

June 1 2012

Homotopy guidelines between p -subgroup categories Bakula (joint w. Matthe Cohen)

G finite group
 p prime $p \mid |G|$

Maybe this can be used
to distinguish between
form/linking systems

\mathcal{L}_G : poset of p -groups $H \leq G$

\mathcal{L}_G^* : \longrightarrow \longrightarrow , $H \neq 1$ ($* \neq 1$) (Brunnet)

Brown $|G|_p \mid \tilde{\chi}(\mathcal{L}_G^*)$

Quillen (78) \mathcal{L}_G^* noncontractible $\Rightarrow \mathcal{O}_p G = 1$

Quillen + Bore $\mathcal{L}_G^{*+rad} \xrightarrow[\text{Bore}]{\cong} \mathcal{L}_G^* \xleftarrow[\text{Quil}]{\cong} \mathcal{L}_G^{*+eab}$

$H \leq G$ is G -radical if $H = \mathcal{O}_p N_G(H)$

Cordary $\mathcal{L}_G^{*+rad} \cong \mathcal{L}_G^{*+eab}$ (homotopy equiv)

$$\sum_{\substack{H \text{ radical} \\ N_G(H)/H}} -\tilde{\chi}(\mathcal{L}_{N_G(H)/H}^*) = \sum_{K \text{ eab}} -\mu(K)$$

(combinatorics)

How could this have been discovered? (counterfactual) ②

Weightings, coweightings, and Euler characteristics (Tom Leinster)

\mathcal{C} finite category

$$\left(|\mathcal{C}(a,b)| \right) \begin{pmatrix} \vdots \\ k \\ \vdots \end{pmatrix} = \begin{pmatrix} p \\ \vdots \\ p \end{pmatrix} \quad \forall a: \sum |\mathcal{C}(a,b)| k^b = p \quad (\text{weighting})$$

$$\left(\dots k_a \dots \right) \left(|\mathcal{C}(a,b)| \right) = (1 \dots 1) \quad \forall b: \sum k_a |\mathcal{C}(a,b)| = 1 \quad (\text{coweighting})$$

If \mathcal{C} has we and cover fun $\text{Ob}(A) \geq \text{supp}(k_a)$ or $\text{Ob}(A) \geq \text{supp}(k^a)$ for

$$\sum k_a = \chi(\mathcal{C}) = \sum k^b$$

FACT: $k^H = -\tilde{\chi}(\mathcal{C}_{N(H)/H})$ weighting for \mathcal{C}_G^*

$k_K = -\tilde{\chi}(\mathcal{C}_{K/K}) = \mu(K)$ coweighting for \mathcal{C}_G^*

The weightings vanish off the radical groups: $\chi(\mathcal{C}_G^{\text{rad}}) = \chi(\mathcal{C}_G^*)$

The coweighting vanishes off the elementary abelian subgroups: $\chi(\mathcal{C}_G^{\text{eab}}) = \chi(\mathcal{C}_G^*)$

So we have inclusions $\text{supp}(k^a) = \{H \mid \tilde{\chi}(\mathcal{C}_{N(H)/H}) \neq 0\} \subseteq \{H \mid H \text{ is } G\text{-rad}\}$

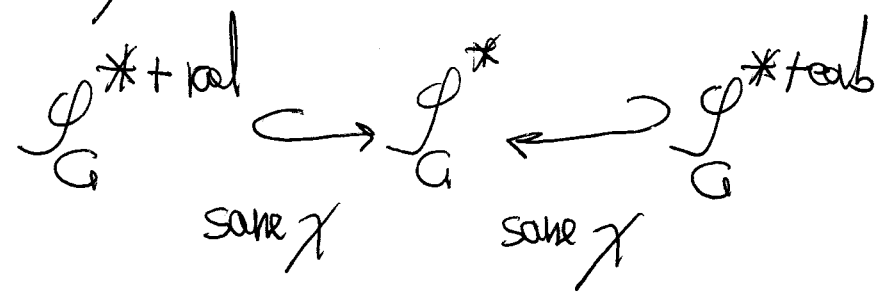
$\mathcal{C}_G^{\text{rad}} \xrightarrow{\text{same weighting}} \mathcal{C}_G^* \xleftarrow{\text{same coweighting}} \mathcal{C}_G^{\text{eab}}$ $\text{supp}(k^a) = \{K \mid \mu(K) \neq 0\} = \{K \mid K \text{ is elementary abelian}\}$

That is all these posets have identical Euler characteristics. $\text{is elementary abelian}$

Lemma \mathcal{C} finite category with full subcat \mathcal{A} . Then $\chi(\mathcal{A}) = \chi(\mathcal{C})$ if $\text{Ob}(\mathcal{A}) \supseteq \text{supp}(k_\bullet)$ or $\text{Ob}(\mathcal{A}) \supseteq \text{supp}(k_\bullet)$ (or $\text{supp}(k_\bullet)$ is a subcategory).
 (and $\mathcal{A} \subseteq \mathcal{C}$ will be a homotopy equivalence)

