Relaxed vertex colorings of simplicial complexes

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Workshop on Algebraic Topology and Combinatorics
Buenos Aires, November 07 – 11, 2011

Outline

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  - Stanley–Reisner rings
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Abstract Simplicial Complexes

Definition (Simplex)

A simplex is the set $D[\sigma]$ of all subsets of a finite set σ.

Definition (ASC)

An Abstract Simplicial Complex is a union of simplices:

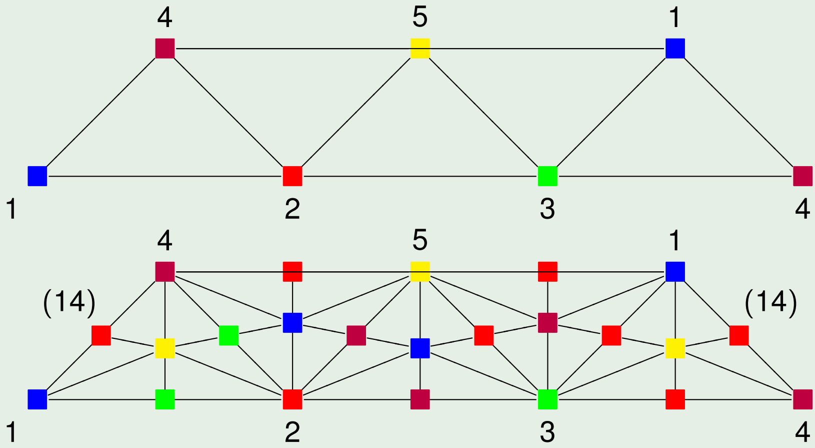
$$K = \bigcup_{\sigma} D[\sigma]$$

$n(K) = \dim K + 1$: number of vertices in maximal simplex of K
$m(K) = |V|$: number of vertices in K

Standard colorings

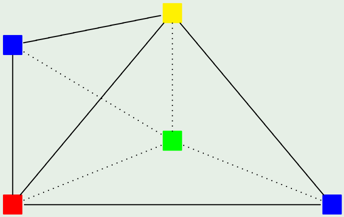
Example (Standard coloring of the Möbius band MB)

Standard coloring of 5-vertex complex MB and its barycentric subdivision using 5 colors



Standard Colorings

Example



Standard coloring of K
= Standard coloring of
1-skeleton of K
$\Longrightarrow$ graph theory

Standard colorings live on the 1-skeleton

Standard coloring of K = Standard coloring of $sk_1(K)$

A coloring of the vertices is a standard coloring of K if and only if K contains no monochrome 1-simplices.

Sudoku as a standard coloring problem

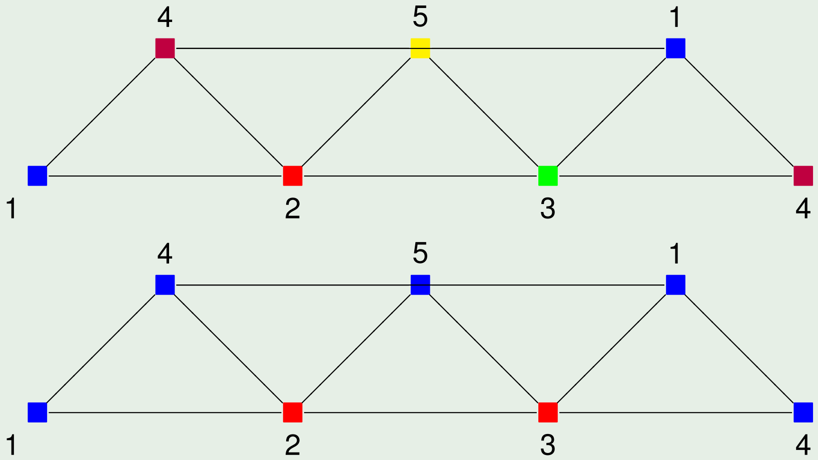
	6							
8	3	4	7	6	1	9	5	2
	4							
	8		4	5	3			
	1		6	8	9			
	5		1	7	2			
	9							
	7							
	2							

