Weightings and coweightings of p-subgroup categories

$\mathscr{C}$ finite category

Q: ? Leinster R. the very answer on $\chi\left(\mathcal{S}_G(X)\right)$?

$a, b \in Ob(\mathscr{C})$ $\quad \mathscr{C}(a,b) \quad |\mathscr{C}(a,b)|$

A weighting $k^{\bullet} : Ob(\mathscr{C}) \to \mathbb{Q}$ is a solution to

$$\left(|\mathscr{C}(a,b)|\right)\begin{pmatrix} \vdots \\ k^b \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \qquad \underset{\text{weighting}}{k^{\bullet} : Ob(\mathscr{C}) \to \mathbb{Q}}$$

A coweighting $k_{\bullet} : Ob(\mathscr{C}) \to \mathbb{Q}$ is a solution to

$$\left(\cdots k_a \cdots\right)\left(|\mathscr{C}(a,b)|\right) = \left(1 \cdots 1\right)$$

$k^{\bullet} : Ob(\mathscr{C}) \to \mathbb{Q}$ weighting $\forall a : \sum_b |\mathscr{C}(a,b)| k^b = 1$

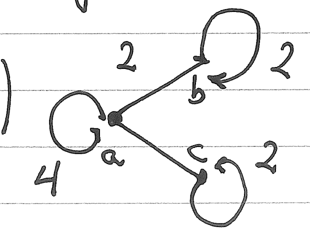
$k_{\bullet} : Ob(\mathscr{C}) \to \mathbb{Q}$ coweighting $\forall b : \sum_a k_a |\mathscr{C}(a,b)| = 1$

Def: If $\mathscr{C}$ has a we and a cowe then

$$\sum k_a = \chi(\mathscr{C}) = \sum k^b$$

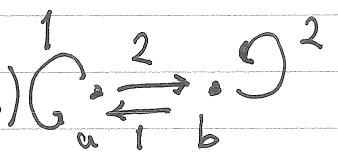
$$\sum_{a,b} k_a |\mathscr{C}(a,b)| k^b = \sum k_a = \sum k^b$$

Example 1) G finite group $\quad \chi(G) = \frac{1}{|G|}$

2) 

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & & & \\ b & & \begin{pmatrix} 4 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} & \\ c & & & \end{array} \qquad k^{\bullet} = \begin{pmatrix} -1/4 \\ 1/2 \\ 1/2 \end{pmatrix} \qquad k_{\bullet} = (1/4 \;\; 1/4 \;\; 1/4)$$

$$\chi = 3/4$$

3) 

$$\begin{array}{c|cc} & a & b \\ \hline a & 1 & 2 \\ b & 1 & 2 \end{array} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

EI-category General information

EI: $\mathcal{C}(a,a)$ is a group

Assume from now on that all categories have wc, cowc, and χ

- Any EI-category has wc, cowc, and χ

- $$\left.\begin{array}{c} \mathcal{C}_1 \underset{L}{\overset{R}{\rightleftarrows}} \mathcal{C}_2 \\ \mathcal{C}_1(Lc_2, c_1) = \mathcal{C}_2(c_2, Rc_1) \end{array}\right\} \Rightarrow \chi(\mathcal{C}_1) = \chi(\mathcal{C}_2)$$

- If $\mathcal{C}$ has an initial or terminal element then $\chi(\mathcal{C}) = 1$.

$\mathcal{C}_0 \subseteq \mathcal{C}$ full subcategory

- $$\left.\begin{array}{c} k^\bullet \text{ wc on } \mathcal{C} \\ \operatorname{supp}(k^\bullet) \subseteq \operatorname{Ob}(\mathcal{C}_0) \end{array}\right\} \Rightarrow \begin{array}{c} k^\bullet|\operatorname{Ob}(\mathcal{C}_0) \text{ is a wc on } \mathcal{C}_0 \\ \chi(\mathcal{C}_0) = \chi(\mathcal{C}) \end{array}$$

