FROBENIUS CATEGORIES OF CHEVALLEY GROUPS

JESPER M. MØLLER

Contents

Abstract. Notes for a course, held in the Biebrza Marshes, Poland, May 6–10, 2010, on Frobenius categories of finite groups of Lie type.

1. Notation

- p is a fixed prime
- n_p is the highest power of p dividing the natural number $n = \prod n_p$

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/home/moller/projects/autplfg/version3/presentation/frobenius.tex.

- $m^\varphi \in N$ and $K^\varphi \subset N$ are the images under the map $M \xrightarrow{\varphi} N$ of the element $m \in M$ or the subset $K \subset M$. The composition of the two maps $M \xrightarrow{\varphi} N \xrightarrow{\psi} P$ is $\varphi\psi$
- **GRP** is the category of groups
- **GRPOID** is the category of groupoids
- **ALG** is the category of algebraic groups
- **FRB** is the category of Frobenius categories

2. Category theory for beginners

$$\begin{tikzcd} & Q \arrow[dr, "\psi"] & \\ P \arrow[ur, "\varphi"] \arrow[rr, "\varphi\psi"'] & & R \end{tikzcd}$$

Let $\mathcal{C}$ be a category with objects P and Q:

- $\mathcal{C}(P,Q)$ is the set of morphisms from P to Q
- $\mathcal{C}(P,Q) \times \mathcal{C}(Q,R) \xrightarrow{\circ} \mathcal{C}(P,R)\colon (\varphi,\psi) \to \varphi \circ \psi$ is composition in $\mathcal{C}$
- $\mathcal{C}(Q) = \mathcal{C}(Q,Q)$ is the monoid of endomorphisms and $\mathcal{C}(Q)^\circ$ the group of automorphisms of Q

2.1. **Isomorphisms and equivalences.**

Definition 2.1. [31] *[nCat] A functor $T\colon \mathcal{C} \to \mathcal{D}$ is*

- *an isomorphism if it is bijective on both objects and morphisms*
- *an equivalence if $\mathcal{C}(Q,R) \to \mathcal{D}(Q^T,R^T)$ is bijective for all objects Q, R of $\mathcal{C}$ and every object of $\mathcal{D}$ is isomorphic to some object in the image of T*

We say that a functor is an equivalence if it is essentially surjective on objects and fully faithful on morphisms.

2.2. **The exterior quotient.** An *left (right) interior structure* in a category $\mathcal{C}$ is a function that to any object P of $\mathcal{C}$ associates a group $\mathcal{I}(P) \leq \mathcal{C}(P)^\circ$ (of "inner" automorphisms of P) such that $\mathcal{I}(P)\circ\varphi \subset \varphi\circ\mathcal{I}(Q)$ $(\varphi\circ\mathcal{I}(Q) \subset \mathcal{I}(P)\circ\varphi)$ for all $\varphi \in \mathcal{C}(P,Q)$. (In case of a left interior structure, this means that for any $p \in \mathcal{I}(P)$ there exists $q \in \mathcal{I}(Q)$ and a commutative diagram

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "\varphi"'] & P \arrow[d, "\varphi"] \\ Q \arrow[r, "q"] & Q \end{tikzcd}$$

in $\mathcal{C}$.) In particular, $\mathcal{I}(P) \lhd \mathcal{C}(P)^\circ$.

Definition 2.2. [37, 1.3] *The* exterior quotient *of the left interior structure $\mathcal{I}$ is the quotient category $\widetilde{\mathcal{C}}$ that has the same objects as $\mathcal{C}$ and morphism sets*

$$\widetilde{\mathcal{C}}(P,Q) = \mathcal{C}(P,Q)/\mathcal{I}(Q), \qquad \widetilde{\mathcal{C}}(P)^\circ = \mathcal{C}(P)^\circ/\mathcal{I}(P) \qquad (\textit{left cosets})$$

Composition in $\widetilde{\mathcal{C}}$ is induced from composition in $\mathcal{C}$ by the rule $\varphi\mathcal{I}(Q) \circ \psi\mathcal{I}(R) = \varphi\psi\mathcal{I}(R)$ for $\varphi \in \mathcal{C}(P,Q)$, $\psi \in \mathcal{C}(Q,R)$.

The exterior quotient *of the right interior $\mathcal{I}$ structure is the quotient category $\widetilde{\mathcal{C}}$ that has the same objects as $\mathcal{C}$ and whose morphism sets are*

$$\widetilde{\mathcal{C}}(P,Q) = \mathcal{I}(P)\backslash\mathcal{C}(P,Q) \qquad \widetilde{\mathcal{C}}(P)^\circ = \mathcal{I}(P)\backslash\mathcal{C}(P)^\circ \qquad (\textit{right cosets})$$

Composition in $\widetilde{\mathcal{C}}$ is induced from composition in $\mathcal{C}$ by the rule $\mathcal{I}(P)\varphi \circ \mathcal{I}(Q)\psi = \mathcal{I}(P)\varphi\psi$ for $\varphi \in \mathcal{C}(P,Q)$, $\psi \in \mathcal{C}(Q,R)$.