SUDOKU is an 8-dimensional simplicial complex with $9+9+9$ maximal simplices. A sudoku problem consists in completing a given partial standard coloring to a full standard coloring of SUDOKU using 9 colors.

$$H_*(\mathrm{SUDOKU};\mathbf{Z}) = H_*(\underbrace{S^1 \vee \cdots \vee S^1}_{28};\mathbf{Z})$$

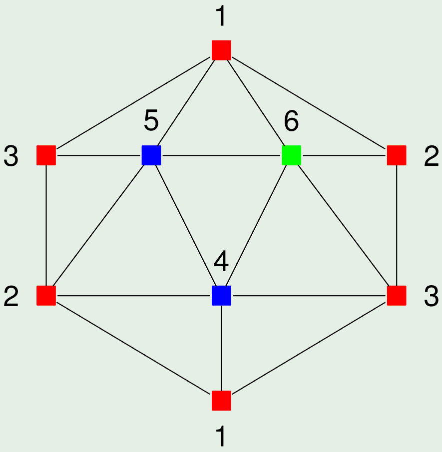
Standard and Relaxed Colorings

Example (Standard and Relaxed coloring of Möbius band MB)



Relaxed colorings

Example (Relaxed Coloring of projective plane P^2)



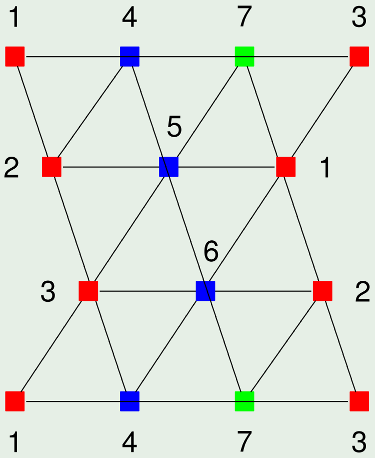
A $(3,2)$-coloring of a triangulation P^2 of the projective plane.

No monochrome 2-dimensional simplices

A standard coloring of P^2 needs 6 colors.

Relaxed coloring

Example (Relaxed coloring of the torus T^2)



A $(3,2)$-coloring of Möbius' minimal triangulation of the torus.

A standard coloring needs 7 colors.

Colorings of Abstract Simplicial Complexes

Let K be an ASC on vertex set V, and P a finite palette of r colors.

Definition ((r, s)-coloring of an ASC)

A (P, s)-coloring (or (r, s)-coloring) of K is a map $f\colon V \to P$ that is at most s-to-1 on all simplices of K.

$f\colon V \to P$ is an (r, s)-coloring if and only if K contains no monochrome s-simplices.

Remark

An (r, s) coloring with

- $s = 1$ is a standard coloring using r colors
- $s > 1$ is a relaxed coloring using r colors

Theorem ((r, s)-colorings live on the s-skeleton)

(r, s)-colorings of K = (r, s)-colorings of $\mathrm{sk}_s(K)$

Coloring the triangulated Poincaré homology 3-sphere

A $(4, 2)$-coloring of the 16-vertex Björner–Lutz triangulation of the Poincaré homology 3-sphere:

$\{1, 2, 4, 9\}, \{1, 2, 4, 15\}, \{1, 2, 6, 14\}, \{1, 2, 6, 15\}, \{1, 2, 9, 14\}, \{1, 3, 4, 12\}, \{1, 3, 4, 15\}, \{1, 3, 7, 10\},$
$\{1, 3, 7, 12\}, \{1, 3, 10, 15\}, \{1, 4, 9, 12\}, \{1, 5, 6, 13\}, \{1, 5, 6, 14\}, \{1, 5, 8, 11\}, \{1, 5, 8, 13\}, \{1, 5, 11, 14\},$
$\{1, 6, 13, 15\}, \{1, 7, 8, 10\}, \{1, 7, 8, 11\}, \{1, 7, 11, 12\}, \{1, 8, 10, 13\}, \{1, 9, 11, 12\}, \{1, 9, 11, 14\}, \{1, 10, 13, 15\},$
$\{2, 3, 5, 10\}, \{2, 3, 5, 11\}, \{2, 3, 7, 10\}, \{2, 3, 7, 13\}, \{2, 3, 11, 13\}, \{2, 4, 9, 13\}, \{2, 4, 11, 13\}, \{2, 4, 11, 15\},$
$\{2, 5, 8, 11\}, \{2, 5, 8, 12\}, \{2, 5, 10, 12\}, \{2, 6, 10, 12\}, \{2, 6, 10, 14\}, \{2, 6, 12, 15\}, \{2, 7, 9, 13\}, \{2, 7, 9, 14\},$
$\{2, 7, 10, 14\}, \{2, 8, 11, 15\}, \{2, 8, 12, 15\}, \{3, 4, 5, 14\}, \{3, 4, 5, 15\}, \{3, 4, 12, 14\}, \{3, 5, 10, 15\}, \{3, 5, 11, 14\},$
$\{3, 7, 12, 13\}, \{3, 11, 13, 14\}, \{3, 12, 13, 14\}, \{4, 5, 6, 7\}, \{4, 5, 6, 14\}, \{4, 5, 7, 15\}, \{4, 6, 7, 11\}, \{4, 6, 10, 11\},$
$\{4, 6, 10, 14\}, \{4, 7, 11, 15\}, \{4, 8, 9, 12\}, \{4, 8, 9, 13\}, \{4, 8, 10, 13\}, \{4, 8, 10, 14\}, \{4, 8, 12, 14\}, \{4, 10, 11, 13\},$
$\{5, 6, 7, 13\}, \{5, 7, 9, 13\}, \{5, 7, 9, 15\}, \{5, 8, 9, 12\}, \{5, 8, 9, 13\}, \{5, 9, 10, 12\}, \{5, 9, 10, 15\}, \{6, 7, 11, 12\},$
$\{6, 7, 12, 13\}, \{6, 10, 11, 12\}, \{6, 12, 13, 15\}, \{7, 8, 10, 14\}, \{7, 8, 11, 15\}, \{7, 8, 14, 15\}, \{7, 9, 14, 15\},$
$\{8, 12, 14, 15\}, \{9, 10, 11, 12\}, \{9, 10, 11, 16\}, \{9, 10, 15, 16\}, \{9, 11, 14, 16\}, \{9, 14, 15, 16\}, \{10, 11, 13, 16\},$
$\{10, 13, 15, 16\}, \{11, 13, 14, 16\}, \{12, 13, 14, 15\}, \{13, 14, 15, 16\}$

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Chromatic Numbers of ASCs

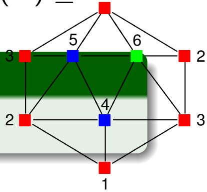
Definition (Chromatic numbers of ASCs)

The s-chromatic number, $\mathrm{chr}^s(K)$, is the least r such that K admits an (r, s) coloring.

- $|V| \geq \mathrm{chr}^1(K) \geq \mathrm{chr}^2(K) \geq \cdots \geq \mathrm{chr}^{1+\dim K}(K) = 1$
- $\mathrm{chr}^s(D[V]) = \left\lceil \frac{|V|}{s} \right\rceil$
- $K \subset K' \implies \mathrm{chr}^s(K) \leq \mathrm{chr}^s(K')$
- $\left\lceil \frac{n(K)}{s} \right\rceil \leq \mathrm{chr}^s(K) \leq \left\lceil \frac{m(K)}{s} \right\rceil$
- K admits an (r, s)-coloring $\implies \left\lceil \frac{n(K)}{s} \right\rceil \leq r \implies n(K) \leq rs$

Example (Chromatic numbers of the ASC P^2)

$\mathrm{chr}^1(P^2) = 6$, $\mathrm{chr}^2(P^2) = 3$, and $\mathrm{chr}^3(P^2) = 1$



