Chromatic polynomials of simplicial complexes

Two open problems

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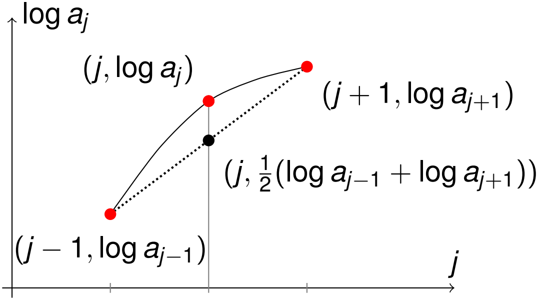
log-concave sequences

Definition 1.1 (LC)

A finite sequence $a_1, a_2, \dots, a_m$ of positive numbers is log-concave (LC) if $a_{j-1}a_{j+1} \leq a_j a_j$ for $1 < j < m$.

$$(a_j)_{j=1}^m \text{ is log-concave} \iff \frac{a_1}{a_2} \leq \frac{a_2}{a_3} \leq \cdots \leq \frac{a_{m-1}}{a_m}$$

$$\iff \frac{\log a_{j-1} + \log a_{j+1}}{2} \leq \log a_j \implies (a_j)_{j=1}^m \text{ is unimodal}$$



Example 1.2

Binomial sequence $j \to \binom{m}{j}$ is LC
$1, 2, 5, 2, 1$ is unimodal but not LC

Falling factorials and Stirling numbers

Two bases for the polynomial ring $\mathbf{Z}[r]$

$$[r]_j = \underbrace{r(r-1)\cdots(r-j+1)}_{j}, \qquad r^j = \overbrace{r \cdot r \cdots r}^{j}$$

falling factorial base (FFB) monomial base (MOB)

$$\boxed{[r]_0, [r]_1, [r]_2, [r]_3, \ldots} \xleftrightarrow[\text{change}]{\text{base}} \boxed{r^0, r^1, r^2, r^3, \ldots}$$

$$[r]_m = \sum_{j=0}^{m} S_1(m,j) r^j \qquad r^m = \sum_{j=0}^{m} S_2(m,j)[r]_j$$

Stirling numbers 1st kind Stirling numbers 2nd kind

$S_2(m,j)$ is the number of partitions of an m-set into j blocks

$[r]_1 = r^1$	$r^1 = [r]_1$	$j \to \lvert S_1(m,j) \rvert$ is LC
$[r]_2 = -r^1 + r^2$	$r^2 = [r]_1 + [r]_2$	$j \to S_2(m,j)$ is LC
$[r]_3 = 2r^1 - 3r^2 + r^3$	$r^3 = [r]_1 + 3[r]_2 + [r]_3$	

Colorings of simplicial complexes

Definition 2.1 (Colorings of simplicial complexes)

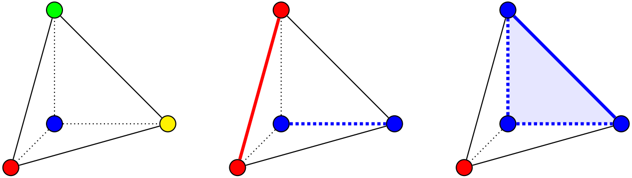
A (weak) (r, d)-coloring of the simplicial complex K is a map

$$\mathrm{col}\colon F^0(K) \to \{1, 2, \ldots, r\}$$

such that

$$|\mathrm{col}(\sigma)| = 1 \implies \dim \sigma < d$$

for all simplices $\sigma \in K$. ($K \neq \emptyset$, $d > 0$.)



$(4, 1)$-coloring $\quad$ $(2, 2)$-coloring $\quad$ $(2, 3)$-coloring

Chromatic numbers of simplicial complexes

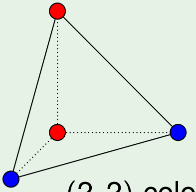
Definition 2.3 (The d-chromatic number of a simplicial complex K)

The d-chromatic number of K, $\mathrm{chr}(K, d)$, is the minimal r so that K admits an (r, d)-coloring.

$$|F^0(K)| \geq \mathrm{chr}(K, 1) \geq \mathrm{chr}(K, 2) \geq \cdots \geq \mathrm{chr}(K, \dim K) \geq 1$$

Example 2.4 (Do we know the chromatic numbers of any complexes?)