Example 2.3. (1) In the category **GRP** of groups, the inner automorphisms $\mathrm{Inn}(G) = Z(G)\backslash G \leq \mathbf{GRP}(G)$ provide a left interior structure. In the left exterior quotient $\widetilde{\mathbf{GRP}}$ the morphism sets and the automorphism groups

$$\widetilde{\mathbf{GRP}}(G,H) = \mathbf{GRP}(G,H)/\mathrm{Inn}(H), \qquad \widetilde{\mathbf{GRP}}(G)^\circ = \mathbf{GRP}(G)^\circ/\mathrm{Inn}(G)$$

consist of left H-conjugacy classes of group homomorphisms of $G \to H$ and outer automorphisms of G. (Observe that morphisms in $\widetilde{\mathbf{GRP}}$ do not have images but still have *kernels*.)
(2) Consider the group $G = G(*)$ as a category with one object $*$. The subgroup $O^pG(*)$ generated by the p'-elements is a left and right interior structure because $O^pG(*)g = gO^pG(*)$ for all $g \in G(*)$ as $O^pG(*)$

is a normal subgroup. The right exterior quotient $\widetilde{G} = O^pG\backslash G$ is the group of right cosets. This is a functorial construction. Any group homomorphism $G_1 \xrightarrow{f} G_2$ sends O^pG_1 to O^pG_2 and so induces a group homomorphism $O^p(G_1)\backslash G_1 \to O^p(G_2)\backslash G_2$. Thus $G \mapsto O^pG\backslash G$ is an endofunctor of the category **GRP** of groups. (We will need this construction in Proposition 11.8.)

(3) Suppose that M is a groupoid. There is a left and right interior structure given by $\mathcal{I}(P) = O^pM(P)$ for any object P of M. In the right exterior quotient groupoid $O^pM\backslash M$ the morphism sets and automorphism groups are $O^pM(P)\backslash M(P,Q)$ and $O^pM(P)\backslash M(P)$ (the p-groups nearest to $M(P)$). This is a functorial construction. For any functor $M_1 \xrightarrow{f} M_2$ and any object P of M_1, we have that $O^pM_1(P)^f \subset O^pM_2(P^f)$, which means that f induces a functor $O^pM_1\backslash M_1 \to O^pM_2\backslash M_2$ between the right exterior quotient groupoids. Thus $M \mapsto O^pM\backslash M$ is en endofunctor of the category **GRPOID** of groupoids. (We will need this construction in subsection 11.1.)

3. Group theory for beginners

Let G be a finite group with element $g \in G$ and subgroups $H, K \leq G$:

- G is a p-group if $|G|_p = |G|$ and a p'-group if $|G|_p = 1$.
- $h^g = g^{-1}hg$ for group elements $g, h \in G$
- Conjugation by $g \in G$ is the inner automorphism $g \in \mathbf{GRP}(G)^\circ$ taking $h \in G$ to h^g.
- H^g is the image of H under conjugation by g
- $H \lhd G$ means that H is a normal subgroup of G, ie $H^g = H$ for all $g \in G$
- $C_G(H) = \{g \in G \mid \forall h \in H\colon h^g = h\}$ is the centralizer of H
- $N_G(H) = \{g \in G \mid H^g = H\}$ is the normalizer of H
- $N_G(H,K) = \{g \in G \mid H^g \leq K\}$ is the transporter set
- $H\backslash G \ni Hg$ is the set of right cosets of $H \leq G$ and $G/H \ni gH$ the set of left cosets[1]
- $O^p(G)$ is the normal subgroup generated by all p'-elements of G [23, p 3]
- $O^{p'}(G)$ is the normal subgroup generated by all p-elements of G [24, p 34]
- $O_p(G)$ is the largest normal p-subgroup of G, the intersection of all Sylow p-subgroups of G [40, 9.1]
- $O_{p'}(G)$ the largest normal p'-subgroup of G [40, 9.1]

3.1. **Index theorems.** We shall need these basic facts about the index of a subgroup. If H and K are subgroups, their commutator is $[H,K] = \langle h^{-1}k^{-1}hk \mid h \in H, k \in K\rangle$, $[H,K] \lhd \langle H,K\rangle$; we say that H normalizes K if $H \leq N_G(K)$ or $[H,K] \leq K$.