Chromatic numbers of cyclic polytopes

The cyclic $2n$-polytope and $(2n+1)$-polytope on $m > n$ vertices

$$\mathrm{CP}(m, 2n), \qquad \mathrm{CP}(m, 2n+1)$$

are n-neighborly. The first chromatic numbers are

$$\mathrm{chr}^s(\mathrm{CP}(m, 2n)) = \left\lceil \frac{m}{s} \right\rceil, \qquad s < n$$

$$\mathrm{chr}^n(\mathrm{CP}(m, 2n)) = \begin{cases} 2 & m \text{ even} \\ 3 & m \text{ odd} \end{cases}$$

$$\mathrm{chr}^s(\mathrm{CP}(m, 2n+1)) = \left\lceil \frac{m}{s} \right\rceil, \qquad s < n$$

$$\mathrm{chr}^n(\mathrm{CP}(m, 2n+1)) = \begin{cases} 4 & n = 1 \\ 3 & n > 1 \end{cases}$$

Chromatic numbers of polyhedra

Definition (Chromatic numbers of polyhedra)

The s-chromatic number of the polyhedron M is the maximum

$$\operatorname{chr}^s(M) = \sup\{\operatorname{chr}^s(K) \mid K \text{ triangulates } M\}$$

$\operatorname{chr}^s(M) = r \iff$ Any triangulation of M can be colored with at most r colors such that there are no monochrome s-simplices.

Example (Chromatic numbers of 2-dimensional polyhedra)

- $\operatorname{chr}^1(M) \geq 5$ and $\operatorname{chr}^2(M) \geq 2$, $M =$ Möbius band.
- $\operatorname{chr}^1(\mathbf{R}P^2) \geq 5$ and $\operatorname{chr}^2(\mathbf{R}P^2) \geq 3$

4-color theorem

$\operatorname{chr}^1(S^2) = 4$, $\operatorname{chr}^2(S^2) = 2$, $\operatorname{chr}^3(S^2) = 1$.

What are the chromatic numbers of $\mathbf{R}P^2$?

Chromatic numbers of spheres

Chromatic numbers of the 3-sphere

$\mathrm{chr}^1(S^3) = \infty$ and $\mathrm{chr}^2(S^3) \geq 4$.

Proof.

- $\mathrm{chr}^1(\mathrm{CP(m,4)}) \to \infty$ for $m \to \infty$
- There is a triangulation ALT of S^3 with $\mathrm{chr}^2(\mathrm{ALT}) = 4$.

The first interesting chromatic number for a sphere is

$$\mathrm{chr}^{\lceil \frac{n}{2} \rceil}(S^n)$$

as $\mathrm{chr}^s(S^n) = \infty$ for $s < \lceil \frac{n}{2} \rceil$.

Speculations

- Is $\mathrm{chr}^2(S^3)$ finite?
- $\mathrm{chr}^1(S^2), \mathrm{chr}^2(S^3), \mathrm{chr}^2(S^4), \mathrm{chr}^3(S^5), \mathrm{chr}^3(S^6), \ldots = 4$?

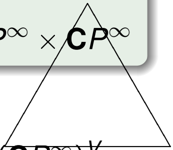
The Davis–Januszkiewicz space of K in three stages

- Let $\mathrm{DJ}(D[V]) = \mathrm{map}(V, \mathbf{C}P^\infty) = \overbrace{\mathbf{C}P^\infty \times \cdots \times \mathbf{C}P^\infty}^{m(K)}$
- For $\sigma \subset V$ consider $\mathrm{DJ}(D[\sigma]) = \mathrm{map}(V, V - \sigma; \mathbf{C}P^\infty, *)$ as the subspace of the σ-axes of $\mathrm{DJ}(D[V]) = \mathrm{map}(V, \mathbf{C}P^\infty)$
- $\mathrm{DJ}(K) = \bigcup_{\sigma \in K} \mathrm{DJ}(D[\sigma]) \subset \mathrm{DJ}(D[V])$

Example

If $K = \partial D[\{1,2,3\}] \subset D[\{1,2,3\}]$ then $\mathrm{DJ}(K)$ is the fat wedge

$$\mathbf{C}P^\infty \times \mathbf{C}P^\infty \times \{*\} \cup \mathbf{C}P^\infty \times \{*\} \times \mathbf{C}P^\infty \cup \{*\} \times \mathbf{C}P^\infty \times \mathbf{C}P^\infty$$



