p-singular elements in finite groups of Lie type of defining char p

Episode 1–6

Episode 1 Frobenius 1907 thm

Episode 2 ~~Euler~~-Euler chars of finite cats and Quillen's 1978 thm

Episode 3 Finite groups of Lie type and the 1971 Borel-Tits thm

Episode 4 ~~Borel-Tits~~ Solomon-Tits 1969 thm

Episode 5 Solomons 1966 polynomial identity

Episode 6 Cross-characteristic case

Frobenius 1907 thm Episode 1

G finite group

p prime number $p \mid |G|$

~~Defn~~ $g \in G$ is p-singular : the order of g ~~is a~~ $1, p, p^2, p^3, \dots$

G_p = set of p-singular elements $= \{g \in G \mid g \text{ is } p\text{-singular}\}$

$$= \bigcup_{S \in Syl_p(G)} S = \{g \in G \mid g^{|G|_p} = e\}$$

Frobenius 1907

$$|G|_p \mid |G_p|$$

Problem If G is a finite group of Lie type in char p, what is $|G_p|$?

In fact Frobenius proved more: If $n \mid |G|$ then $n \mid |\{g \in G \mid g^n = e\}|$

$n \mid |G| \Rightarrow n \mid |\{g \in G \mid g^n = e\}|$

Two cases where answer is known:

Example (Stanley : Combinatorics) Symmetric group $\Sigma_n = G$

Conjugacy classes in $\Sigma_n / \Sigma_n \longleftrightarrow$ partitions of n $\quad n_1^{e_1} n_2^{e_2} \cdots \longleftrightarrow \underbrace{(n_1)\cdots(n_1)}_{e_1}\underbrace{(n_2)\cdots(n_2)}_{e_2}$

~~Conjugacy classes of p-singular elements~~ $(\Sigma_n)_p / \Sigma_n \longleftrightarrow$ p-partitions of n

$$\sum_{n \geq 1} |(\Sigma_n)_p| \frac{X^n}{n!} = \exp\left(X + \frac{X^p}{p} + \frac{X^{p^2}}{p^2} + \frac{X^{p^3}}{p^3} + \cdots\right)$$

$$|(\Sigma_n)_p| = \begin{cases} p=2 & 1, 2, 4, 16, 56, 256, 1072, \dots \\ p=3 & 1, 1, 3, 9, 21, 81, 351 \end{cases}$$

Thm $|SL_n(\mathbb{F}_q)_p| = |SL_n(\mathbb{F}_q)|_p^2 \quad q = p^e$

What about finite groups of Lie type?

Episode 2 Leinster–Euler char of finite cats and Quillen's thm 1978

$\mathcal{C}$ finite category $a,b \in Ob(\mathcal{C})$ $\mathcal{C}(a,b)$ = morphisms $a \to b$

$\mathcal{C}(a,a) = \mathcal{C}(a)$

$$\zeta(\mathcal{C}) = \left(|\mathcal{C}(a,b)| \right)_{(a,b) \in Ob(\mathcal{C})}$$

$$(\zeta(\mathcal{C})) \begin{pmatrix} \vdots \\ k^b \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \qquad (\cdots k_a \cdots)\,(\zeta(\mathcal{C})) = (1 \cdots 1)$$

we $\qquad\qquad$ cowe

$\mathcal{C}$ has a we $k^\bullet$ and a cowe $k_\bullet$.

<u>Defn</u> (Tom Leinster 2008) $\mathcal{C}$ has Euc if

$$\chi(\mathcal{C}) = \sum k^b = \sum k_a$$

We are interested in $\mathcal{C} = \mathcal{S}_G^{p+*}$, $\mathcal{O}_G^p$

$\mathcal{S}_G^{p+*}$ = Brown poset of nontrivial p-subgroups $H \le K \Leftrightarrow H \subseteq K$

$\mathcal{O}_G^p$ = orbit cat of p-subgroups $\mathcal{O}_G(H,K) = \{g \in G \mid H^g \le K\}/K$ $\mathcal{O}_G(H) = N_G(H)/H$