$K = D[4]$



$(2, 2)$-coloring

$$\mathrm{chr}(K, 1) = 4$$
$$\mathrm{chr}(K, 2) = 2$$
$$\mathrm{chr}(K, 3) = 2$$
$$\mathrm{chr}(D[m], d) = \lceil \tfrac{m}{d} \rceil$$

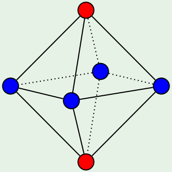
Chromatic numbers of triangulable manifolds

Definition 2.5 (The d-chromatic number of a compact manifold M)

$\mathrm{chr}(M,d) = \sup\{\mathrm{chr}(K,d) \mid K \text{ triangulates } M\}$

$$\infty \geq \mathrm{chr}(M,1) \geq \mathrm{chr}(M,2) \geq \cdots \geq \mathrm{chr}(M,\dim M) \geq 1$$

Example 2.6 (Do we know the chromatic numbers of any manifolds?)



$|K| = S^2$

$\mathrm{chr}(S^2,2) \geq$

$\mathrm{chr}(K,2) = 2$

Is there a triangulation K of S^2 with $\mathrm{chr}(K,2) > 2$?

Theorem 2.7 (The 4-color theorem = chromatic numbers of S^2)

$\mathrm{chr}(S^2,1) = 4$ *and* $\mathrm{chr}(S^2,2) = 2$

Chromatic numbers of S^3

Problem 1: What are the chromatic numbers of S^3?

- $\mathrm{chr}(S^3, 1) = \infty$ FOR SURE
- $\mathrm{chr}(S^3, 2) = \infty$ PRESUMABLY
- $\mathrm{chr}(S^3, 3) < \infty$ UNKNOWN

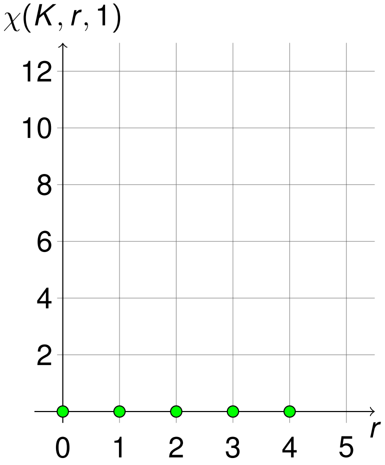
The standard triangulation $K = \partial D[5]$ of S^3 has $\mathrm{chr}(K, 3) = 2$.
There exists a triangulation K, $f(K) = (18, 143, 250, 125)$, of S^3 with 3-chromatic number $\mathrm{chr}(K, 3) = 3$.
Does there exist a triangulation K of S^3 with 3-chromatic number $\mathrm{chr}(K, 3) > 3$?

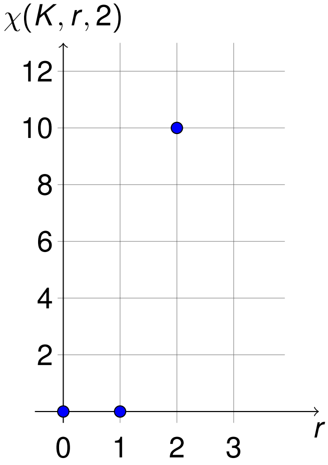
Theorem 2.8 (Chromatic numbers of spheres)