FACT: Weighting for \mathcal{C} $k^H = -\tilde{\chi}(\mathcal{C}_{N_G(H)/H}^*)$
 $k^K = -\mu(K) = -\tilde{\chi}(\mathcal{C}_{N_G(K)/K}^*)$

$\text{supp}(k^*) = \{H \mid -\tilde{\chi}(\mathcal{C}_{N_G(H)/H}^*) \neq 0\} \subseteq \{H \mid \mathcal{C}_{N_G(H)/H}^* \neq * \} = \{H \mid H \text{ abelian}\}$
 $\text{supp}(k_\bullet) = \{K \mid \mu(K) \neq 0\} = \{K \mid K \text{ is elementary abelian}\}$



Idea Look at other categories of groups, eg \mathcal{F} fusion system and consider inclusions

$$\mathcal{F}^{\text{supp}(k^*)} \subseteq \mathcal{F} \supseteq \mathcal{F}^{\text{supp}(k_\bullet)}$$

should they be homotopy equivalences?

k^* weighting (\mathcal{P})

k_K co-weighting (\mathcal{P}^*) ③

$$-\tilde{\chi}(\varphi_{\mathcal{O}_G(H)}^*)$$

$$-\mu(K) = -\tilde{\chi}(\varphi_K^{(1,K)})$$

$$|N_G(H)|^{-1} \sum_{X \in C_{N_G(H)}(\varphi_{\mathcal{O}_G(H)}^*)} \tilde{\chi}(\varphi_{C_{N_G(H)}(X/H)}^*)$$

$$\frac{-\mu(K)}{|I_G(K)|} \frac{-\mu(K)}{|I_G(K)|}$$

↑ This is a weighting but it is not clear what is the support of this weighting

?

— " —

$$\frac{-\tilde{\chi}(\varphi_{\tilde{I}_G(H)}^*)}{|I_G(H)|}$$

?

$$\frac{-\tilde{\chi}(\varphi_{\tilde{I}_G(H)}^*)}{|L_G(H)|}$$

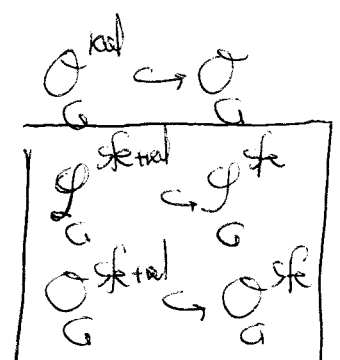
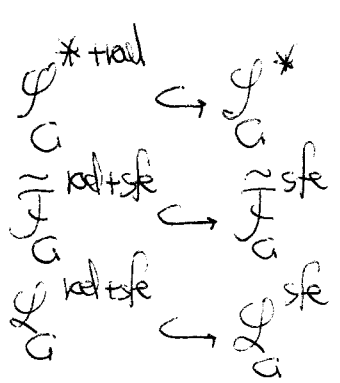
?

Question: $\varphi_{\mathcal{O}_G(H)} = 1$
 $\tilde{\chi}(\varphi_{\mathcal{O}_G(H)}^*) \neq 0 \Rightarrow H$ is G -radical
 $\varphi_{\tilde{I}_G(H)} = 1$

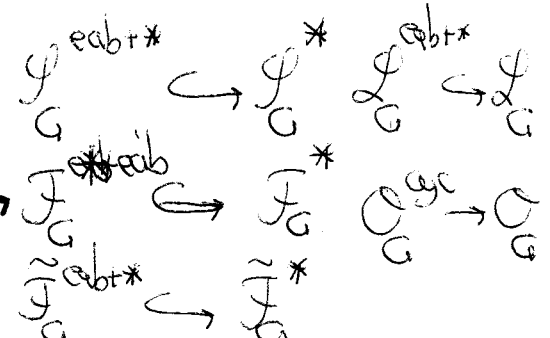
$$\frac{-\mu(K)}{|L_G(K)|}$$

$$\frac{-\tilde{\chi}(\varphi_{\mathcal{O}_G(H)}^*) \tilde{\chi}(\varphi_{\tilde{I}_G(H)}^*) \neq 0 \Rightarrow H \text{ is } \mathbb{F}_q\text{-radical}}{|O_G(H)|}$$

$$\begin{cases} |G|^{-1} K = 1 & \varphi_{\mathcal{O}_G(H)} = 1/K \\ \frac{-1}{|G|} |K| & K > 1 \text{ cyclic} \\ 0 & K \text{ non-cyclic} \end{cases}$$



will try to prove this one \rightarrow



The All of these inclusions - except for $\mathcal{Q}_G^{cyc} \rightarrow \mathcal{Q}_G^!$ - are homotopy equivalences.

Corollary $\mathcal{F}^* \rightarrow \tilde{\mathcal{F}}^*$ is a homotopy equiv

Proof $\cong \begin{matrix} \uparrow & \uparrow \cong \\ \mathcal{F}^{*+orb} & \tilde{\mathcal{F}}^{*+orb} \end{matrix}$

What about proofs?