Proposition 3.1. [40, 1.3.11, 1.3.14, 1.4.4] *Let H and K be subgroups of G and put $H \cdot K = \{hk \mid h \in H, k \in K\}$.*

1. $|H \cdot K||H \cap K| = |H||K|$
2. *If $L \leq G$, and L contains at least one of the groups H or K then $(H \cdot K) \cap L = (H \cap L) \cdot (K \cap L)$*
3. *$|G : H \cap K| \leq |G : H||G : K|$ with equality if the indeces $|G : H|$ and $|G : K|$ are coprime.*
4. *If $H \cdot K = K \cdot H$, then $HK = \langle H, K\rangle = KH$ is a subgroup.*
5. *If H normalizes K then HK is a subgroup, $K \lhd HK$, $H \cap K \lhd H$, and $HK/K \cong H/H \cap K$.*

Proposition 3.2. *The following conditions are equivalent for a normal subgroup $N \lhd G$:*

1. *N contains a Sylow p-subgroup of G*
2. *N contains all p-elements of G*
3. *N contains $O^{p'}(G)$*
4. *N has p'-index*

3.2. **Sylow subgroups.** Sylow's theorem is the king of finite group theory. Sylow proved his theorem without knowing what a group is!

Definition 3.3. *A Sylow p-subgroup is a p-subgroup of index prime to p.*

Theorem 3.4 (Sylow theorem)**.** [40, 1.6.16] *Let G be any finite group.*

1. *G has a Sylow p-subgroup*
2. *Every p-subgroup of G is contained in a Sylow p-subgroup.*

[1] In a *right* coset, Hg, the element from G is on the right and the set, $H\backslash G$, of right cosets is a right G-set.

(3) *All Sylow p-subgroups are conjugate*
(4) *The number of Sylow p-subgroups (the length of the Sylow p-subgroup) is congruent to* 1 *modulo p.*

Corollary 3.5 (Sylow subgroups of (normal) subgroups and factor groups)**.** [40, 1.6.18, 5.2.14][23, Lemma 4.4] *Let P be a Sylow p-subgroup of G, $H \leq G$ a subgroup, and $N \lhd G$ a normal subgroup.*

(1) *$H^g \cap P$ is a Sylow p-subgroup of H^g for some $g \in G$*
(2) *$N \cap P$ is a Sylow p-subgroup of N and PN/N is a Sylow p-subgroup of G/N.*
(3) *$G = N_G(N \cap P) \cdot N$ (The Frattini argument)*

Proof. (1) Let S be a Sylow p-subgroup of H. There is an element g of G so that S^g is contained in the Sylow p-subgroup P of G (Theorem 3.4). Now $S^g \leq H^g \cap P \leq H^g$. Since S^g is a maximal p-subgroup of H^g, we actually have $S^g = H^g \cap P$. This subgroup is a Sylow p-subgroup of H^g.
(2) This follows from the index formulas of Proposition 3.1.
(3) $N \cap P$ is a Sylow p-subgroup of N. Let g be any element of G. Then $(N \cap P)^g \leq N^g = N$, so $(N \cap P)^g$ is also a Sylow p-subgroup of N, and thus $(N \cap P)^g = (N \cap P)^n$ for some $n \in N$. This means that gn^{-1} normalizes $N \cap P$ and that $g = (gn^{-1})n \in N_G(N \cap P) \cdot N$. □

By induction on the length of a subnormal chain from H to G it follows that if H is subnormal in G, then $H \cap P$ is a Sylow p-subgroup of H.

Lemma 3.6. *If the normal subgroup $N \lhd G$ has p-power index, then all Sylow p-subgroups of G are conjugate by elements of N.*

Proof. We need to prove that $N \to G \to N_G(P)\backslash G$ is surjective. The image of this map is $N \cap N_G(P)\backslash N$. Thus we need to show that $|N : N \cap N_G(P)| = |G : N_G(P)|$. This follows from the index formula $|G : N \cap N_G(P)| = |G : N||G : N_G(P)|$ which holds since N and $N_G(P)$ have coprime indices (Proposition 3.1.(3)). □

3.3. **From centralizers to normalizers.** We introduce the K-normalizer of a subgroup.

Definition 3.7. *For any $Q \leq G$ and $K \leq \mathbf{GRP}(Q)^\circ$ the K-normalizer $N_G^K(Q)$ is the inverse image of K under the homomorphism $N_G(Q) \to \mathbf{GRP}(Q)^\circ$. Q is K-normal in G if $N_G^K(Q) = G$.*

By the very definition there is a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & C_G(Q) \arrow[r] \arrow[d, equal] & N_G^K(Q) \arrow[r] \arrow[d, hook] & K \cap \mathbf{GRP}(Q)^\circ \arrow[d, hook] \\ 1 \arrow[r] & C_G(Q) \arrow[r] & N_G(Q) \arrow[r] & \mathbf{GRP}(Q)^\circ \end{tikzcd}$$

with exact rows. In particular, $C_G(Q) = N_G^1(Q) \lhd N_G^K(Q) \leq N_G^{\mathbf{GRP}(Q)^\circ}(P) = N_G(Q)$. If $H \leq K \leq \mathbf{GRP}(Q)^\circ$, then $N_G^H(Q) \leq N_G^K(Q)$, and if $H \lhd K$, then $N_G^H(Q) \lhd N_G^K(Q)$. If K contains the inner automorphisms of Q, then $Q \lhd N_G^K(Q)$.

