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Homotopy equivalences between p-subgroup categories

(joint w. Matthew Gelvin)

G finite group

p prime $p \mid |G|$

$\mathcal{S}_G$: poset of p-groups $H \leq G$

$\mathcal{S}_G^*$: ——— " ———, $H \neq 1$ $(* = \neq 1)$ (Brown poset)

Maybe this can be used to distinguish between fusion/linking systems

Brown $|G|_p \mid \tilde{\chi}(\mathcal{S}_G^*)$

Quillen (78) $\mathcal{S}_G^*$ noncontractible $\Rightarrow O_p G = 1$

Quillen + Bouc

$$\mathcal{S}_G^{*+rad} \xhookrightarrow[\text{Bouc}]{\simeq} \mathcal{S}_G^* \xhookleftarrow[\text{Quillen}]{\simeq} \mathcal{S}_G^{*+eab}$$

$H \leq G$ is G-radical if $H = O_p N_G(H)$

Corollary

$$\mathcal{S}_G^{*+rad} \simeq \mathcal{S}_G^{*+eab} \quad \text{(homotopy theory)}$$

$$\sum_{H \text{ radical}} -\tilde{\chi}(\mathcal{S}^*_{N_G(H)/H}) = \sum_{K \text{ eab}} -\mu(K) \quad \text{(combinatorics)}$$

How could this have been discovered? (counterfactual history) 2

Weightings, coweightings, and Euler characteristics (Tom Leinster)

$\mathcal{C}$ finite category

$$\begin{pmatrix} |\mathcal{C}(a,b)| \end{pmatrix}\begin{pmatrix} \vdots \\ k^b \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \qquad \forall a : \sum |\mathcal{C}(a,b)|\, k^b = 1 \quad \text{(weighting)}$$

$$\begin{pmatrix} \cdots & k_a & \cdots \end{pmatrix}\begin{pmatrix} |\mathcal{C}(a,b)| \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} \qquad \forall b : \sum k_a |\mathcal{C}(a,b)| = 1 \quad \text{(coweighting)}$$

If $\mathcal{C}$ has we and cowe then

$$\sum k_a = \chi(\mathcal{C}) = \sum k^b$$

Lemma If $\mathcal{A} \subseteq \mathcal{C}$ and $\mathrm{Ob}(\mathcal{A}) \supseteq \mathrm{supp}(k_\bullet)$ or $\mathrm{Ob}(\mathcal{A}) \supseteq \mathrm{supp}(k^\bullet)$ then

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FACT: $k^H = -\tilde{\chi}(\mathcal{S}^*_{N_G(H)/H})$ weighting for $\mathcal{S}^*_G$

$k_K = -\tilde{\chi}(\mathcal{S}^{(1,K)}_K) = \mu(K)$ coweighting for $\mathcal{S}^*_G$

The weighting vanishes off the radical groups: $\chi(\mathcal{S}^{\mathrm{rad}}_G) = \chi(\mathcal{S}^*_G)$

The coweighting vanishes off the elementary abelian subgroups: $\chi(\mathcal{S}^{\mathrm{eab}+*}_G) = \chi(\mathcal{S}^*_G)$

So we have inclusions $\mathrm{supp}(k^\bullet) = \{H \mid \tilde{\chi}(\mathcal{S}^*_{N_G(H)/H}) \neq 0\} \subseteq \{H \neq 1 \mid H \text{ is } G\text{-rad}\}$

$\mathrm{supp}(k_\bullet) = \{K \mid \mu(K) \neq 0\}$

$= \{K \neq 1 \mid K$ is elementary abelian$\}$.

$$\mathcal{S}^{\mathrm{rad}}_G \xrightarrow{\text{same weighting}} \mathcal{S}^*_G \xleftarrow{\text{same coweighting}} \mathcal{S}^{\mathrm{eab}+*}_G$$

~~that are~~ and all these posets have identical Euler characteristics.