$\mathrm{chr}(S^d, \lceil d/2 \rceil) = \infty$ *when* $d \geq 3$ *PRESUMABLY*

Red-necked Grebe

$\chi(K, r, d)$ is the number of (r, d)-colorings of K

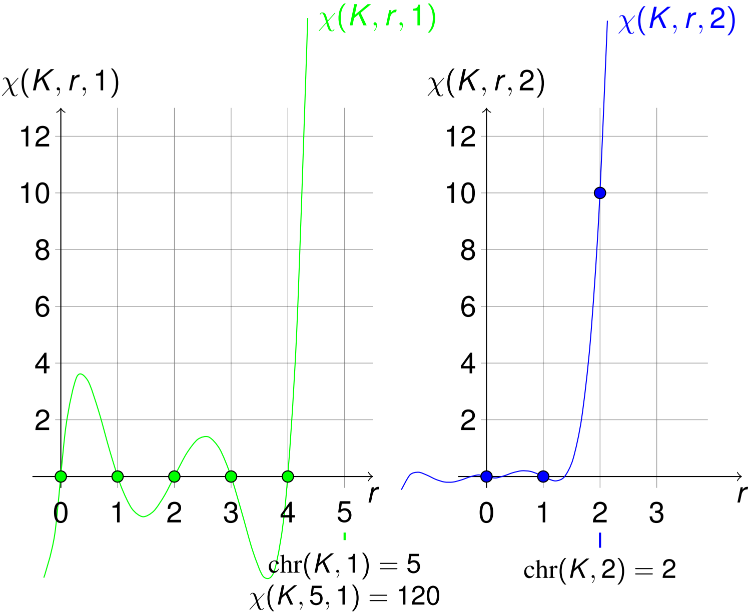




$\mathrm{chr}(K, 1) = 5$
$\chi(K, 5, 1) = 120$

$\mathrm{chr}(K, 2) = 2$
$\chi(K, 2, 2) = 10$

$r \to \chi(K, r, d)$ is the d-chromatic polynomial of K



Compute the number $\chi(K,r,d)$ of (r,d)-colorings of K!

Definition 2.9 (Simplical Stirling numbers)

$S(K,j,d)$ is the number of partitions of $F^0(K)$ into j blocks containing only K-simplices of dimension $< d$.

- $S(K,j,d) = S_2(m,j)$ when $K = \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet$
- $K_1 \subseteq K_2 \implies S(K_1,j,d) \geq S(K_2,j,d)$ when $F^0(K_1) = F^0(K_2)$
- $S_2(m,j) \geq S(K,j,d) \geq S(D[m],j,d)$ with equality for $j = \underbrace{|F^0(K)|,\ldots,|F^0(K)|-d+1}_{d}$, $m = |F^0(K)|$
- $S(K,|F^0(K)|-d,d) = S_2(|F^0(K)|,|F^0(K)|-d) - f^d(K)$
- $S(K,j,d) = 0$ for $0 < j < \mathrm{chr}(K,d)$
- $\mathrm{chr}(K,d) = \min\{j \mid S(K,j,d) > 0\}$

Colorings and equivalence relations

$$\begin{array}{ccc} S(K,j,d)[r]_j & & S(K,j,d) \\ \begin{array}{c}(r,d)\text{-colorings of } K \\ \mathrm{col}\colon K \to D[r] \\ \text{with } |\mathrm{col}(F^0(K))| = j\end{array} & \xrightarrow{[r]_j\text{-to-1}} & \begin{array}{c}\text{Partitions of } F^0(K) \text{ into } j \\ \text{blocks without } d\text{-simplices}\end{array}\end{array}$$

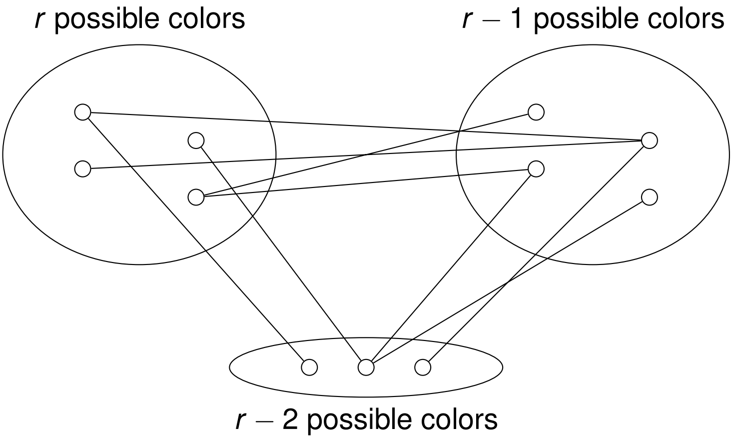
$$\boxed{\mathrm{col}\colon F^0(K) \to \{1,\ldots,r\}} \longrightarrow \boxed{\coprod\{v \in F^0(K) \mid \mathrm{col}(v) = k\}}$$