Any injective group homomorphism $\varphi \colon Q \to P$ induces an isomorphism $\mathbf{GRP}(Q)^\circ \to \mathbf{GRP}(Q^\varphi)^\circ$ taking $k \in \mathbf{GRP}(Q)^\circ$ to $k^\varphi = \varphi^{-1}k\varphi$. Let $K^\varphi \leq \mathbf{GRP}(Q^\varphi)^\circ$ be the image of $K \leq \mathbf{GRP}(Q)^\circ$ under this isomorphism.

$$\begin{tikzcd} Q \arrow[r, "\varphi", "\cong"'] \arrow[loop below, "K"] & Q^\varphi \arrow[loop below, "K^\varphi"] \end{tikzcd}$$

Proposition 3.8. *Assume that $Q \leq P \leq G$ and $K \leq \mathbf{GRP}(Q)^\circ$.*

(1) *$N_P^K(Q)^\varphi = N_{P^\varphi}^{K^\varphi}(Q^\varphi)$ for any group automorphism $\varphi \in \mathbf{GRP}(G)^\circ$*
(2) *$N_P^K(Q)^\varphi \leq N_P^{K^\varphi}(Q^\varphi)$ for any injective group homomorphism $\varphi \colon Q \cdot N_P^K(Q) \to P$*

Proof. (1) This is clear as φ is an automorphism of G.
(2) If $g \in P$ normalizes Q and conjugation by g is the automorphism $k \in K$ of Q, then g lies in $N_P^K(Q)$ so g^φ lies in P, g^φ normalizes Q^φ, and conjugation by g^φ is the automorphism k^φ of Q^φ. □

4. Algebraic group theory for beginners

In this section, $\mathbf{F}$ is an algebraically closed field of finite characteristic r.

An affine variety is the common set of zeros in $\mathbf{F}^n$ for a finite set of polynomials. The affine varieties are the closed sets in the Zariski topology on $\mathbf{F}^n$. The affine space $\mathbf{F}^n$ itself is an irreducible affine variety.

The set

$$\mathrm{GL}_n(\mathbf{F}) = \{(x_{ij}) \in \mathbf{F}^{n^2} \mid \det(x_{ij}) \neq 0\}$$

is a principal open set in the irreducible affine variety $\mathbf{F}^{n^2}$. It is therefore itself an affine variety [26, p 10], namely the variety in $\mathbf{F}^{n^2+1} \ni (d, x_{ij})$ defined by the equation $d\det(x_{ij}) - 1 = 0$. The group operations can be seen to be morphisms of varieties. $\mathrm{GL}_n(\mathbf{F})$ is the basic example of a an algebraic group[26, Chp II].

Definition 4.1. *An algebraic group over $\mathbf{F}$ is a variety that is also a group such that the group operation maps are morphisms of algebraic varieties. An algebraic group morphism between algebraic groups is a morphism of varieties that is also a group homomorphism.*

Examples of algebraic groups:

- $\mathrm{GL}_n(\mathbf{F})$
- Zariski closed subgroups of $\mathrm{GL}_n(\mathbf{F})$, such as $\mathrm{SL}_n(\mathbf{F})$
- Products of algebraic groups are algebraic groups
- The multiplicative group $\mathbf{F}^\times = \mathrm{GL}_1(\mathbf{F})$ (with endomorphism monoid $\mathbf{ALG}(\mathbf{F}^\times) = \mathbf{Z}$).
- The additive group
$$\mathbf{F}^+ = U_2(\mathbf{F}) = \left\{\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} \mid t \in \mathbf{F}\right\} \leq \mathrm{GL}_2(\mathbf{F}), \qquad \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & t+u\\ 0 & 1\end{pmatrix}$$
(with automorphism group $\mathbf{ALG}(\mathbf{F}^+)^\circ = \mathbf{F}^\times$).
- Diagonal matrices $T_n(\mathbf{F}) = \left\{\begin{pmatrix}s_1 & 0 & 0\\ 0 & s_2 & 0\\ 0 & 0 & s_3\end{pmatrix} \mid s_1, \ldots, s_n \in \mathbf{F}^\times\right\} \cong \mathbf{F}^\times \times \cdots \times \mathbf{F}^\times$
- Unitriangular matrices $U_n(\mathbf{F}) = \left\{\begin{pmatrix}1 & * & *\\ 0 & 1 & *\\ 0 & 0 & 1\end{pmatrix}\right\} \leq \mathrm{GL}_n(\mathbf{F})$
- Triangular matrices $B_n(\mathbf{F}) = T_n(\mathbf{F})U_n(\mathbf{F}) = \left\{\begin{pmatrix}s_1 & * & *\\ 0 & s_2 & *\\ 0 & 0 & s_3\end{pmatrix}\right\} \leq \mathrm{GL}_n(\mathbf{F})$