- $K \subset K' \implies \mathrm{DJ}(K) \subset \mathrm{DJ}(K')$
- $\bigvee_V \mathbf{C}P^\infty = \mathrm{DJ}(\mathrm{sk}_0(K)) \subset \mathrm{DJ}(K) \subset \mathrm{DJ}(D[V]) = (\mathbf{C}P^\infty)^V$

Vector bundles over Davis–Januszkiewicz spaces

Definition (The canonical vector bundle λ_K)

The canonical vector bundle λ_K over $\mathrm{DJ}(K)$ is the restriction

$$\begin{array}{ccc} \lambda_K & \longrightarrow & \lambda \times \cdots \times \lambda \\ \downarrow & & \downarrow \\ \mathrm{DJ}(K) & \hookrightarrow & \mathbf{C}P^\infty \times \cdots \times \mathbf{C}P^\infty \end{array} \qquad \dim \lambda_K = m(K)$$

to $\mathrm{DJ}(K)$ of the product of the tautological complex line bundles.

Theorem (The canonical vector bundle ξ_K)

There exists a short exact sequence of vector bundles

$$0 \to \xi_K \to \lambda_K \to \mathbf{C}^{m(K)-n(K)} \to 0$$

where $\dim \xi_K = n(K)$.

Colorings = Stable splittings of vector bundles

Assume that $n(K) \leq rs$. The following are equivalent:

- K admits an (r, s)-coloring
- There exists a lift in either of the diagrams

$$\begin{array}{ccc} & & BU(s)^r \\ & \overset{?}{\nearrow} & \downarrow \oplus \\ \mathrm{DJ}(K) & \xrightarrow[\lambda_K]{} & BU \end{array} \qquad \begin{array}{ccc} & & BU(s)^r \\ & \overset{?}{\nearrow} & \downarrow \oplus \\ \mathrm{DJ}(K) & \xrightarrow[\xi_K]{} & BU(rs) \end{array}$$

- There exist r vector bundles $\lambda_1, \ldots, \lambda_r$ over $\mathrm{DJ}(K)$ such that $\dim \lambda_j \leq s$ and

$$\lambda_K = \bigoplus_{1 \leq j \leq r} \lambda_j$$

in $K(\mathrm{DJ}(K))$.

A failed proof of the 4-color theorem

Theorem (The 4-color theorem)

$\mathrm{chr}^1(K) \leq 4$ *for all triangulations K of S^2.*

Failed Proof.

$$\begin{array}{ccc} & & BU(1)^4 \\ & \overset{?}{\nearrow} & \downarrow \oplus \\ \mathrm{DJ}(K) & \xrightarrow[\lambda_K]{} & BU \end{array} \qquad \begin{array}{ccc} & & BU(1)^4 \\ & \overset{?}{\nearrow} & \downarrow \oplus \\ \mathrm{DJ}(K) & \xrightarrow[\xi_K]{} & BU(4) \end{array}$$

□

Colorings of other compact surfaces? $\left\lfloor \frac{7+\sqrt{49-24\chi}}{2} \right\rfloor$. The 5-color theorem.

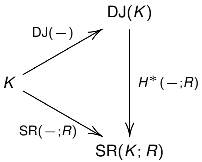
Stanley–Reisner algebras

Definition (The Stanley–Reisner algebra of K)

$\mathrm{SR}(K;R) = R[V]/\left(\prod \tau \mid \tau \in D[V] - K\right)$ is the quotient of the polynomial algebra on V (in degree 2) by the monomial ideal generated by the (minimal) non-simplices of K.