Theorem 2.10 (The *d*-chromatic polynomial of *K*)

The number of (r,d)-colorings of K is

$$\chi(K,r,d) = \sum_{j=\mathrm{chr}(K,d)}^{|F^0(K)|} S(K,j,s)[r]_j$$

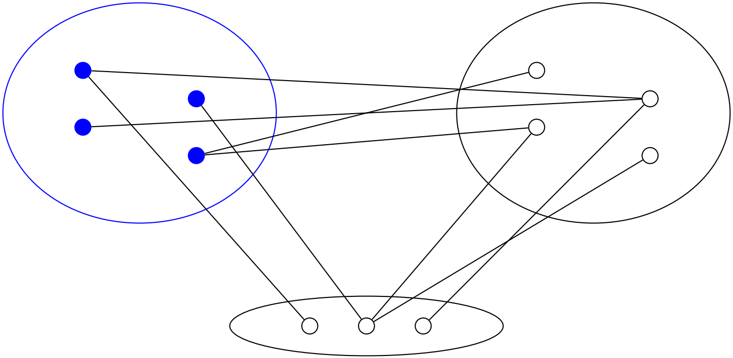
Colorings and equivalence relations



3 blocks with no d-simplices can be colored in $[r]_3$ ways from a palette of r colors

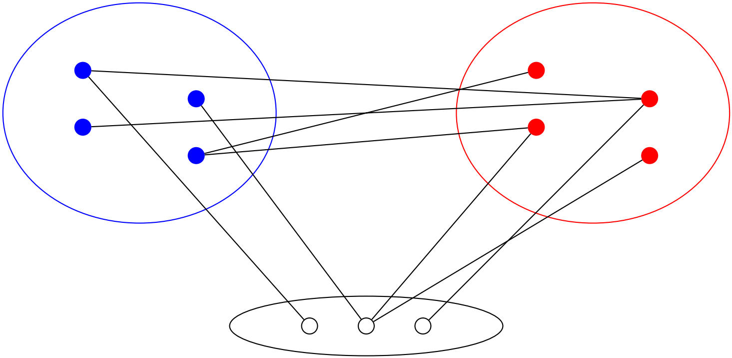
$$\boxed{\chi(K,r,d) = \sum_{j=\mathrm{chr}(K,d)}^{|F^0(K)|} S(K,j,d)[r]_j}$$

Colorings and equivalence relations



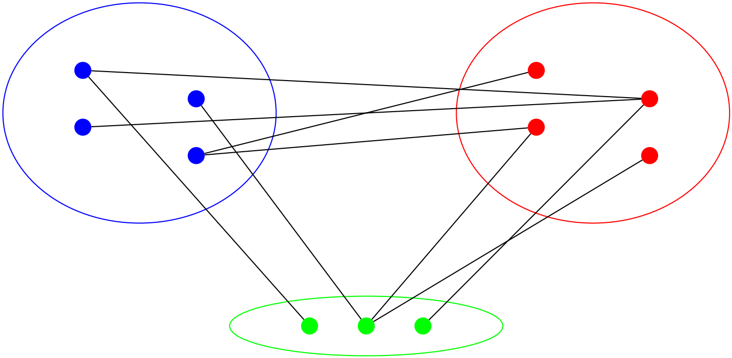
$$\boxed{\chi(K,r,d) = \sum_{j=\mathrm{chr}(K,d)}^{|F^0(K)|} S(K,j,d)[r]_j}$$

Colorings and equivalence relations



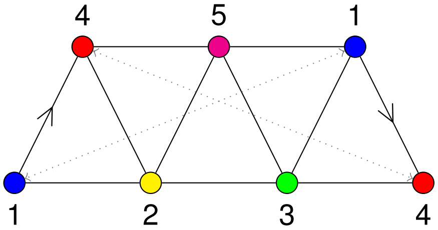
$$\boxed{\chi(K,r,d) = \sum_{j=\mathrm{chr}(K,d)}^{|F^0(K)|} S(K,j,d)[r]_j}$$

