What about χ_2?

$$\chi_2(M_g) \leq \left\lceil \tfrac{1}{2}\chi_1(M_g) \right\rceil$$

Exact answer:

$\chi_2(S^2) = 2$ $\quad \chi_2(M_1) = \chi_2(N_1) = \chi_2(N_2) = 3$

Only cases when $\chi_2(M)$ is known

$\chi_2(\dot{M}_1) = \chi_2(\dot{N}_1) = 3 \Rightarrow$

$T_1 \# \mathbb{R}P^2 = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$

$\chi_2(S_g) \geq 3$ for $g \geq 1$

Theoretical result

The 2-chromatic number of a surface can get arbitrarily large.

Kündgen & Ramamurthi

$$\sup\{\chi_2(M) \mid M^2\} = \infty$$

Show me a surface ~~with the~~ M with $\chi_2(M) \geq 4$!

There is an ~~nonorien~~ orientable surface M of $g=20$ with $\chi_2 \geq 4$

— " — nonorientable — " — N_{26} of $g=26$ with $\chi_2 \geq 4$.

The same lower bound holds for $\dot{M}_{20}$ and $\dot{N}_{26}$ $\quad \chi_2(\dot{M}_{20}) \geq 4$ $\quad \chi_2(\dot{N}_{26}) \geq 4$

<u>Prop</u> $\chi_2(M_g) \geq 4$ if $g \geq 20$ and $\chi_2(N_g) \geq 4$ if $g \geq 26$

<u>Question 1</u> Is $\chi_2(M_g) = 3$ for all g with $1 \leq g \leq 19$?

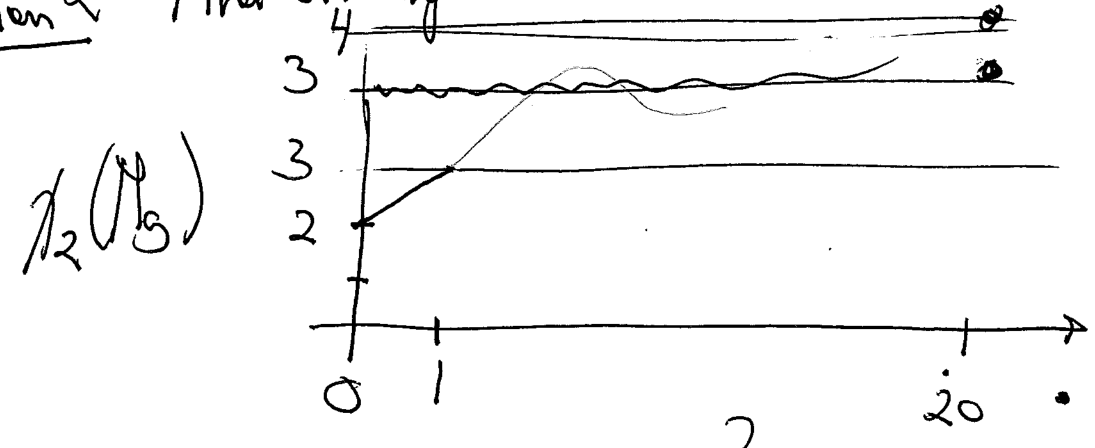
Is $\chi_2(N_g) = 3$ — " — $1 \leq g \leq 25$ No

concrete example

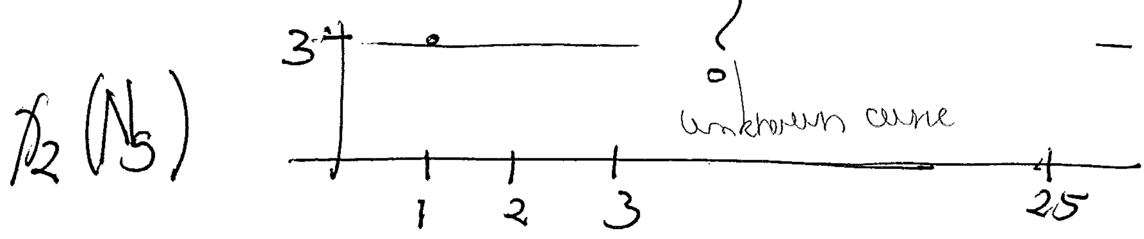
<u>Question 2</u> Find an explicit triangulation K of a surface with $\chi_2(K) \geq 5$

