Bob et Jean, Paris July 3-4 2014

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~~Weightings and coweightings~~ Euler characteristics of for p-subgroup categories

G finite group of order $|G|$ — $\mathcal{S}^*_G$ poset of nontrivial p-...

p prime number $p \mid |G|$ — $\mathcal{O}_G$ orbit category of p-subgroups

$|G|_p$ p-part of group order

$$G_p = \bigcup_{S \in \mathrm{Syl}_p G} S = \{g \in G \mid g \text{ has } p\text{-power order}\}$$

$$O_p(G) = \bigcap_{S \in \mathrm{Syl}_p G} S$$ The p-part $|G|_p$ is involved in

Frobenius Thm 1907: $|G|_p \mid |G_p|$ ($\Leftrightarrow$ Sylow thm)

Brown Thm 1975: $|G|_p \mid \tilde{\chi}(\mathcal{S}^*_G)$

Alpern AWC_p 1987 involves

$$z_p(G) = |\{\chi \in Irr(G) \mid |G|_p \mid \chi(1)\}|$$

1) From Frobenius to Brown

2) Brown's thm with a group action

3) Re-interpretation: AWC_p predicts how the abelian p'-subgroups act on the elementary abelian subgroups, on $\mathcal{S}^{*+eab}_G$

4) Related results $\mathcal{S}^{*+eab}_G \hookrightarrow \mathcal{S}^*_G \hookleftarrow \mathcal{S}^{*+rad}_G$, $\mathcal{T}_G$

②

Weightings and coweightings of finite categories (Tom Leinster)

$\mathcal{C}$ finite category $\quad \zeta(\mathcal{C}) = \left(|\mathcal{C}(a,b)|\right)_{a,b\in Ob(\mathcal{C})}$

$$\cancel{\zeta}\left(|\mathcal{C}(a,b)|\right)\begin{pmatrix} \vdots \\ k^a \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \qquad (\cdots k_b \cdots)\left(|\mathcal{C}(a,b)|\right) = (1 \cdots 1)$$

Weighting $k^{\bullet} : Ob(\mathcal{C}) \to \mathbb{Q}$ $\forall b: \sum_a |\mathcal{C}(a,b)| k^a = 1$

Coweighting $k_{\bullet} : Ob(\mathcal{C}) \to \mathbb{Q}$ $\forall a: \sum_b k_b |\mathcal{C}(a,b)| = 1$

Defn If $\mathcal{C}$ has a weighting and a coweighting $\sum k^a = \chi(\mathcal{C}) = \sum k^b$

Properties

Finite EI-categories have Euler characteristics.

$$\boxed{\mathcal{C}_1 \underset{L}{\overset{R}{\rightleftarrows}} \mathcal{C}_2 \quad \chi(\mathcal{C}_1) = \chi(\mathcal{C}_2)}$$

Properties

Exmp If $\mathcal{C}$ has an initial or terminal element then $\chi(\mathcal{C}) = 1$.

Exmp (Quillen) If $\mathcal{S}$ is a poset then $\chi(\mathcal{S}) = \chi(B\mathcal{S})$

If $O_p(G) \neq 1$ then $\chi(\mathcal{S}_G^*) = 1$. $\quad H \leq H \cdot O_p(G) \geq O_p(G)$

Strong Quillen conjecture: $O_p(G) = 1 \Leftrightarrow \chi(\mathcal{S}_G^*) = 1$

Only need to calculate $\chi(\mathcal{S}_G)$ for G with $O_pG = 1$ (p-reduced G)

Euler char. of $\mathcal{S}^*_G$

Weighting $\quad k^H = -\tilde{\chi}(\mathcal{S}^*_{N_G(H)/H}) = -\tilde{\chi}(\mathcal{S}^*_{O_G(H)})$

or $-\mu(K)$?

Coweighting $\quad k_K = \begin{cases} (-1)^r p^{\binom{r}{2}} \mu(K) & K = \overbrace{p \times \cdots \times p}^{r} \text{ el. ab.} \\ 0 & \text{otherwise} \end{cases}$

$k_K = -\mu(K)$

The weighting is concentrated at the p-radical $H \leq G$ ($H = O_p N_G(H)$)
The coweighting at the elementary abelian subgroups.