(both with Edward) 2a

<u>Lemma</u> $\mathcal{C}$ finite category with full subcat $\mathcal{A}$. Then ~~$\chi(\mathcal{A}) = \chi(\mathcal{C})$~~ If $Ob(\mathcal{A}) \supseteq supp(k_\bullet)$ (weighting) or $Ob(\mathcal{A}) \supseteq supp(k^\bullet)$ (coweighting) then $\chi(\mathcal{A}) = \chi(\mathcal{C})$.
(and $\mathcal{A} \hookrightarrow \mathcal{C}$ should be a homotopy equivalence)

FACT: Weighting for $\mathcal{S}_G^*$ $\quad k^H = -\tilde{\chi}(\mathcal{S}^*_{N_G(H)/H})$

Co- ——— " ——— $\quad k_K = -\mu(K) = -\tilde{\chi}(\mathcal{S}_K^{(1,K)})$

$supp(k^\bullet) = \{H \mid -\tilde{\chi}(\;) \neq 0\} \subseteq \{H \mid \mathcal{S}^*_{N_G(H)/H} \not\simeq *\} = \{H \mid H \text{ radical}\}$
$supp(k_\bullet) = \{K \mid \mu(K) \neq 0\} = \{K \mid K \text{ is elementary abelian}\}$

$$\mathcal{S}_G^{*+\text{rad}} \hookrightarrow \mathcal{S}_G^* \hookleftarrow \mathcal{S}_G^{*+\text{eab}}$$
same χ $\qquad$ same χ

<u>Idea</u> Look at other categories of groups, eg $\mathcal{F}$ fusion system and consider inclusions
$$\mathcal{F}^{supp(k^\bullet)} \subseteq \mathcal{F} \supseteq \mathcal{F}^{supp(k_\bullet)}$$
should they be homotopy equivalences?

$\mathcal{C}$	k^* weighting $(\mathcal{C})$	k_K coweighting $(\mathcal{C}^*)$
$\mathcal{S}^*_G$	$-\tilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G(H)})$	$-\mu(K) = -\tilde{\chi}(\mathcal{S}^{(1,K)}_K)$
$\cancel{\mathcal{T}_{G^*}}$ $\mathcal{F}^*_G$	$\lvert N_G(H)\rvert^{-1} \sum_{x \in C_{N_G(H)}(H)} -\tilde{\chi}(\mathcal{S}^*_{C_{N_G(H)}(x)/H})$ ↑ there is a weighting but it is not clear what is the support of this weighting	$\frac{-\mu(K)}{\lvert \mathcal{F}_G(K)\rvert}$ $\quad \frac{-\mu(K)}{\lvert \mathcal{T}_G(K)\rvert}$
$\tilde{\mathcal{F}}^*_G$	?	— " —
$\tilde{\mathcal{F}}^{sfc}_G$	$\frac{-\tilde{\chi}(\mathcal{S}^*_{\tilde{\mathcal{F}}_G(H)})}{\lvert \tilde{\mathcal{F}}_G(H)\rvert}$	?
$\mathcal{L}^{sfc}_G$	$\frac{-\tilde{\chi}(\mathcal{S}^*_{\tilde{\mathcal{F}}_G(H)})}{\lvert \mathcal{L}_G(H)\rvert}$	?
$\mathcal{L}^*_G$	? Quillen: $\tilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G(H)}) \neq 0 \Rightarrow H$ is G-radical $\quad O_p\mathcal{O}_G(H)=1$	$\frac{-\mu(K)}{\lvert \mathcal{L}_G(K)\rvert}$
$\mathcal{O}_G$	$\frac{-\tilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G(H)})}{\lvert \mathcal{O}_G(H)\rvert}$ $\tilde{\chi}(\mathcal{S}^*_{\tilde{\mathcal{F}}_G(H)}) \neq 0 \Rightarrow H$ is $\tilde{\mathcal{F}}_G$-radical $\quad O_p\tilde{\mathcal{F}}_G(H)=1$	$\mathcal{O}_G(1,K) = G/K$ $\begin{cases} \lvert G\rvert^{-1} & K=1 \\ \frac{p-1}{p}\frac{\lvert K\rvert}{\lvert G\rvert} & K>1 \text{ cyclic} \\ 0 & K \text{ noncyclic} \end{cases}$