Colorings and equivalence relations



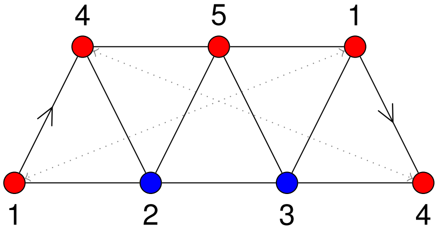
$$\chi(K,r,d) = \sum_{j=\mathrm{chr}(K,d)}^{|F^0(K)|} S(K,j,d)[r]_j$$

The two chromatic polynomials of a 2-complex



$$\chi(\mathrm{MB}, r, 1)$$

$$\boxed{\begin{gathered} r^5 - 10r^4 + 35r^3 - 50r^2 + 24r^1 \\ [r]_5 \\ \mathrm{chr}(\mathrm{MB}, 1) = 5 \end{gathered}}$$



$$\chi(\mathrm{MB}, r, 2)$$

$$\boxed{\begin{gathered} r^5 - 5r^3 + 5r^2 - r^1 \\ 5[r]_2 + 20[r]_3 + 10[r]_4 + [r]_5 \\ \mathrm{chr}(\mathrm{MB}, 2) = 2 \end{gathered}}$$

Chromatic polynomials of graphs

Example 2.11 (Specialization to graphs)

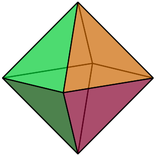
An $(r, 1)$-coloring of K is an r-coloring of the simple graph K^1, and the 1-chromatic number of K is the graph chromatic number of K^1.

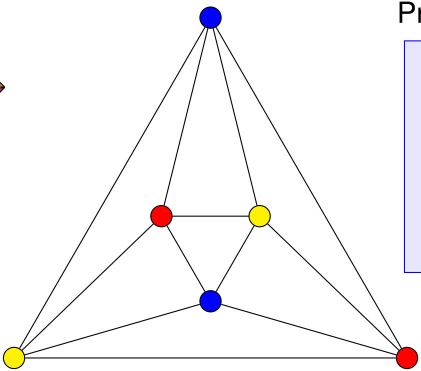
$$\overbrace{\sum_{j=1}^{|F^0(K)|} b_j r^j}^{\text{monomial basis (MOB)}} = \chi(K, r, 1) = \overbrace{\sum_{j=1}^{|F^0(K)|} a_j [r]_j}^{\text{falling factorial basis (FFB)}}$$

Properties of 1-chromatic polynomials

MOB is $\pm$	The MOB coefficients (b_j) alternate in sign
MOB is LC	The MOB coefficients $(\lvert b_j \rvert)$ are LC
No roots < 0	$\chi(K, r, 1)$ has no roots < 0
m, $m-1$ vals	$\chi(K, m, 1) > e\chi(K, m-1, 1)$, $m = \lvert F^0(K) \rvert$
FFB is LC	The FFB coefficients (a_i) are LC UNKNOWN

Example of a 1-chromatic polynomial





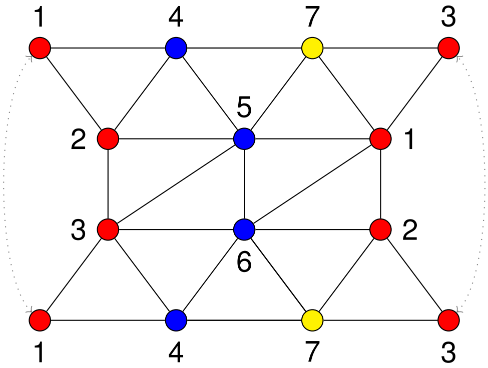
Properties of $\chi(\mathrm{OG}, r, 1)$

MOB is $\pm$	Yes
MOB is LC	Yes
No roots < 0	Yes
m, $m-1$ vals	Yes
FFB is LC	Yes

1-chromatic polynomial in MOB and FFB

$$\begin{aligned}\chi(\mathrm{OG}, r, 1) &= -64r^1 + 154r^2 - 137r^3 + 58r^4 - 12r^5 + r^6 \\ &= [r]_3 + 3[r]_4 + 3[r]_5 + [r]_6\end{aligned}$$