Must exist

Return to this later



No orientable surface is known



Chromatic numbers of spheres (higher chromatic numbers)

$\chi_2(S^2) = 2$ 4CT. ~~Assume~~ Look at S^d with $d \geq 3$

$\chi_1(S^3) = \infty$ - in fact

Lemma ~~that~~ $\chi_s(S^d) = \infty$ if $1 \leq s \leq \lceil d/2 \rceil$

Proof Use fact there are embedded simplicial complexes with high χ.

$\partial CP(m, d+1)$ boundary of cyclic polytope

Moment curve $\square$

The known triangulations of S^3 have $\chi_3 \leq 4$. Conjecture: $\chi_3 = 4$.

For S^3 ~~the~~ $\chi_2(S^3)$ is the only interesting number — what is $\chi_2(S^3)$?

The topological version involves the complete graph K_5

Is it finite?

Thm $\chi_3^3(S^3) \geq 5$.

$f = (167, 1579, 2824, 1412)$ smallest known example Martin Tancer

In the literature goes on find examples when $\chi_2 = 4$

(Altshuler: peculiar combinatorial 3-sphere)

Relation between χ_1 and χ_2

$\partial CP(4,m)$ boundary of cyclic polytope in $\mathbb{R}^4$ on m-vertices

$$\chi_2(\partial CP(4,m)) = \begin{cases} 2 & m \text{ even} \\ 3 & m \text{ odd} \end{cases}$$

but $\chi_2(\partial CP(4,m)) = m$

The difference can be arbitrarily large

Exercise Determine the function $d \longrightarrow \chi_d(S^d)$.

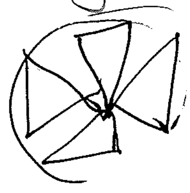
$(a+b)+c=0$
if $a=b$ then $c=0$
$abc\neq 0$

Return to surfaces

K & R say that there's a surf H with $\chi_2(H) \geq 5$.

Where is it? Thousands of surfaces from Frank's library

they all have $\chi_2 \leq 4$. Need:



One possibility is to use Steiner Triple Systems as ~~colorings~~ There is some info about colorings. Simplicial Complex

$\mathbb{F}_{2^d}$ $\quad PG(2^d) = \{(a,b,c) \in \mathbb{F}_{2^d}^\times \mid a+b+c=0, \cancel{abc\neq 0}\}$ on $V = \mathbb{F}_{2^d}^\times$

STS: 2-dim simplicial complex on 2^d-1 points so that every 1-simplex is contained in precisely 1 2-simplex

complete 1-skeleton Exercise Find $\chi_2(PG(2^d))$! but clearly:

FACTS: $\chi_2(PG(2^d)) \to \infty$ for $d \to \infty$, $\chi_2(PG(2^d)) < d$

$\chi_2(PG(16)) = 3$

$\chi_2(PG(128)) \geq \chi_2(PG(64)) = 5$

$2^7 \qquad 2^6$

$64 = 2^6$
But 6 is not a prime!

$2^3 = 8$
$2^4 = 16$
$2^5 = 32$
$2^6 = 64$
$2^7 = 128$

But $PG(2^d)$ is only half a surface. A ~~transversal is a~~ surface polynomial $P = \sum a_i X^i$ is a polynomial inducing a permutation of $\mathbb{F}_{2^d}^\times$ such that $PG(2^d) \cup PG(2^d)^P$ is a surface. (orientable or not)