Definition 4.2. *A torus is an algebraic group isomorphic to $T_n(\mathbf{F}) = \mathbf{F}^\times \times \cdots \times \mathbf{F}^\times$ for some n*

The abelian group of algebraic group homomorphisms from an n-torus $T_n(\mathbf{F})$ to the multiplicative group $\mathbf{F}^\times$ is the free finitely generated abelian group (lattice)

$$\mathbf{ALG}(T_n(\mathbf{F}), \mathbf{F}^\times) = \mathbf{Z}\{e_1, \ldots, e_n\}$$

generated by the n coordinate functions $e_i \colon T_n(\mathbf{F}) \to \mathbf{F}^\times$ given by $s^{e_i} = s_i$ for $s = (s_1, \ldots, s_n) \in T_n(\mathbf{F})$. The linear combination $\sum n_i e_i \in \mathbf{Z}\{e_1, \ldots, e_n\}$ corresponds to the homomorphism $T_n(\mathbf{F}) \to \mathbf{F}^\times \colon s \mapsto \prod s_i^{n_i}$; in particular, $s^{e_j - e_i} = s_j s_i^{-1}$.

Theorem 4.3 (Existence of maximal tori)**.** *Any algebraic group contains a maximal torus and any two maximal tori are conjugate.*

For instance, $T_{n+1}(\mathbf{F})$ is a maximal torus in $\mathrm{GL}_{n+1}(\mathbf{F})$ and

$$T_{n+1}(\mathbf{F}) \cap \mathrm{SL}_{n+1}(\mathbf{F}) = \langle\, \mathrm{diag}(s_1, s^{-1}, 1, \ldots, 1), \ldots, \mathrm{diag}(1, \ldots, s_n, s_n^{-1}) \mid s_1, \ldots, s_n \in \mathbf{F}^\times \rangle$$

is a maximal torus of $\mathrm{SL}_{n+1}(\mathbf{F})$

4.1. Root systems.

Let E be a Euclidean space over $\mathbf{R}$, ie E is a finite-dimensional real vector space with an inner product $(\cdot, \cdot)$. Let α be a nonzero vector. The orthogonal projection of β on α is $\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ and the *orthogonal reflection* across the hyperplane $\alpha^\perp$ perpendicular to α is the isometry

$$r_\alpha(\beta) = \beta - \langle \beta, \alpha\rangle\alpha, \qquad \beta \in E,$$

where we let $\langle \beta, \alpha\rangle = 2\frac{(\beta,\alpha)}{(\alpha,\alpha)} = 2\frac{(\beta,\alpha)}{|\alpha|^2}$. (Then the inner product $(\alpha, \beta) = 2\langle \alpha, \beta\rangle|\alpha|^2$.)

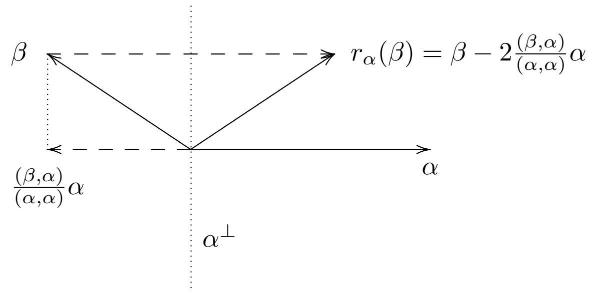


FIGURE 1. Reflection of β across $\alpha^\perp$ changes β in the direction of α

Definition 4.4. *A root system in E is a set $\Sigma \subset E$ of nonzero vectors (called roots) in E such that $r_\alpha(\Sigma) = \Sigma$ for all $\alpha \in \Sigma$.*

Suppose that Σ is a root system. Σ is

reduced: if $\mathbf{R}\alpha \cap \Sigma = \{-\alpha, \alpha\}$ for all $\alpha \in \Sigma$ (Note that $-\alpha = r_\alpha(\alpha) \in \Sigma$.)
crystallographic: if the numbers $\langle \beta, \alpha \rangle$ are all integers (the Cartan integers)
irreducible: if there is no partition $\Sigma = \Sigma_1 \cup \Sigma_2$ of Σ into nonempty orthogonal subsets

The *Weyl group* of Σ is the group of isometries $W(\Sigma) = \langle r_\alpha \mid \alpha \in \Sigma \rangle$ generated by the reflections associated to the elements of Σ. Since $r_\alpha(\Sigma) = \Sigma$, the Weyl group $W(\Sigma)$ acts on the root system Σ.