Avoid on \mathcal{Q}_G

Like for \mathcal{J}_G^* we have both a weighting and a co-weighting - so

$$\frac{1 + (g-1) \sum_{C \text{ cycle}} |C|}{p|G|} = \sum_{[H]} \frac{-\tilde{\chi}(\mathcal{J}_{\mathcal{Q}(H)}^*)}{|\mathcal{Q}_G(H)|} \quad H \text{ Graded}$$

What about proofs?

Proof uses **A** Boer's Key (independent interest)

A Theorem (Boer, Connes, Renault) **B** Any identification of weight/co-weight of \mathcal{F} at \mathcal{H}

Let \mathcal{J} be a finite poset and \mathcal{A} a subset. Suppose that either

(a) $x < \mathcal{J}$ is ^{non} contractible for ~~all~~ x ⁱⁿ ~~outside~~ \mathcal{A} , or,

(b) $\mathcal{J}_{<x}$ \xrightarrow{u} \xrightarrow{v}

Boer's theorem generalizes to finite EI-categories.

Def \mathcal{C} category, x and y objects of \mathcal{C}

(a) $x // \mathcal{C}$ is the category of nonisomorphic morphisms from x $\begin{matrix} \searrow & \rightarrow & \swarrow \\ & y & \\ \end{matrix}$ y_2

(b) $\mathcal{C} // y$ \xrightarrow{u} \xrightarrow{v} $\begin{matrix} x_1 & & x_2 \\ & \searrow & \swarrow \\ & y & \end{matrix}$

Example $\text{supp}(\cdot // \mathcal{C}) = \{x \in \text{Ob}(\mathcal{C}) \mid x // \mathcal{C} \text{ is noncontractible}\}$

$\text{supp}(\mathcal{C} // \cdot) = \{y \in \text{Ob}(\mathcal{C}) \mid \mathcal{C} // y \text{ is noncontractible}\}$

Thm (Bourb for finite EI-categories)

Let \mathcal{C} be a finite EI-category \mathcal{A} a full subcategory closed under isomorphism.
 The inclusion $\mathcal{A} \hookrightarrow \mathcal{C}$ is a homotopy equivalence iff $\mathcal{C}(\mathcal{A})$
~~is~~ $\text{supp}(\cdot // \mathcal{C})$ or $\text{supp}(\mathcal{C} // \cdot)$.

Example

$$J_G^{*+ab} \hookrightarrow J_G^*$$

REMEMBER EXAMPLE!
 (using coweighting)

Need to show: $\mathcal{A} // J_G^*$ is noncontractible $\Rightarrow \mathcal{A}$ is elementary abelian
 So what is $\mathcal{A} // J_G^*$? $J_G^* // K$

B. Identification of weightings/coweightings for EI-categories.

Thm Let \mathcal{C} be a finite EI-category. Then

$$k_x = \frac{-\tilde{\chi}(x // \mathcal{C})}{|\mathcal{C}(x)|} \quad k_y = \frac{-\tilde{\chi}(\mathcal{C} // y)}{|\mathcal{C}(y)|}$$

is a weighting and a coweighting for \mathcal{C} .

Example continued

Condition \mathcal{C} finite EI-cat

$$\sum_{x \in \mathcal{C}} \frac{-\tilde{\chi}(x // \mathcal{C})}{|\mathcal{C}(x)|} = \chi(\mathcal{C}) = \sum_{y \in \mathcal{C}} \frac{-\tilde{\chi}(\mathcal{C} // y)}{|\mathcal{C}(y)|}$$

$$\frac{-\mu(\mathcal{C} // K)}{|\mathcal{C}(K)|} = \frac{-\tilde{\chi}(J_G^* // K)}{|\mathcal{C}(K)|}$$

Guess:

$$J_G^* // K \cong \mathcal{C} // K$$

True for $\mathcal{C} // K$
 $\mathcal{C} \rightarrow \mathcal{C} \rightarrow \mathcal{C} \rightarrow \mathcal{C}$

This is true ~~and~~ Note that $J_G^* // K$ does not depend on J_G !

C. Fixed point

Lemma

$\mathcal{F}_K^{(T,K)}$

is contractible $\Rightarrow K$ is simply connected

Conclusion
Im

$\mathcal{F}^{*+eab} \hookrightarrow \mathcal{F}^*$ is a homotopy equivalence.

Three identities weighting

co-weighting

$$\sum -\tilde{\pi}(\varphi_{G(H)}^*) = \sum -\mu(K)$$

φ_G

$$\sum \frac{-\tilde{\pi}(\varphi_{G(H)}^*)}{|N_G(H)|} = \sum \frac{-\mu(K)}{|N_G(K)|}$$

J_G

$$\sum \frac{-\tilde{\pi}(\varphi_{G(H)}^*)}{|O_G(H)|} = \frac{1 + (p-1) \sum |C|}{p|G|}$$

φ_G