$g =$

12.24

The Euler char of this is

$$\sum_{g \in G} \chi\left(C_X(g) / C_G(g)\right) = \chi_2(\Delta, G)$$

$\mathcal{S}_G^{p+*}$ = poset of

Thévenaz
Knörr-Robinson
Atiyah-Segal
Counting stuff
~~Alexy~~ Adem + Cohen

Change p to S
More emphasis on own results!

$$k(G) \geq k_{p'}(G) \geq z_p(G)$$

(G, p)

AWC$_p$
KRC$_p$

Really $N_G(P)$

$$|G \, C_G(P)| \longrightarrow \mathcal{S}_G^{p+*} \longrightarrow \mathcal{S}_G^{p+*}/G \ni PG$$

$$|G \, C_G(\sigma)| \rightarrow \Delta(\mathcal{S}_G^{p+*}) \longrightarrow \Delta(\mathcal{S}_G^{p+*})/G \ni \sigma G$$

$k(G) = \#(\text{conjugacy classes in } G) = \#(\text{irr. } \mathbb{C}G\text{-modules})$

$k_{p'}(G) = \#(\text{\_\_\_\_\_\_\_\_ of } p'\text{-order}) = \#(\text{irr } \overline{\mathbb{F}}_pG\text{-modules})$

$z_p(G) = \#(\text{irreducible } \mathbb{C}G\text{-modules of dim a multiple of } |G|_p) =$

$\#(\text{irr } \overline{\mathbb{F}}_pG\text{-modules})$

Example $G = SL_3(\mathbb{F}_2)$ $\quad p=2 \quad |G| = 168 = 8\cdot 21$

$$k(G) = \quad \underset{6}{k(G)} \geq \underset{4}{k_{p'}(G)} \geq \underset{1}{z_p(G)}$$

$$\mathcal{S}_G^{p+*}/G = [C_2, \; C_2\times C_2, \; C_2 \times C_2, \; C_4, \; D_8$$

$$N_G(P)/P \quad \cancel{C_2} \times C_2 \quad \Sigma_3 \quad \Sigma_3 \quad C_2 \quad 1$$

$$z_p(N_G(P)/P) \quad 0 \quad 1 \quad 1 \quad 0 \quad 1$$

$$\sum_{P \in \mathcal{S}_G^{p+*}/G} z_p(N_G(P)/P) = 3 = k_{p'}(G) - z_p(G)$$

orbit subgroups

	$\Delta(\mathcal{S}_G^{p+*})_d/G$	$C_G(\sigma)$
$d=0$	5	$3\times D_8, \; 2\times \Sigma_4$
$d=1$	9	$\cancel{2\times (C_2\times C_2)}, \; 7\times D_8$
$d=2$	5	$\cancel{2\times (C_2\times C_2)}, \; 3\times D_8$

$$\sum_{\sigma \in \Delta(\mathcal{S}_G^{p+*})/G} (-1)^{d(\sigma)} k(C_G(\sigma)) = -k(D_8) + 2k(\Sigma_4) = -5 + 2\times 5 = 5$$
$$= k(G) - z_p(G)$$

$$|G|_p \,|\, \tilde{\chi}(\mathcal{S}_G^{p+*})$$

Also: $\chi(\Delta(\mathcal{S}_G^{p+*})/G) = 5+9+5 = 1 \qquad |G|_p \,|\, \tilde{\chi}(\mathcal{S}_G^{p+*}) = -8$

$$\chi(\mathcal{S}_G^{p+*}) = \sum_{\sigma \in \Delta/G} (-1)^{d(\sigma)} |G : C_G(\sigma)| = -|G : D_8| + 2|G : \Sigma_4|$$
$$= -21 + 2\cdot 7 = -7$$

cosmic coincidence

$AWC_p(G)$ (1986) $\sum_{\sigma} z_p(N_G(\sigma)/\sigma) = k_{p'}(G) z_p(G)$

$KRC_p(G)$ (1988) $\sum (-1)^{d(\sigma)} k(C_G(\sigma)) = k(G) z_p(G)$

Equivalent

$\cancel{AWC_p(G) \Leftrightarrow KRC_p(G)}$

$\forall P \leq G : AWC_p(N_G(P)/P) \Leftrightarrow \forall P \leq G : KRC_p(G)$

$AWC_p(G) + KRC_p(G)$

AWC_p OK

- G ~~p-solvable~~ solvable or p-solvable
- $G = \Sigma_n$
- G finite grp of Lie type in char p (not in cross char case)
- $GL_n(\mathbb{F}_q)$ for all q
- G with abelian 2-Sylow