A *fundamental system* for the reduced, crystallographic root system Σ is a linearly independent set of roots $\Pi \subset \Sigma$ such that $\Sigma = \Sigma^+ \cup \Sigma^-$ where Σ^+ (Σ^-) is the set of roots expressible as nonnegative (nonpositive) linear combinations of the fundamental roots. The information contained in the *Cartan matrix* is the matrix $(\langle \alpha, \beta \rangle)_{\alpha,\beta \in \Pi}$. is conveniently displayed in the *Dynkin diagram* is the graph whose nodes are the fundamental roots and with $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ edges connecting $\alpha, \beta \in \Pi$, $\alpha \neq \beta$ [13, 3.4]. There is no edge between perpendicular fundamental roots. In case two fundamental roots are not perpendicular and not of equal length, then there is an inequality sign between them. The relation between the lengths of two fundamental roots and the angle between them is given by

$$\langle \alpha, \beta \rangle |\beta|^2 = 2(\alpha, \beta) = 2(\beta, \alpha) = \langle \beta, \alpha \rangle |\alpha|^2, \qquad \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4\frac{(\alpha, \beta)(\beta, \alpha)}{(\alpha, \alpha)(\beta, \beta)} = 4\frac{(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)} = 4\cos^2 \angle(\alpha, \beta)$$

The number of edges connecting α and β in the Dynkin diagram, $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$, is the product of two integers and is ≥ 0 and ≤ 4 and, in fact, ≤ 3 since α and β are linearly independent so not proportional. This leaves very few possibilities for the angles between and relative lengths of two fundamental roots [4, VI.§1.3]. Therefore the Dynkin diagram determines the root system completely. (Figure 2 shows three examples of 2-dimensional root systems with Cartan matrices and Dynkin diagrams. The root system $A_1 \times A_1$ could be added to that picture. The Weyl group for B_2 is the symmetry group of the square given by the roots, the dihedral group of order 8. The Weyl group for G_2 is the symmetry group of the hexagon given by the short roots, the dihedral group of order 12.)

Here are some useful relations

$$\langle \alpha, \beta \rangle = 2\frac{(\alpha, \beta)}{|\beta|^2} = (\alpha, \beta^\vee) \quad (\beta^\vee = \frac{2\beta}{|\beta|^2}), \qquad (\alpha, \beta) = \frac{1}{2}|\beta|^2 \langle \alpha, \beta \rangle, \qquad \langle \alpha, \sum b_i \beta_i \rangle = \frac{\sum b_i \langle \alpha, \beta_i \rangle |\beta_i|^2}{|\sum b_i \beta_i|^2}$$

connecting the bilinear form $(-,-)$ and the function $\langle -,- \rangle$ which is linear only in the first variable.

In the above definition the inner product comes out of the blue. Let V be any real vector space, α an element of V, and $\alpha^\vee$ an element of the dual of V, a real form on V. The endomorphism

$$r_{\alpha,\alpha^\vee}(\beta) = \beta - (\beta)\alpha^\vee \alpha, \qquad \beta \in V,$$

fixes the hyperplane $\ker \alpha^\vee$ and has order 2 if $(\alpha)\alpha^\vee = 2$. Let us call such a map a *reflection.* (If V has an inner product, then $\alpha^\vee = 2\frac{(-,\alpha)}{(\alpha,\alpha)}$ is a possibility.)

Lemma 4.5. [4, VI.§1] *Let V be a real vector space and $\Sigma \subset V$ a finite set of nonzero vectors. Suppose that*

(1) *Σ spans V*
(2) *for each $\alpha \in \Sigma$ there exists a linear form $\alpha^\vee$ on V such that $(\alpha)\alpha^\vee = 2$ and $r_{\alpha,\alpha^\vee}$ stabilizes Σ*