<u>Quillen 1978</u> $O_p(G) \neq 1 \Rightarrow \tilde{\chi}(\mathcal{S}_G^{p+*}) = 0$

<u>Martin & JMM 2012</u>

$$\frac{1}{|G|}|G_p| = \chi(\mathcal{O}_G^p) = \frac{1}{|G|} \sum_{H \le G} -\tilde{\chi}(\mathcal{S}^{p+*}_{\mathcal{O}_G(H)})|H|$$

Formula for $|G_p|$:

$$|G_p| = \sum_{H \le G} -\tilde{\chi}(\mathcal{S}^{p+*}_{\mathcal{O}_G(H)})|H| = \sum_{[H]} -\tilde{\chi}(\mathcal{S}^{p+*}_{\mathcal{O}_G(H)})|H|\,|G : N_G(H)|$$

Only H with $H = O_p N_G(H)$ needed radical

Radical and parabolic subgroups of finite groups of Lie type and the Borel-Tits thm

q is a power of prime p. $q = p^e$

(Σ, Π) irreducible crystallographic root system with fundamental roots Π $\Sigma = \Sigma^- \cup \Sigma^+$

$\overline{K}$ algebraic group over $\overline{\mathbb{F}}_p$ with root system Σ

$K = \Sigma(q) = C_{\overline{K}^F} O^{p'}\{x \in \overline{K} \mid x^F = x\}$ (untwisted case)

Example: $\boxed{SL_n(\mathbb{F}_q), SO_{2n+1}(\mathbb{F}_q), Sp_{2n}(\mathbb{F}_q), SO_{2n}(\mathbb{F}_q), \ldots}$

For every $J \subseteq \Pi$ (set of fundamental roots) there are subgroups P_J, U_J, L_J of $K = P_\Pi = L_\Pi$ such that

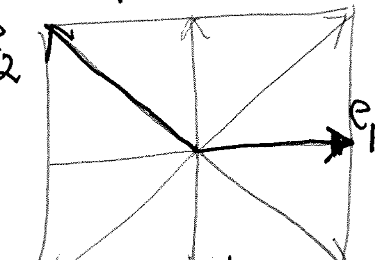
fame.pdf p21
sum.pdf p26, p32

parabolic $\rightarrow$ $P_J = N_K(U_J)$ $\quad U_J = O_p(P_J)$ (U_J is radical)

$$P_J = U_J \rtimes L_J \supseteq P_\emptyset = \underbrace{U_\emptyset}_{U} \rtimes \underbrace{L_\emptyset}_{H} \qquad U_\emptyset \in Syl_p(K)$$

$\Sigma(q) = P_\Pi = L_\Pi$

Borel-Tits thm [1971] The U_J, $J \subseteq \Pi$, are a complete set of representatives for the p-radical subgroups of $\Sigma(q)$.



Root system

Σ
Π

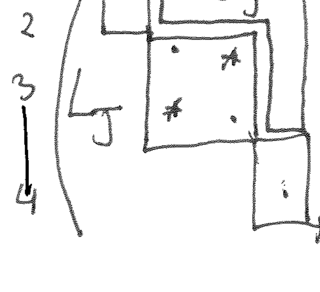
Cor. 3.1.5
Borel-Tits: 3.13
Thm 2.6.5

For the p-radical U_J

$$\mathcal{O}_{\Sigma(q)}(U_J) = N_K(U_J)/U_J = P_J/U_J = L_J$$

The formula is

$$|\Sigma(q)_p| = \sum_{J \subseteq \Pi} \overbrace{-\tilde{\chi}(\mathcal{S}_{L_J}^{p+*})}^{\text{what is this?}} |U_J| \, |P_\Pi : P_J|$$