$KRC_p(G)$

$KRC_p(G)$ OK

- G solvable
- G finite grp of Lie type in char p

$KRC_p(G)$ look like an Euler characteristic

Believed to be a cosmic coincidence

AWC_p for p-group P:

$z_p(P) = \begin{cases} 1 & P = 1 \\ 0 & P \neq 1 \end{cases}$

$\sum_{1 \leq Q \leq P} z_p(N_P(Q)/Q) = z_p(1) = 1$
$= k_{p'}(P) = k_{p'}(P) z(P)$

$AWC_p(G) \Rightarrow AWC_p(G/O_p(G))$

for any normal p-subgrp $P \triangleleft G$

Some of us are reminded about

$\operatorname{hocolim}_{\sigma \in \Delta(\mathcal{S}_G^{p+*})/G} BC_G(\sigma) \xrightarrow{H_*\mathbb{F}_p} BG$ (Dwyer)

Also:

Webb Thm: $\chi(\mathcal{S}_G^{p+*}/G) = 1$

Brown Thm: $|G|_p \mid \chi(\mathcal{S}_G^{p+*})$

Equivariant Euler characteristic

$\Delta \circlearrowleft G$ $\quad$ Δ G-simplicial complex

$$\chi_r(\Delta,G)=\frac{1}{|G|}\sum_{X\in Hom(\mathbb{Z}^r,G)}\chi(C_\Delta(X))$$

$r=1$ $\quad \chi_1(\Delta,G)=\frac{1}{|G|}\sum_{g\in G}\chi(\Delta^g)=\chi(\Delta/G)$

$r=2$ $\quad \chi_2(\Delta,G)=\frac{1}{|G|}\sum_{gG\in G/G}\chi_1(C_\Delta(g),C_G(g))=\frac{1}{|G|}\sum\chi(C_\Delta(g)/C_G(g))$

$$=\dim K^0_G(\Delta)\otimes\mathbb{C}-\dim_{\mathbb{C}} K^1_G(\Delta)\otimes\mathbb{C}$$

$$K^*_G(X)\otimes\mathbb{C}=\Big[\bigoplus_{g\in G}K(X^g)\otimes\mathbb{C}\Big]=\Big[\bigoplus_{g\in G}H^*(X^g,\mathbb{C})\Big]^G$$

In general:

$$\chi_r(\Delta,G)=\frac{1}{|G|}\sum_{X\in Hom(\mathbb{Z}^r,G)}\ \sum_{\sigma\in C_\Delta(X)}(-1)^{d(\sigma)}$$

$$=\dots$$

$$=\sum_{\cancel{X\in Hom(\mathbb{Z}^{r-1},G)/G}\ \ \sigma G\in\Delta/G}(-1)^{d(\sigma)}\,|Hom(\mathbb{Z}^{r-1},C_G(\sigma))/C_G(\sigma)|$$

In particular f $r=2$:

$$\chi_2(\Delta,G)=\sum_{\sigma G\in\Delta/G}(-1)^{d(\sigma)}\,k(C_G(\sigma))$$

$\chi_1(\{\cdot\},G)=1$

$\chi_2(\{\cdot\},G)=k(G)$

$\chi_r(\{\cdot\},G)=|Hom(\mathbb{Z}^{r-1},G)/G|$

Back to $KRC_p(G)$: Let $\Delta = \Delta(\mathcal{S}_G^{p+*})$ and we get the $KRC_p(G)$

$$\chi_2(\mathcal{S}_G^{p+*}, G) = k(G) - z_p(G)$$

$KRC_p(G)$ is more topological than $AWC_p(G)$

Can replace $\mathcal{S}_G^{p+*}$ by $\mathcal{S}_G^{p+rad+*}$ (look back at $G = SL_3(F_2)$)