Prop $\quad \mathcal{S}^{*+\text{rad}}_G \xhookrightarrow{\simeq} \mathcal{S}^*_G \xleftarrow{\simeq} \mathcal{S}^{*+\text{eab}}_G$

$$\sum_{1 \lneq H \leq G} -\tilde{\chi}(\mathcal{S}^*_{\cancel{O_G(H)}\, N_G(H)/H}) = \chi(\mathcal{S}^*_G) \overset{?}{=} \sum \pm \mu(K)$$

$$\boxed{\cancel{1+} \sum_{1 \leq H \leq G} \tilde{\chi}(\mathcal{S}^*_{N_G(H)/H}) = \overset{-1}{\cancel{0}}}$$

Example $\quad p = 2$

n	4	5	6	7	8	9	10	11
$\chi(\mathcal{S}^*_{A_n})$	1	5	-15	-175	65	5121	15105	-55935

Thm $\quad \mathcal{S}^{*+p+\text{rad}}_G \hookrightarrow \mathcal{S}^{*+p}_G \hookleftarrow \mathcal{S}^{*+p+\text{eab}}_G$

are htpy equivalences. $\quad \mathcal{S}^{\text{sfc}+\text{rad}}_G \hookrightarrow \mathcal{S}^{\text{sfc}}_G$

sfc: $p \nmid |C_G(H) : C_H(H)|$

Euler char of $\mathcal{O}_G$

Weighting $\quad k^H = -\frac{1}{|G|}\tilde{\chi}(\mathcal{S}^*_{N_G(H)/H})\cdot|H|$

Coweighting $\quad k_K = \begin{cases} \frac{1}{|G|} & K=1 \\ \frac{1}{|G|}(1-\frac{1}{p})|K| & K>1 \text{ cyclic} \\ 0 & \text{otherwise} \end{cases}$

The weighting is concentrated at the p-radical $H \le G$.
The cow. at the cyclic subgroups.

$\mathcal{O}^{cyc}_G \hookrightarrow \mathcal{O}_G$ NOT a h.e.

Pointed out to me by Sune P. Reeh

$\chi(\mathcal{O}_G) \in \mathbb{Z}[|G|_{p'}^{-1}]$

$\chi(\mathcal{O}_G) \in \mathbb{Z}_{(p)}$

Prop $\quad \chi(\mathcal{O}_G) = \frac{|G_p|}{|G|}$, $\quad \mathcal{O}^{rad}_G \hookrightarrow \mathcal{O}_G$ is a h.e.

$$\boxed{\cancel{|G_p|} + \sum_{1 \le H \le G} \tilde{\chi}(\mathcal{S}^*_{N_G(H)/H})|H| = \cancel{0}\ -|G_p|}$$

This is the GLOBAL RELATION

Fundamental equation

$$\boxed{\sum_{[H]\in[\mathcal{O}_G]} \frac{\tilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G(H)})}{|\mathcal{O}_G(H)|} = -\frac{|G_p|}{|G|}}$$

Reformulation:

$$\underbrace{|G_p| + \tilde{\chi}(\mathcal{S}^*_G)}_{\text{divisible by } |G|_p} + \sum_{[H]\neq 1} \underbrace{\frac{|G|}{|N_G(H):H|_{p'}}}_{\text{divisible by } |G|_p} \underbrace{\frac{\tilde{\chi}(\mathcal{S}^*_{N_G(H)/H})}{|N_G(H):H|_p}}_{\text{integer by induction}}$$

Thm (Brown) $\quad |G|_p \mid \tilde{\chi}(\mathcal{S}^*_G)$

Thm $\quad \mathcal{O}^{p+rad}_G \hookrightarrow \mathcal{O}^p_G$ is a h.e. $\quad \mathcal{O}^{p+sfc+rad}_G \hookrightarrow \mathcal{O}^{sfc}_G$

$\mathcal{O}^{p+cyc}_G \hookrightarrow \mathcal{O}^p_G$ is NOT a h.e.