The Solomon-Tits theorem

Episode 4

<u>Thm</u> (1969) The $\mathcal{S}^{p+*}_{\Sigma(q)}$ is a spherical poset: The topological realization of $\mathcal{S}^{p+*}_{\Sigma(q)}$ is a wedge of $|\Sigma(q)|_p$ spheres of dimension $|\Pi|-1$

$$-\tilde{\chi}(\mathcal{S}^{p+*}_{L_\Pi}) = (-1)^{|\Pi|}\,|P_\Pi|_p$$

Generalization: (proved by gut-feelings)

$$-\tilde{\chi}(\mathcal{S}^{p+*}_{L_J}) = (-1)^{|J|}\,|L_J|_p$$

QLS III p 66
Replace L_J by $O^{p'}L_J = H_J$

It follows that

$$-\tilde{\chi}(\mathcal{S}^{p+*}_{L_J})\,|U_J| = (-1)^{|J|}\,|U_J|\,|L_J|_p = (-1)^J\,|P_J|_p = (-1)^J\,|\Sigma(q)|_p$$

The formula is now

$$|\Sigma(q)_p| = \sum_J \cancel{-\tilde{\chi}(\mathcal{S}^{p+*}}\,(-1)^J\,|\Sigma(q)|_p\,|P_\Pi : P_J|$$

$$= |\Sigma(q)|_p \sum_J (-1)^J\,|P_\Pi : P_J| =$$

$$= \cancel{|\Sigma(q)|_p \sum_J (-1)^J \frac{|P_\Pi : P_\emptyset|}{|P_J : P_\emptyset|}}$$

Episode 5 Solomon's 1966 polynomial identity

Thm (Solomon) $$\sum_{J\subseteq\Pi} (-1)^{J}\, |P_\Pi : P_J| = q^{|\Sigma_\Pi^+|} = |\Sigma(q)|_p$$

As $|\Sigma(q)|_p = q^{|\Sigma_\Pi^+|}$ we conclude

$$|\Sigma(q)_p| = |\Sigma(q)|_p \sum (-1)^J |P_\Pi : P_J| = |\Sigma(q)|_p^2$$

$|G : N_G(H)|$

Another polynomial identity

general fact

$$1 = \sum_{1\le H\le G} -\tilde{\chi}\left(\mathcal{S}^{\wedge+*}_{O_G(H)}\right)^{G=\Sigma(q)} = \sum_{J\subseteq\Pi} -\tilde{\chi}\left(\mathcal{S}^{\wedge+*}_{L_J}\right) |P_\Pi : P_J|$$

$$-\tilde{\chi}\left(\mathcal{S}^{\wedge+*}_{L_J}\right) = (-1)^{|J|} q^{|\Sigma_J^+|} \qquad = \sum_{J\subseteq\Pi} (-1)^J q^{|\Sigma_J^+|} |P_\Pi : P_J|$$

Why are they called polynomial identities?

Stanley 37

A crystallographic reflection group $W = W_\Pi$ has degrees:

$$\mathbb{Q}[t_1,\dots,t_n]^W = \mathbb{Q}[f_1,\dots,f_n] \qquad D(W) = \{|f_i|\}$$

q-analogue of d

q-integer

The degree polynomial of $W = W_\Pi$ is

$$DP_\Pi(q) = \prod_{d_i\in D(W)} \frac{q^{d_i}-1}{q-1} = \prod [d_i]_q = |P_\Pi : P_\emptyset|$$

More general: $$DP_J(q) = \prod_{d_i\in D(W_J)} [d_i]_q = |P_J : P_\emptyset|$$

So $$|P_\Pi : P_J| = \frac{|P_\Pi : P_\emptyset|}{|P_J : P_\emptyset|} = \frac{DP_\Pi(q)}{DP_J(q)}$$

Simply: $$|P_J : P_\emptyset| = \prod_{d\in \text{degrees } W_J} (d)_q$$

Episode 6

Crosscharacteristic case $p \nmid q$

for $SL_n(\mathbb{F}_q)$

$$\left| SL_n(\mathbb{F}_q)_p / SL_n(\mathbb{F}_q) \right| \doteq \sum_{\lambda \vdash n} T(\lambda) \prod_{b \in \lambda} (q^b - 1)_p$$

$T(\lambda)$ = number of elements in Σ_n of cycle type λ