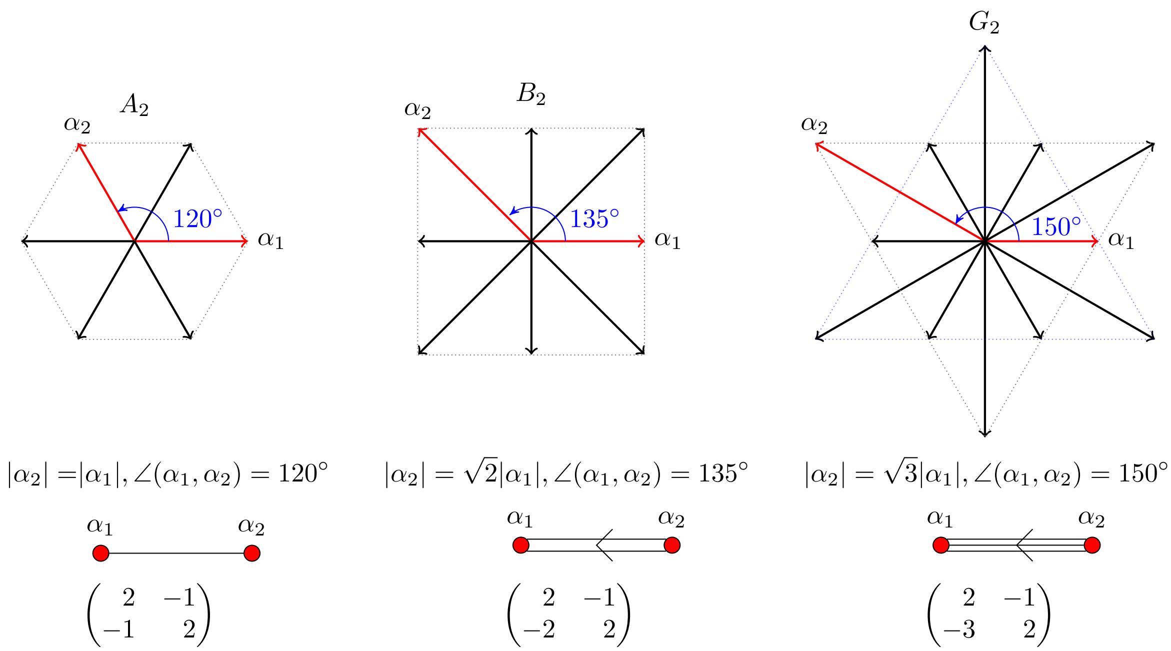


FIGURE 2. Root systems A_2, B_2, and G_2

(3) $(\Sigma)\alpha^\vee \subset \mathbf{Z}$ *for all* $\alpha \in \Sigma$.

Then there is an inner product $(\cdot, \cdot)$ *on* V *such that* $r_{\alpha,\alpha^\vee}$ *is the orthogonal reflection across* $\alpha^\perp$ *and* Σ *is a crystallographic root system in the Euclidean space* $(V, (\cdot, \cdot))$*.*

In fact, $\alpha^\vee$ is uniquely determined by α and we can write just r_α instead of $r_{\alpha,\alpha^\vee}$.

Example 4.6 (The root system A_n)**.** Let $e_1, \ldots, e_{n+1}$ be the standard orthonormal basis for $\mathbf{R}^{n+1}$, $n \geq 2$, with the usual inner product. The set of $n(n+1)$ vectors

$$A_n = \{e_j - e_i \mid 1 \leq i, j \leq n+1, i \neq j\} \subset \mathbf{Z}\{e_1, \ldots, e_{n+1}\}$$

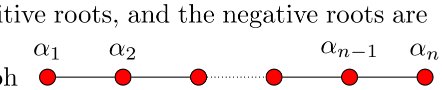
is a (reduced, crystallographic, irreducible) root system. The reflection $r_{e_j - e_i}$ across $(e_j - e_i)^\perp$

$$r_{e_j-e_i}(\beta) = \beta - (\beta_j - \beta_i)(e_j - e_i) = \beta + (\beta_i - \beta_j)e_j + (\beta_j - \beta_i)e_i$$

interchanges the ith and jth coordinate for any vector $\beta \in \mathbf{R}^{n+1}$. The Weyl group $W(A_n) \leq \mathrm{GL}_{n+1}(\mathbf{Z}) \leq \mathrm{GL}_{n+1}(\mathbf{R})$ generated by these reflections is the permutation group Σ_{n+1} on the $n+1$ coordinates. (The rank of the root system A_n, the dimension of the vector space spanned by A_n, is n.) The set

$$\Pi = \{e_2 - e_1, \ldots, e_{n+1} - e_n\}$$

is a fundamental system, $A_n^+ = \{e_j - e_i \mid 1 \leq i < j \leq n+1\}$ are the positive roots, and the negative roots are $A_n^- = \{e_j - e_i \mid 1 \leq j < i \leq n+1\}$. The Dynkin diagram is the linear graph



with Π as vertex set.