Theorem (Davis–Januszkiewicz)

$\mathrm{SR}(K;R) = H^*(\mathrm{DJ}(K);R)$



- If $V = \{v_1, v_2, v_3\}$ then
  - $\mathrm{SR}(D[V];R) = R[v_1, v_2, v_3]$
  - $\mathrm{SR}(\partial D[V];R) = R[v_1, v_2, v_3]/\langle v_1 v_2 v_3\rangle$
- $K \subset K' \implies \mathrm{SR}(K;R) \twoheadleftarrow \mathrm{SR}(K';R)$
- $R[V] \twoheadrightarrow \mathrm{SR}(K) = \lim(P(K)^{\mathrm{op}}; \mathrm{SR}(D[\sigma])) \subset \prod_{\sigma \in K} R[\sigma]$

Colorings and the Stanley–Reisner algebra

Theorem (Stanley–Reisner recognition of colorings)

The partition $V = V_1 \cup \cdots \cup V_r$ *is an* (r,s)*-coloring of* $K \iff$

$$\prod_{v \in V} (1 + v) = \prod_{1 \le j \le r} c_{\le s}(V_j)$$

in $\mathrm{SR}(K; \mathbf{Z})$.

Theorem (Colorings = Factorizations of symmetric polynomials)

K *admits an* (r,s)*-coloring* $\iff$ *there exist* r *elements* $c_1, \ldots, c_r$ *of* $\mathrm{SR}(K; \mathbf{Z})$ *such that* $\deg(c_j) \le 2s$ *and*

$$\prod_{v \in V} (1 + v) = \prod_{1 \le j \le r} c_j$$

in $\mathrm{SR}(K; \mathbf{Z})$.

The Stanley–Reisner ring of $P2$ and C_5

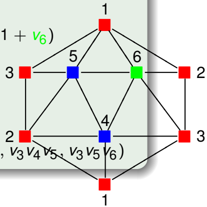
Example (A $(3,2)$-coloring of $P2$)

Since $[1,2,3,4,5,6]$ is a $(3,2)$-coloring, the identity

$$\prod_{1\le i\le 6}(1+v_i) = (1+v_1+v_2+v_3+v_2v_3+v_1v_3+v_1v_2)(1+v_4+v_5+v_4v_5)(1+v_6)$$

holds in the Stanley–Reisner ring for $P2$

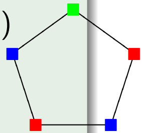
$$SR(P^2;\mathbf{Z}) = \mathbf{Z}[v_1,\dots,v_6]/(v_1v_2v_3, v_1v_2v_5, v_1v_3v_6, v_1v_4v_5, v_1v_4v_6, v_2v_3v_4, v_2v_4v_6, v_2v_5v_6, v_3v_4v_5, v_3v_5v_6)$$



Example (A $(3,1)$-coloring of C_5)

$$SR(C_5;\mathbf{Z}) = \mathbf{Z}[v_1,\dots,v_5]/(v_1v_3, v_1v_4, v_2v_4, v_2v_5, v_3v_5)$$

$$\prod_{1\le i\le 5}(1+v_i) = (1+v_1+v_3)(1+v_2+v_4)(1+v_5)$$



Another failed proof of the 4-color theorem

Theorem (The 4-color theorem)

$\mathrm{chr}^1(K) \le 4$ *for all triangulations* K *of* S^2*.*

Failed Proof.

Let K be a triangulation of S^2 with vertex set V. There exist 4 elements $c_1, c_2, c_3, c_4 \in \mathrm{SR}(K;\mathbf{Z})$ of degree ≤ 2 so that

$$\prod_{v \in V} (1+v) = c_1 c_2 c_3 c_4$$

in $\mathrm{SR}(K;\mathbf{Z})$. □

Summary

What we learned today

- An (r, s)-coloring is a coloring of the vertices by r colors so that at most s vertices of any simplex has the same color
- (r, s)-colorings depend only on the s-skeleton
- (r, s)-coloring is equivalent to splitting the canonical vector bundle over the Davis–Januszkiewicz space
- (r, s)-coloring is equivalent to factorizing the total Chern class of the canonical vector bundle in the Stanley–Reisner ring

Questions to think about

Questions

- Is $\mathrm{chr}^2(S^3) = 4$?
- Is $\mathrm{chr}^n(S^{2n-1}) = 4$ for all $n \geq 2$?
- Is $\mathrm{chr}^n(S^{2n}) = 4$ for all $n \geq 1$?
- Is it possible to find a topological proof of the 4-color theorem?
- Is it possible to compute the chromatic numbers of the compact surfaces?
- Is there a connection between the face numbers and the chromatic numbers (as in the 6-color theorem)?