- $$\left.\begin{array}{c} k_\bullet \text{ cowc on } \mathcal{C} \\ \operatorname{supp}(k_\bullet) \subseteq \operatorname{Ob}(\mathcal{C}_0) \end{array}\right\} \Rightarrow \chi(\mathcal{C}_0) = \chi(\mathcal{C})$$

$$\mathcal{C}(L0, c) = 0(0, Rc) \qquad \mathcal{C} \underset{L}{\overset{R}{\rightleftarrows}} 0$$

$$1(Lc, 1) = \mathcal{C}(c, R1) \qquad 1 \underset{L}{\overset{R}{\rightleftarrows}} \mathcal{C}$$

Group theory

G finite group $\quad p$ prime

$O_p G$ = smallest normal p-subgroup of $G = \bigcap_{S \in Syl_p G} S$

$H \leq G$ subgroup

H is p-elementary abelian $\quad H = C_p^r = C_p \times \cdots \times C_p$

H is p-cyclic $\quad H = C_{p^r}$

H is p-radical $\quad H = O_p N_G H$ or $O_p(N_G(H)/H) = 1$

$N_G(H,K) = \{ g \in G \mid H^g \leq K \}$ transporter set

$\boxed{\mathcal{S}_G}$ poset of p-subgroups

$\mathcal{S}_G$ has 1 as initial element

$\mathcal{S}^*_G$ poset of non identity p-subgroups

Prop: Suppose G has a normal p-subgroup $N \neq 1 \Rightarrow \chi(\mathcal{S}^*_G) = 1$

Proof: If $H, K \leq G$ and $K \geq N$ then $HN \leq K \Leftrightarrow H \leq K$

$$\mathcal{S}^*_G \underset{L}{\overset{R}{\rightleftarrows}} \mathcal{S}_{\geq N} \qquad \begin{array}{l} RH = NH \\ LK = K \end{array}$$

$$\chi(\mathcal{S}^*_G) = \chi(\mathcal{S}_{\geq N}) = 1 \text{ as } N \text{ is initial in } \mathcal{S}_{\geq N} \qquad \square$$

Consequence $H \leq G$ H not p-radical $\Rightarrow \tilde{\chi}(\mathcal{S}^*_{N_G(H)/H}) = 0$

$(\tilde{\chi}(\mathcal{C}) = \chi(\mathcal{C}) - 1)$

Strong Quillen conjecture:

$$O_p G \neq 1 \Leftrightarrow \tilde{\chi}(\mathcal{S}^*_G) = 0$$

Consequence H is p-radical $\Leftrightarrow \tilde{\chi}(\mathcal{S}^*_{N_G(H)/H}) \neq 0$

Example

n	4	5	6	7	8	9	10	11	12
$\chi(\mathcal{S}^*_{A_n})$	1	5	-15	-175	65	5121	15105	-55935	-288255

what is the continuation?

Weighting for $\mathcal{S}^*_G$

$$k^H = -\tilde{\chi}\left(\mathcal{S}^*_{N_G(H)/H}\right)$$

$k^\bullet$ picks out the p-radical subgroups $\quad \mathrm{supp}(k^\bullet) \subseteq \mathcal{S}^{*+\mathrm{rad}}_G$

coweighting for $\mathcal{S}^*_G$

$$k_K = -\mu(K) = \begin{cases} (-1)^r p^{\binom{r}{2}} & K = C_p^r \text{ is el. ab} \\ 0 & \text{otherwise} \end{cases}$$

$k_\bullet$ picks out the elem. ab subgroups $\quad \mathrm{supp}(k_\bullet) = \mathcal{S}^{*+\mathrm{eab}}_G$

$$\mathcal{S}^{*+\mathrm{rad}}_G \hookrightarrow \mathcal{S}^*_G \hookleftarrow \mathcal{S}^{*+\mathrm{eab}}_G$$

$$\underbrace{\sum_{1 \neq H \leq G} -\tilde{\chi}\left(\mathcal{S}^*_{N_G(H)/H}\right)} = \chi\left(\mathcal{S}^*_G\right) = \sum_{1 \neq K \leq G} -\mu(K)$$

What does $\chi(\mathcal{S}^*_G)$, $\chi(\mathcal{F}^*_G)$ measure

$$\boxed{\sum_{1 \leq H \leq G} \tilde{\chi}\left(\mathcal{S}^*_{N_G(H)/H}\right) = -1}$$

$$\underbrace{\qquad\qquad}$$

$$\tilde{\chi}\left(\mathcal{S}^*_G\right) = \sum_{1 \leq [K] \leq G} \frac{-\mu(K)}{|N_G(K)|} |G|$$