$\mathcal{S}^{*+rad}_G \hookrightarrow \mathcal{S}^*_G$

$\tilde{\mathcal{F}}^{rad+sfc}_G \hookrightarrow \tilde{\mathcal{F}}^{sfc}_G$

$\mathcal{L}^{rad+sfc}_G \hookrightarrow \mathcal{L}^{sfc}_G$

$\mathcal{O}^{rad}_G \hookrightarrow \mathcal{O}_G$

$\boxed{\begin{array}{l} \mathcal{L}^{sfc+rad}_G \hookrightarrow \mathcal{L}^{sfc}_G \\ \mathcal{O}^{sfc+rad}_G \hookrightarrow \mathcal{O}^{sfc}_G \end{array}}$

$\mathcal{S}^{eab+*}_G \hookrightarrow \mathcal{S}^*_G$ $\quad \mathcal{L}^{eab+*}_G \hookrightarrow \mathcal{L}_G$

will try to prove this one → $\mathcal{F}^{*eab}_G \hookrightarrow \mathcal{F}^*_G$ $\quad \mathcal{O}^{cyc}_G \to \mathcal{O}_G$

$\tilde{\mathcal{F}}^{eab+*}_G \hookrightarrow \tilde{\mathcal{F}}^*_G$

<u>Thm</u> All of these inclusions – except for $\mathcal{O}_G^{cyc} \hookrightarrow \mathcal{O}_G$! – are homotopy equivalences.

<u>Corollary</u> $\mathcal{F}^* \to \tilde{\mathcal{F}}^*$ is a homotopy equivalence

<u>Proof</u>

$$\begin{tikzcd} \mathcal{F}^* \arrow[r] & \tilde{\mathcal{F}}^* \\ \mathcal{F}^{*+orb} \arrow[u, "\simeq"] \arrow[r, equal] & \tilde{\mathcal{F}}^{*+orb} \arrow[u, "\simeq"'] \end{tikzcd}$$

~~What about proofs~~ ?

$\boxed{\text{A word on } \mathcal{O}_G}$ Like for $\mathcal{S}_G^*$ we have both a weighting and a coweighting – so

$$\frac{1+(p-1)\sum_C |C|}{p^{|G|}} = \sum_{[H]} \frac{-\tilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G(H)})}{|\mathcal{O}_G(H)|}$$

C cyclic, H G-radical

<u>What about proofs</u> ? Proof uses A. Bouc theory (independent interest) for EI-cats
no posets B. An identification of weighting/coweighting for EI cats
C. Final ingredient: The inclusion $\mathcal{A} \hookrightarrow \mathcal{S}$ is a h.e. if

A. <u>Theorem</u> (Bouc, Comptes Rendus) Let $\mathcal{S}$ be a finite poset and $\mathcal{A}$ a subposet. ~~Suppose that either~~

(a) $\mathcal{S}_{x<}$ is non contractible only for ~~all~~ x ~~outside~~ in $\mathcal{A}$, or,

(b) $\mathcal{S}_{<x}$ ——"——

Bouc's theorem generalizes to finite EI-categories.

<u>Def</u> $\mathcal{C}$ category, x and y objects of $\mathcal{C}$

(a) $x//\mathcal{C}$ is the category of nonisomorphic morphisms from x

$$\begin{tikzcd} & x \arrow[dl] \arrow[dr] & \\ y_1 \arrow[rr, dashed] & & y_2 \end{tikzcd}$$

(b) $\mathcal{C}//y$ ——"—— to y

$$\begin{tikzcd} x_1 \arrow[dr] & & x_2 \arrow[dl] \\ & y & \end{tikzcd}$$

~~Example~~ $\mathrm{supp}(\bullet//\mathcal{C}) = \{x \in Ob(\mathcal{C}) \mid x//\mathcal{C} \text{ is noncontractible}\}$