Euler char of the fusion category $\mathcal{F}_G^*$ and $\tilde{\mathcal{F}}_G^*$ ⑤

Thm $\chi(\mathcal{F}_G^*) = \frac{1}{|G|} \sum_{\substack{x \in G \\ p \nmid |x|}} \tilde{\chi}(\mathcal{S}^*_{\mathcal{E}(x)})$

$\chi(\mathcal{F}_G^*) \in \mathbb{Z}_{(p)}$ (only need to invert $|G|_{p'}$)

Thm
$\mathcal{F}_G^{*+\text{eab}} \hookrightarrow \mathcal{F}_G^*$ is a htpy equiv.

$\mathcal{F}_G^* \longrightarrow \tilde{\mathcal{F}}_G^*$ is a htpy equivalence

$\chi(\mathcal{F}_G^*) \geq 0$ always if there is a central p-subgp then $\chi(\mathcal{F}_G^*) = 1$

Is $\chi(\mathcal{F}_G^*) \leq 2$ always?

Thm $\tilde{\mathcal{F}}_{(G)}^{*+\text{eab}} \hookrightarrow \tilde{\mathcal{F}}^*$

$\tilde{\mathcal{F}}^{\text{sfc}+\text{rad}} \hookrightarrow \tilde{\mathcal{F}}^{\text{sfc}}$

Thm $\mathcal{L}^{\text{sfc}+\text{rad}} \hookrightarrow \mathcal{L}^{\text{sfc}}$

$\mathcal{L}$ centric linking system associated to fusion category $\mathcal{F}$.

$\mathcal{L}_G^{*+\text{eab}} \hookrightarrow \mathcal{L}_G^*$

Brown's thm ~~with an again~~ revisited

$G \circlearrowleft A$ A finite abelian p'-group

$\mathcal{S}^*_G \circlearrowleft A$

$C_{\mathcal{S}^*_G}(A) \supset C_G(A)$

What we know about these numbers

$$\boxed{1 + \sum_{\substack{1 \le H \le G \\ H \text{ is p-radical}}} \tilde{\chi}(C_{\mathcal{S}^*_{N_G(H)/H}}(A)) = 0}$$

Thm $|C_G(A)_p| \mid \tilde{\chi}(C_{\mathcal{S}^*_G}(A))$

This is the reason
Here are some interesting numbers!

Proof $\mathcal{O}_{C_G(A)} \hookrightarrow C_{\mathcal{O}_G}(A)$ have the same Euler characteristic

$KRC_p(G)$:

$$z_p(G)|G| + \sum_{\substack{A \le G \\ A \text{ abelian} \\ p \nmid |A|}} \tilde{\chi}(C_{\mathcal{S}^*_G}(A))\, GRP(A) = 0$$

Note: $|\mathrm{Irr}(G)|\,|G|$
$= \sum GRP(A)$

$\forall G: KRC_p(G) \Leftrightarrow \forall G: AWC_p(G)$ $\frac{1}{|G|}\sum |C_G(A)_p|\, GRP(A) \in \mathbb{Z}$

Prop $KRC_p(G)$ is true if $O_p(G) \ne 1$.

Proof $z_p(G) = 0$ and $C_{\mathcal{S}^*_G}(A) \simeq *$ for all A

$KRC_p(G)$ $GL_n(\mathbb{F}_q)$ $q = p^e$

$M_{11}, M_{12}, M_{22} - M_{24}$

Cor $KRC_p(G)$ holds for $N_G(P)/P$ for all $G \in \mathcal{S}^{p\text{-rad}}_G \Rightarrow AWC_p(G)$

Example Mathieu group M_{11}

$$-z_p(G)|G| = \sum -\tilde{\chi}(C_{\mathcal{S}^*_G}(A))\,GRP(A)\,|G:N_G(A)|$$

M_{11} $p=2$ $|M_{11}| = 7920$

A	1	3	5	11	3x3
$\tilde{\chi}(C_{\mathcal{S}^*_G}(A))$	-496	8	-1	-1	-1
GRP	1	8	24	120	48
$\|G:N_G(A)\|$	1	220	396	144	55
	-496	14080	-9508	-17280	-2640

The sum of these numbers is $-15840 = -z_2(M_{11})|M_{11}|$

Is there a category with (co)we of the form $-\tilde{\chi}(C_{\mathcal{S}^*_G}(A))GRP(A)$?