$\Delta_d(\mathcal{S}_G^{p+rad+*})/G$		$C_G(\sigma)$	$\sum_{\sigma G \in \Delta(\mathcal{S})/G} k(C_G(\sigma))$
$d=0$	3	$D_8,\ 2\times\Sigma_4$	
$d=1$	2	$2\times D_8$	$= -k(D_8) + 2k(\Sigma_4) = 5$

$KRC_p(G)$ when $O_p(G) \neq 1$:

$\mathcal{S}_G^{p+*} \underset{G}{\simeq} 1$ so $\tilde{\chi}_2(\mathcal{S}_G^{p+*}, G) = \tilde{\chi}_2(1, G) = |Hom(\mathbb{Z}^2, G)|/|G| = |Hom(\mathbb{Z}^2, G)/G|$

$= k(G)$

Ito's thm implies that $z_p(G) = 0$. So $KRC_p(G)$ is true.

- FALSE proof of the $AWC_p(G)$:

$KRC_p(G \times C_p) \underset{\text{equivalent}}{\overset{\text{conjecture}}{\Longleftrightarrow}} AWC_p(G \times C_p) \Rightarrow AWC_p(G)$

This is where I went off a tangent.

$\chi_1(\mathcal{S}_G^{p+*}, G) = 1$

$\chi_2(\mathcal{S}_G^{p+*}, G) = k(G) - z_p(G)$

- $\chi_3(\mathcal{S}_G^{p+*}, G) = ?$

Maybe $KRC_p(G)$ is just first layer of a series of conjectures!

Work here with reduced Euc $\tilde{\chi}_r$

Examples of $\tilde{\chi}_r(\mathcal{S}_G^{p+*}, G)$ $\qquad p = \mathrm{char}(\mathbb{F}_q)$

$$1 \pm \sum \tilde{\chi}_{r+1}(\mathcal{S}^{p+*}_{GL_n^\pm(\mathbb{F}_q)}, GL_n^\pm(\mathbb{F}_q)) \frac{x^n}{n} = \prod_{0 \le j \le r} (1 + (\pm 1)^j q^{r-j} x)^{(-1)^j \binom{r}{j}}$$

$$1 - \sum \tilde{\chi}_{r+1}(\mathcal{S}^{p+*}_{Sp_{2n}(\mathbb{F}_q)}, Sp_{2n}(\mathbb{F}_q)) \frac{x^n}{n} = \prod_{\substack{0 \le j \le r \\ j \not\equiv r \bmod 2}} (1 - q^{r-j} x)^{\binom{r}{j}}$$

For $GL_n(\mathbb{F}_q)$ and $r = 3$

$$\prod_{0 \le j \le 3} = \frac{(1 - q^3 x)}{(1 - q^2 x)^3} \frac{(1 - qx)^3}{(1 - x)}$$

Some of these have occurred before in maths as Hasse-Weil zeta functions.

Reduced equiv. Euc

$$\tilde{\chi}_r(\Delta, G) = \chi_r(\Delta, G) - \chi_r(P, G) = \chi_r(\Delta, G) - |\mathrm{Hom}(\mathbb{Z}^r, G)|/|G|$$

p-primary ~~Equiv~~. Euler characteristics

$$\chi_r^p(\Delta, G) = \cancel{\Delta} \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}_p^{r-1}, G)} \chi(C_\Delta(X))$$

<u>Thm</u> $\chi_r^p(\Delta, G)$ is the Euc of $\Delta_{hG} = (\Delta \times EG)/G$ computed in Morava $K(r)$-theory at p.

$$1 \pm \sum \chi_{r+1}^p(\mathcal{S}_{GL_n^\pm(\mathbb{F}_q)}, GL_n^\pm(\mathbb{F}_q)) \frac{x^n}{n} = \exp\left(-\sum_{n \geq 1} (q^n - (\pm 1)^n)_p^r \frac{x^n}{n}\right)$$