$\mathrm{supp}(\mathcal{C}//\bullet) = \{y \in Ob(\mathcal{C}) \mid \mathcal{C}//y \text{ is noncontractible}\}$

<u>Thm</u> (Bouc for finite EI-categories)

Let $\mathcal{C}$ be a finite EI-category and $\mathcal{A}$ a full subcategory closed under isomorphism. The inclusion $\mathcal{A} \hookrightarrow \mathcal{C}$ is a homotopy equivalence if $Ob(\mathcal{A})$ ~~contains~~ contains $supp(\cdot // \mathcal{C})$ or $supp(\mathcal{C} // \cdot)$.

$\boxed{\text{Example}}$ $\mathcal{F}_G^{*+ab} \hookrightarrow \mathcal{F}_G^{*}$ REMEMBER EXAMPLE! (using coweighting)

Need to show: ~~$H\backslash\backslash \mathcal{F}_G^*$~~ $\mathcal{F}_G^*//K$ is noncontractible $\Rightarrow$ ~~H~~ K is elementary abelian

So what is ~~$H\backslash\backslash\mathcal{F}_G^*$~~ $\tilde{\mathcal{F}}_G^* // K$?

B. Identification of weighting/coweighting for EI-categories.

<u>Thm</u> Let $\mathcal{C}$ be a finite EI-category. Then

$$k^{x} = \frac{-\tilde{\chi}(x//\mathcal{C})}{|\mathcal{C}(x)|} \qquad k_y = \frac{-\tilde{\chi}(\mathcal{C}//y)}{|\mathcal{C}(y)|}$$

is a weighting and a coweighting for $\mathcal{C}$.

<u>Corollary</u> $\mathcal{C}$ finite EI-cat

$$\sum_{x \in [\mathcal{C}]} \frac{-\tilde{\chi}(x//\mathcal{C})}{|\mathcal{C}(x)|} = \chi(\mathcal{C}) = \sum_{y \in [\mathcal{C}]} \frac{-\tilde{\chi}(\mathcal{C}//y)}{|\mathcal{C}(y)|}$$

<u>Example continued</u>

$$\frac{-\mu(\mathcal{S}_K^{(1,K)})}{|\mathcal{F}_G(K)|} = \frac{-\tilde{\chi}(\mathcal{F}_G^* // K)}{|\mathcal{F}_G(K)|}$$

Guess: $\mathcal{F}_G^* // K \cong \mathcal{S}_K^{(1,K)}$

All have the same $\mathcal{C}//K$ — True for $\mathcal{L}_G \to \mathcal{F}_G \to \mathcal{F}_G \to \mathcal{L}_G$

This is true ~~and~~ Note that $\boxed{\tilde{\mathcal{F}}_G^* // K \text{ does not depend on } \mathcal{F}_G}$!

C. Final ingredient

<u>Lemma</u> $\mathcal{S}_K^{(\ell,K)}$ is noncontractible $\Rightarrow$ K is elementary abelian

<u>~~Conclusion~~</u>
<u>Thm</u> $\mathcal{F}^{*+eab} \hookrightarrow \mathcal{F}^{*}$ is a homotopy equivalence.

Three identities weighting coweighting

$$\sum -\tilde{\chi}\left(\mathcal{S}^*_{\mathcal{O}_G(H)}\right) = \sum -\mu(K) \qquad \mathcal{S}_G$$

$$\sum \frac{-\tilde{\chi}\left(\mathcal{S}^*_{\mathcal{O}_G(H)}\right)}{|N_G(H)|} = \sum \frac{-\mu(K)}{|N_G(K)|} \qquad \mathcal{T}_G$$

$$\sum \frac{-\tilde{\chi}\left(\mathcal{S}^*_{\mathcal{O}_G(H)}\right)}{|\mathcal{O}_G(H)|} = \frac{1+(p-1)\sum |C|}{p|G|} \qquad \mathcal{O}_G$$