Example of a 2-chromatic polynomial



Properties of $\chi(\mathrm{MT}, r, 2)$

MOB is $\pm$	No
MOB is LC	No
No roots <0	No
m, $m-1$ vals	No
FFB is LC	Yes

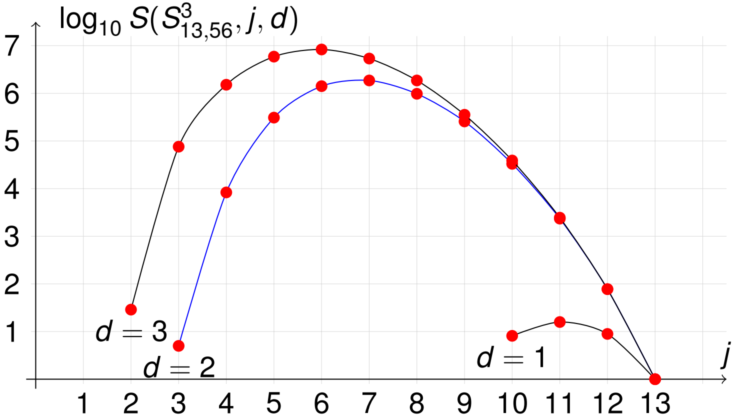
2-chromatic polynomial in MOB and FFB

$$\begin{aligned}\chi(\mathrm{MT}, r, 2) &= 6r - 21r^2 + 7r^3 + 21r^4 - 14r^5 + r^7 \\ &= 84[r]_3 + 231[r]_4 + 126[r]_5 + 21[r]_6 + [r]_7\end{aligned}$$

Problem 2: Are the simplicial Stirling numbers

$$j \to S(K, j, d), \qquad \mathrm{chr}(K, d) \le j \le |F^0(K)|$$

LC for fixed K and d? (Only property that might generalize!)



$\mathrm{chr}(K, 1) = 10$

$\mathrm{chr}(K, 2) = 3$

$\mathrm{chr}(K, 3) = 2$

Topology meets combinatorics

Theorem 2.12 (Equivalent conditions for colorability)

- *K admits an (r,d)-coloring*
- *There exists a lift such that*

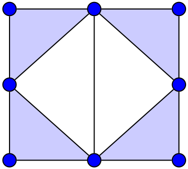
Davis–Januszkiewicz space

$$
\begin{tikzcd}
 & \overbrace{BU(d)\times\cdots\times BU(d)}^{r} \arrow[d, "\lambda_d\times\cdots\times\lambda_d"] \\
\mathrm{DJ}(K) \subseteq \underbrace{BU(1)\times\cdots\times BU(1)}_{|F^0(K)|} \arrow[ur, dotted] \arrow[r, hook, "\lambda_1\times\cdots\times\lambda_1"] & BU
\end{tikzcd}
$$

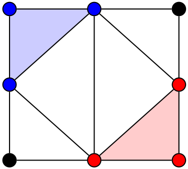
is homotopy commutative

- $\chi(K,r,d) > 0$

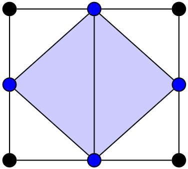
The d-chromatic lattice



$T \notin L(K,2)$



$T \in L(K,2), |\pi(T)| = 2$



$T \in L(K,2), |\pi(T)| = 1$

Definition 3.1

The d-chromatic lattice, $L(K,d)$, is the partially ordered set of monochrome subsets of $F^d(K)$ of the form

$$M^d(\mathrm{col}) = \{\sigma \in F^d(K) \mid |\mathrm{col}(\sigma)| = 1\} \subseteq F^d(K)$$

for some map $\mathrm{col}\colon F^0(K) \to \{1, \ldots, |F^0(K)|\}$.

- $L(K,d)$ is a finite lattice with $\widehat{0} = \emptyset$ and $\widehat{1} = F^d(K)$
- μ is the Möbius function of $L(K,d)$
- $|\pi(T)|$ is the number of connected components of $T \in L(K,d)$

Theorem 3.2 (Relating simplicial and usual Stirling numbers)

$$\chi(K,r,d) = \sum_{T \in L(K,d)} \mu(\widehat{0},T) r^{|\pi(T)|}$$

$$S(K,j,d) = \sum_{T \in L(K,d)} \mu(\widehat{0},T) S_2(|\pi(T)|,j)$$

'Dehn–Sommerville relations' for simplicial Stirling numbers of manifold?
$L(K,d)$ is graded for $d = 1$ but not for $d > 1$.