$\{(a,b,c) \mid a+b+c=0\} \cup \{(P(a),P(b),P(c)) \mid a+b+c=0\} =$ surface

Then we'll have

$$\chi_2\left(PG(2^d) \cup PG(2^d)^P\right) \geq \chi_2(PG(2^d))$$

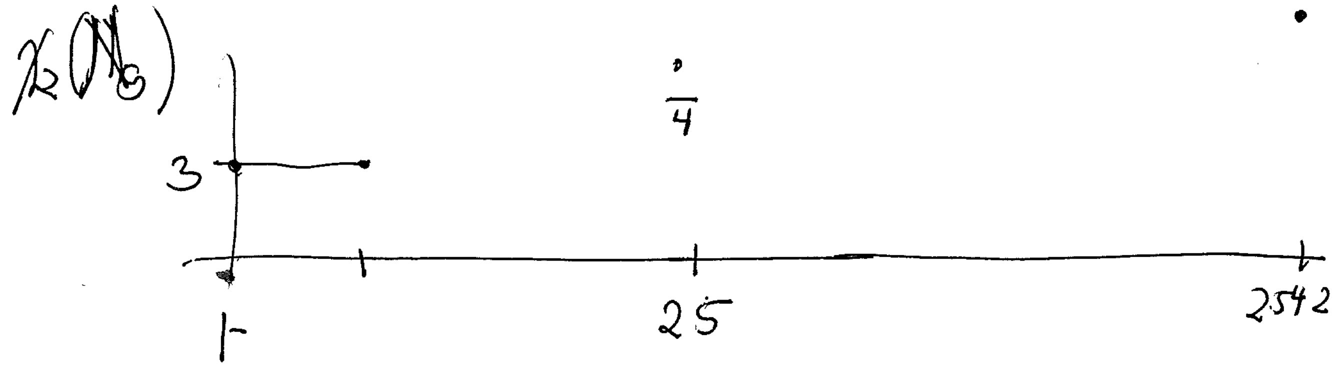
$$128 = 2^7$$

~~Ringel (d=5) Theorem~~

There is a polynomial $P = X^7$ such that $PG(2^7) \cup PG(2^7)^P$
$d=7$

is a nonorientable surface of genus 2542 on 127 vertices

$f =$



Question: Is it really the fact
$\chi_2(N_g) \leq 5$ for $g < 2542$?

Higher dimensional Chromatic polynomials

$\chi(K, r, s) = \#$ of (r,s)-colorings of K

$$\chi(K,r,s) = \sum_{i=0}^{m} S(K,i,s)[r]_i \qquad \sum b_i r^i = \chi(K,r,s) - \sum a_i [r]_i$$

Stirling numbers of K: # of equivalence classes on $V(K)$ with i classes not containing any s-simplex of K.

If $\dim K = 1$ we have a graph. $a_i(K)$

$$\sum b_i^{(K)} r^i = \chi(K,r,1) = \sum S(K,i,1)[r_i]$$

MF monomial form — FF form falling factorials

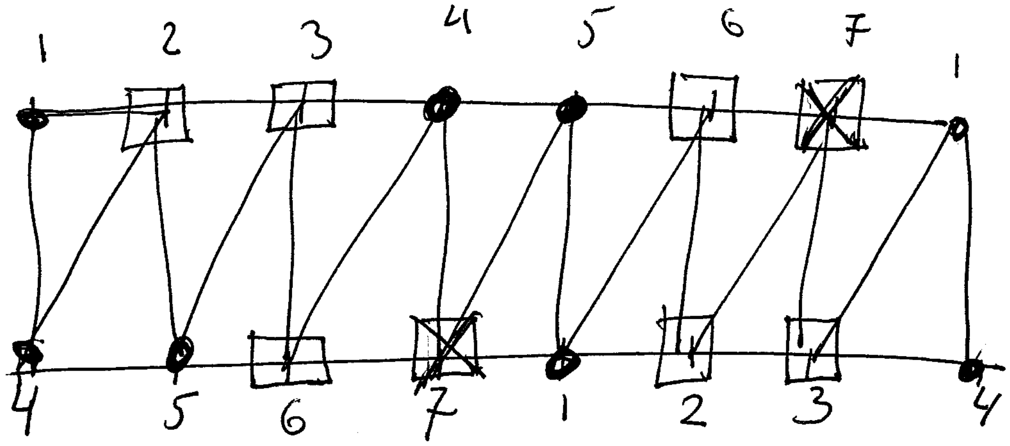
1) The ~~MF~~ coef $b_i^{(K)}$ are alternate in signs ✓