Thm (Bouc + Quillen) $\mathcal{S}^{*+\mathrm{rad}}_G \hookrightarrow \mathcal{S}^*_G \hookleftarrow \mathcal{S}^{*+\mathrm{eab}}_G$ are homotopy equivalences

$\mathcal{O}_G$ orbit category

$\mathcal{O}_G(H,K) = N_G(H,K)/K$ = left G-maps $G/H \to G/K$

$\mathcal{O}_G(1,K) = G/K$

weighting $\quad k^H = \dfrac{-\tilde{\chi}(\mathcal{S}^*_{N_G(H)/H})}{|G:H|} = \dfrac{-\tilde{\chi}(\mathcal{S}^*_{N_G(H)/H})}{|G|}|H|$

$\mathrm{supp}(k^\bullet) \subseteq \mathcal{O}_G^{rad}$

coweighting

$$k_K = \begin{cases} \frac{1}{|G|} & K = 1 \\ \frac{p-1}{p}\frac{1}{|G:K|} & K > 1 \text{ cyclic} \\ 0 & \text{otherwise} \end{cases}$$

$\mathrm{supp}(k_\bullet) \subseteq \mathcal{O}_G^{cyc}$

$$\mathcal{O}_G^{rad} \hookrightarrow \mathcal{O}_G \hookleftarrow \mathcal{O}_G^{cyc}$$

Using coweightings $\quad \chi(\mathcal{O}_G) = \sum_{1 \le K \le G} k_K = \dfrac{|G_p|}{|G|}$ is the probability that a random element in G has p-power order

$G_p = \{g \in G \mid g \text{ has } p\text{-power order}\} = \{g \in G \mid |g| \mid |G|_p\}$

Using weightings

$$\boxed{\sum_{1 \le H \le G} |H| \tilde{\chi}(\mathcal{S}^*_{N_G(H)/H}) = -|G_p|}$$

Consequence:

Frobenius 1907 $\quad |G|_p \mid |G_p|$

Brown 1975 $\quad |G|_p \mid \tilde{\chi}(\mathcal{S}_G^{*})$

With above identity and induction over group order we get

Frobenius' thm $\Leftrightarrow$ Brown's theorem

Thm $\mathcal{O}_G^{\mathrm{rad}} \hookrightarrow \mathcal{O}_G$ is a homotopy equivalence

$\mathcal{O}_G^{\mathrm{cyc}} \hookrightarrow \mathcal{O}_G$ is not a homotopy equivalence (when G is a p-group - what if G is not a p-group)

Prop $|G|_{p'}\,\chi(\mathcal{O}_G) \in \mathbb{Z}$

$\mathcal{F}_G$ the fusion category of G

$\mathcal{F}_G(H,K) = C_G(H)\backslash N_G(H,K)$ = group homomorphisms $H \xrightarrow{\subseteq} K$ given by conjugation with a $g \in G$

The trivial subgroup is initial

$\mathcal{F}_G^*$ fusion category of non identity p-subgroups

weighting for $\mathcal{F}_G^*$

$$k^H = \frac{1}{|G|} \sum_{x \in C_G(H)} -\tilde{\chi}\left(\mathcal{S}^*_{C_{N_G(H)}(x)/H}\right)$$

$$\operatorname{supp}(k^\circ) \underset{\neq}{\subseteq} \{H \mid \exists x \in C_G(H) : H \in \mathcal{S}_p C_{N_G(H)}(x)\}$$

coweighting for $\mathcal{F}_G^*$

$$k_K = \frac{-\mu(K)}{|G|} |C_G(K)|$$

$$\operatorname{supp}(k_\circ) = \mathcal{F}_G^{*+\mathrm{eab}}$$

Using weighting

$$\chi(\mathcal{F}_G^*) = \frac{1}{|G|} \sum_{\substack{x \in G \\ (p,|x|)=1}} \tilde{\chi}(\mathcal{S}^*_{C_G(x)})$$

mean value of $\tilde{\chi}(\mathcal{S}^*_\circ)$ over element centralizers

Using coweighting

$$\tilde{\chi}(\mathcal{F}_G^*) = \sum_{1 \le [K] \le G} \frac{-\mu(K)}{|N_G(K)|} |C_G(K)|$$