Theorem 3.3

The reduced Euler characteristic of the open interval $(\widehat{0},\widehat{1})$ *in* $L(K,d)$ *is*

$$\sum_{j=\mathrm{chr}(K,d)}^{|F^0(K)|} (-1)^{j-1}(j-1)! S(K,j,d)$$

Integer sequences of Euler characteristics

The reduced Euler characteristics of $L(D[m], d)(\widehat{0}, \widehat{1})$ for $m - d = 2, 3, 4, \ldots$ are

$$
\begin{aligned}
d = 1: &\quad 2, -6, 24, -120, 720, -5040, 40320, -362880, \ldots \\
d = 2: &\quad 3, -6, 0, 90, -630, 2520, 0, -113400, 1247400, \ldots \\
d = 3: &\quad 4, -10, 20, -70, 560, -4200, 25200, -138600, \ldots \\
d = 4: &\quad 5, -15, 35, -70, 0, 2100, -23100, 173250, -1051050, \ldots \\
d = 5: &\quad 6, -21, 56, -126, 252, -924, 11088, -126126, \ldots \\
d = 6: &\quad 7, -28, 84, -210, 462, -924, 0, 42042, -630630, \ldots \\
d = 7: &\quad 8, -36, 120, -330, 792, -1716, 3432, -12870, \ldots \\
d = 8: &\quad 9, -45, 165, -495, 1287, -3003, 6435, -12870, 0, \ldots
\end{aligned}
$$

The first sequence is the sequence $(-1)^{m-1}(m-1)!$. The second sequence is A009014 from The On-Line Encyclopedia of Integer Sequences (OES). The remaining 6 sequences don't match any sequences of the OES.

Weighted colorings

Let $w\colon F^0(K) \to \mathbf{N}$ be a weight function on the vertices. The weight of a simplex $\sigma \in K$ is the sum

$$w(\sigma) = \sum_{v \in \sigma} w(v)$$

of the weights of its vertices. (Special case: $w = 1$.)

Definition 4.1 (Weighted (r, d)-coloring of K)

A $(r, w \leq d)$-coloring of K is a function

$$\mathrm{col}\colon F^0(K) \to \{1, 2, \ldots, r\}$$

such that $|\mathrm{col}(\sigma)| = 1 \implies w(\sigma) \leq d$ for all simplices $\sigma \in K$.

Definition 4.2 (Weighted s-chromatic number of K)

The weighted d-chromatic number of K, $\mathrm{chr}(K, w \leq d)$, is the minimal r so that K admits an $(r, w \leq d)$-coloring.

Weighted chromatic polynomials

Definition 4.3 (Weighted simplicial Stirling numbers)

$S(K, j, w \le d)$ is the number of partitions of $F^0(K)$ with j classes containing only simplices $\sigma \in K$ of weight $w(\sigma) \le d$.

Theorem 4.4 (Weighted d-chromatic polynomial)

The number of weighted (r, d)-colorings of K is

$$\chi(K, r, w \le d) = \sum_{j=\mathrm{chr}(K,w\le d)}^{|F^0(K)|} S(K, j, w \le d)[r]_j$$

Problem 2: Are the weighted simplicial Stirling numbers

$$j \to S(K, j, w \le d), \qquad \mathrm{chr}(K, w \le d) \le j \le |F^0(K)|$$

LC for fixed K, w, and d?

Colorings as simplicial maps

Definition 4.5

An (r, d)-coloring of K is a simplicial map

$$\mathrm{col}\colon K \to D[r]$$

such that $\dim\{\sigma \in K \mid \mathrm{col}(\sigma) = j\} < d$ for $1 \leq j \leq r$

Definition 4.6

An (L, d)-coloring of K is a simplicial map

$$\mathrm{col}\colon K \to L$$

such that $\dim \mathrm{col}^{-1}(v) < d$ for all vertices v in L.