2) $\chi(K,r,1)$ has no negative roots except 0 ✓

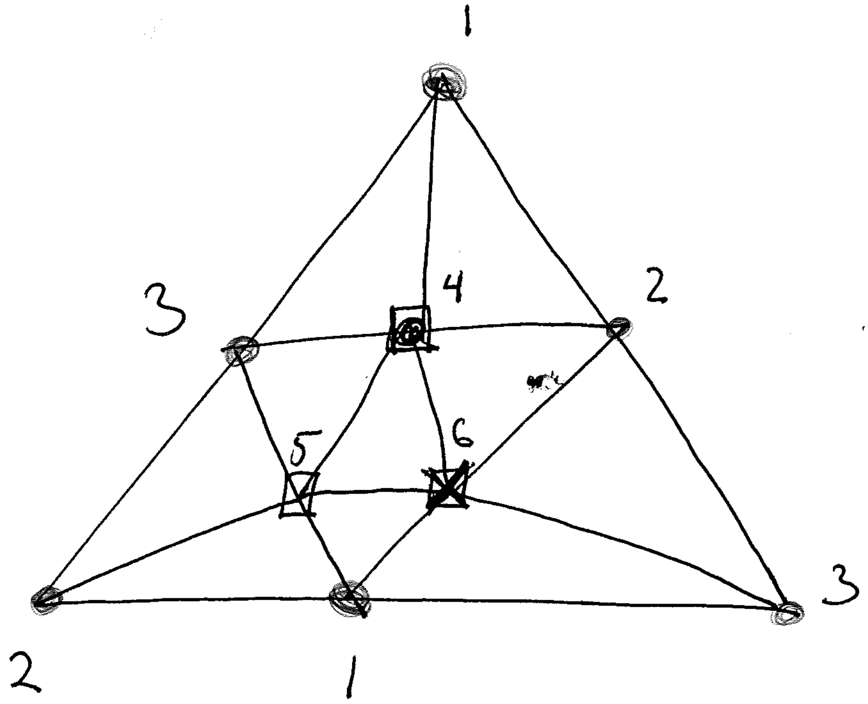
3) $\chi(K,r,1) > e\,\chi(K,r-1,1)$ ✓

4) The coeff $a_i(K)$ are log-concave. Conjecture (not well known) ✓ THEY ARE IN THOUSANDS OF CASES

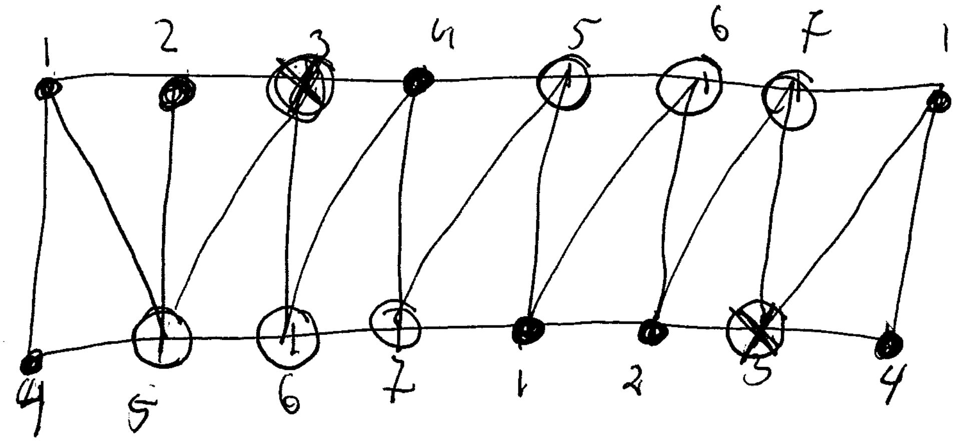
2-chromatic polynom of a triangulated torus that does not satisfy 1, 2, 3 — but 4) — THE ONLY property that might survive! But unknown



$S^1 \times S^1$



$\mathbb{RP}^2$



torus