(cfr formula for $\tilde{\chi}(\mathcal{S}^*_G)$)

Consequence

Prop $|G|_p \cdot \chi(\mathcal{F}^*_G) \in \mathbb{Z}$

Thm $\mathcal{F}^{*+\mathrm{rad}}_G \hookrightarrow \mathcal{F}^*_G$ is a homotopy equivalence

$\mathcal{F}^*_G \longrightarrow \widetilde{\mathcal{F}}^*_G$ is a homotopy.

Thm $\chi(\mathcal{F}^*_G) = \chi(\widetilde{\mathcal{F}}^*_G)$ and $\mathcal{F}^*_G \to \widetilde{\mathcal{F}}^*_G$ is a htpy equiv

Questions

1) Does the homotopy type of $\mathcal{F}^*_G$ determine $\mathcal{F}^*_G$?

(Then we'd have (?) that $\mathcal{F}^{\mathrm{rad}}_G$ determines $\mathcal{F}_G$)

2) $\chi(\mathcal{F}^*_G) \geq 0$ always

3) $\chi(\mathcal{F}^*_G) \leq 2$ always?

4) $|G|_p \cdot \longrightarrow \boxed{\text{finite space?}} \longrightarrow B\mathcal{F}^*_G$

5) $\sum \chi(\mathcal{F}^*_{\Sigma_{p^n+j}}) x^n$

Alperin conjecture (P) p.197 i Thevenaz' paper

$z_p(G)$: number of irreducible complex representations whose dimension is a multiple of $|G|_p$

$$\sum_{\substack{[x] \\ (p,|x|)=1}} \tilde{\chi}\left(\mathcal{S}^*(G)^x / C_G(x)\right) = -z_p(G)$$

Burnside orbit counting lemma. Suppose $S\times G \to S$ is a G-action on a set S. Then the number of orbits is

$$|S/G| = \frac{1}{|G|}\sum_{s\in S}|C_G(s)| = \sum_{s\in S}\frac{1}{|G:C_G(s)|}$$

Proof For any $s\in G$, the size of the orbit is $|sG| = |G:C_G(s)|$

or $|C_G(s)|\,|sG| = |G|$.

If $t_1, t_2 \in sG$ are in the same orbit then the order of the centralizers is constant: $|C_G(t_1)| = |C_G(s)| = |C_G(t_2)|$.

The sum of centralizer orders over one orbit is

$$\sum_{t\in sG}|C_G(t)| = \sum_{t\in sG}|C_G(s)| = |sG|\,|C_G(s)| = |G|$$

The sum of centralizer orders over all element in S is thus

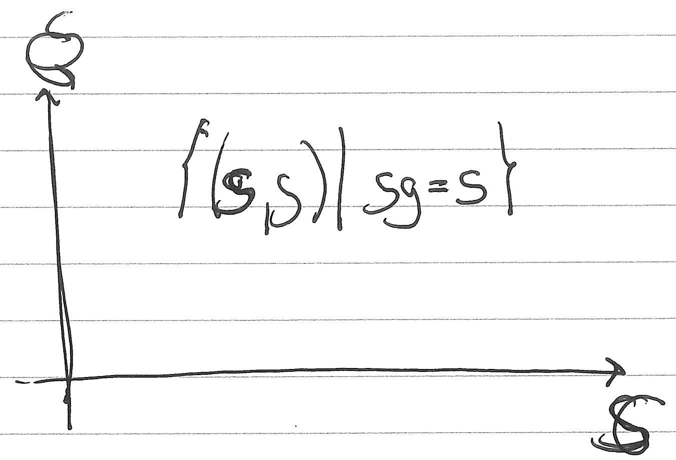
$$\sum_{s\in S}|C_G(s)| = |G/S|\,|G|$$

$k(G)$, the class number of G, is the number of conjugacy classes of elements of G. Let G act on G by conjugation $G\times G\to G : (s,g)\to s^g$ so that $k(G)$ is the number of orbits. Then we see that

$$k(G)=\sum_{s\in G}\frac{1}{|G:C_G(s)|}$$

More on Burnside:

$S\times G\to S$



$$\sum_{g\in G}|C_S(g)| = |\{(s,s)\mid sg=s\}| = \sum_{s\in S}|C_G(s)|$$

$$= \sum_{sG\in S/G}|C_G(s)|\,|sG| = \sum_{sG\in S/G}|C_G(s)|\cdot|G:C_G(s)|$$

$$= |G|\,|S/G|$$

$$C_{\varphi^*_G}(\langle x, g\rangle)$$

$$= \{H \leq G \mid H^{\langle x, g\rangle} = H\}$$

$$= \{H \leq G \mid \langle x, g\rangle \leq N_G(H)\}$$

G acts on poset $\mathcal{S}$ $\quad \mathcal{S} \times G \to \mathcal{S}$, $(s,g) \to s^g$

$g \in G$

$C_{\mathcal{S}}(g) = \{ s \in \mathcal{S} \mid s^g = s \}$ is a subposet of $\mathcal{S}$

$C_G(g)$ acts on $C_{\mathcal{S}}(g)$: If $x \in C_G(g)$ and $s \in C_{\mathcal{S}}(g)$ then $s^{xg} = s^{gx} = s^x$ so $s^x \in C_{\mathcal{S}}(g)$

There is a quotient simplicial complex

$$|C_{\mathcal{S}}(g)| / C_G(g)$$

We want to count the number of simplices of any dimension $d \geq 0$ in this complex. So we want to count

$$|C_{\mathcal{S}}(g)|_d / C_G(g)$$

If $\sigma \in |C_{\mathcal{S}}(g)|_d$ then $\sigma = s_0 < s_1 < \cdots < s_d$ for $s_i \in C_{\mathcal{S}}(g)$

Burnside counting lemma says that

$$\left| \, |C_{\mathcal{S}}(g)|_d / C_G(g) \, \right| = \frac{1}{|C_G(g)|} \sum_{x \in C_G(g)} \left| C_{|C_{\mathcal{S}}(g)|_d}(x) \right|$$

σ is fixed by x iff each $s_i \in C_{\mathcal{S}}(\langle x, g \rangle)$. So

$$C_{|C_{\mathcal{S}}(g)|_d}(x) = |C_{\mathcal{S}}(\langle x, g\rangle)|_d$$

$$= \frac{1}{|C_G(g)|} \sum_{x \in C_G(g)} |C_{\mathcal{S}}(\langle x, g \rangle)|_d$$

Now we have

$$\chi\left({|C_\varphi(g)|} / {C_G(g)} \right) =$$

<u>Proposition</u>

$$\frac{1}{|C_G(g)|} \sum_{x \in C_G(g)} \chi\left(C_\varphi(\langle x, g\rangle)\right) = \chi\left({|C_\varphi(g)|} / {C_G(g)} \right)$$

$$C_\varphi(\langle x, g\rangle) = C_{C_\varphi(g)}(x)$$

Alperin conjecture

$$-z_p(G) = \sum_{[g]} \tilde{\chi}\left({C_{\varphi^*_G}(g)} / {C_G(g)} \right)$$

$$= \sum_{[g]} \chi\left({C_{\varphi^*_G}(g)} / {C_G(g)} \right) - 1$$

$$= \sum_{[g]} \chi\left({C_{\varphi^*_G}(g)} / {C_G(g)} \right) - k(G)$$

$$= \sum_{[g]} \frac{1}{|C_G(g)|} \sum_{x \in C_G(g)} \chi\left(C_{\varphi^*_G}(\langle x, g\rangle)\right) - k(G)$$

some kind of weighting ? but weighting trivial

$$k(G) - z_p(G) = \sum_{[g]} \sum_{x \in C_G(g)} \frac{1}{|C_G(g)|} \chi\left(C_{\varphi^*_G}(\langle x, g\rangle)\right)$$

what the weightings lie only on p-regular elements

This is a function on G but an inherent of G

This is 1 if g is a p-regular $p \nmid |g|$