Example 4.7 (The root system B_n)**.** Let $e_1, \ldots, e_n$ be the standard orthonormal basis for $\mathbf{R}^n$, $n \geq 2$, with the usual inner product. The set of $2n^2$ vectors

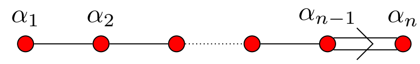
$$B_n = \{\pm e_j \pm e_i \mid n \geq j, i \geq 1, i \neq j\} \cup \{\pm e_i \mid 1 \leq i \leq n\} \subset \mathbf{Z}\{e_1, \ldots, e_n\}$$

is a (reduced, crystallographic, irreducible) root system. The reflections r_{e_i} and $r_{e_j+e_i}$

$$r_{e_i}(\beta) = \beta - 2\beta_i e_i, \qquad r_{e_j+e_i}(\beta) = \beta - (\beta_j + \beta_i)(e_j + e_i) = \beta - (\beta_j + \beta_i)e_j - (\beta_j + \beta_i)e_i$$

changes sign on the ith coordinate and swaps the jth and ith coordinates and changes their signs of any vector $\beta \in \mathbf{R}^n$. The Weyl group $W(B_n) = C_2 \wr \Sigma_n$ consists of coordinatewise permutations and sign changes. The set

$$\Pi = \{e_2 - e_1, \ldots, e_n - e_{n-1}, e_1\}$$

is a fundamental system and $B_n^+ = \{e_j \pm e_i \mid n \geq j > i \geq 1\} \cup \{e_i \mid 1 \leq i \leq n\}$ are the positive roots. The Dynkin diagram is the graph  with Π as vertex set.

Two root systems are isomorphic if there is a bijective map between them preserving the forms $\langle\cdot,\cdot\rangle$ [24, Definition 1.8.4].

Theorem 4.8 (Classification of irreducible reduced crystallographic root systems)**.** [4, VI.§4] [24, Table 1.8, Theorem 1.8.7] *Two irreducible reduced crystallographic root systems are isomorphic if and only if their Dynkin diagrams are isomorphic. Any irreducible reduced crystallographic root is isomorphic to precisely one of the root systems A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 4)$, E_6, E_7, E_8, F_4 or G_2.*

There is a unique element $w_0 \in W(\Sigma)$ so that $w_0(\Pi) = -\Pi$ [13, Proposition 2.2.6]. This element has order 2 and the linear transformation $-w_0$ is an automorphism of Σ. When $w_0 \neq -1$, which happens for A_n $(n > 1)$, D_{2n+1}, and E_6, $\rho = -w_0$ is the unique automorphism of Π of order 2 [24, Remark 1.8.9].

4.2. **Root groups.** We shall illustrate the idea of *root groups* by the simple example of $\mathrm{GL}_3(\mathbf{F})$.

Example 4.9 (Root subgroups of $\mathrm{GL}_3(\mathbf{F})$)**.** Consider the algebraic group $\mathrm{GL}_3(\mathbf{F})$ and let

$$T_3(\mathbf{F}) = \left\{ \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{pmatrix} \middle| \; s_1, s_2, s_3 \in \mathbf{F}^\times \right\}$$

be the standard maximal torus. For $i \neq j$, $1 \leq i, j \leq 3$, put

$$x_{ij}(t) = I + te_{ij} \in \mathrm{GL}_3(\mathbf{F}), \qquad t \in \mathbf{F},$$

where I is the identity matrix and e_{ij} the matrix with 1 in the ith row and jth column and 0 elsewhere, eg

$$x_{12}(t) = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad t \in \mathbf{F}$$

The map x_{12} is clearly injective and as

$$x_{12}(t)x_{12}(u) = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t+u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = x_{12}(t+u),$$

x_{12} is an isomorphism between $\mathbf{F}^+$ and the subgroup

$$X_{12} = \left\{ \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \; t \in \mathbf{F} \right\}$$

of $\mathrm{GL}_3(\mathbf{F})$. Also, the computation

$$\begin{pmatrix} s_1^{-1} & 0 & 0 \\ 0 & s_2^{-1} & 0 \\ 0 & 0 & s_3^{-1} \end{pmatrix} \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{pmatrix} = \begin{pmatrix} 1 & s_2 s_1^{-1} t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & s^{e_2 - e_1} t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

shows that the maximal torus $T_3(\mathbf{F})$ normalizes the subgroup X_{12}.

Similarly for all six roots groups X_{ij}. Conjugating the parameterization $x_{ij}(t)$ of X_{ij} by an element s of the maximal torus

$$x_{ij}(t)^s = x_{ij}(s^{e_j - e_i} t), \qquad s \in T_3(\mathbf{F}), \quad t \in \mathbf{F},$$

amounts to reparameterization by the factor $s^{e_j - e_i}$. We may visualize this by the commutative diagram

$$\begin{tikzcd} X_{ij} \arrow[r, "s"] & X_{ij} \\ \mathbf{F}^+ \arrow[u, "x_{ij}"] \arrow[r, "s^{e_j - e_i}"'] & \mathbf{F}^+ \arrow[u, "x_{ij}"'] \end{tikzcd}$$

of algebraic groups isomorphisms.

Let $\Sigma = \{X_{12}, X_{13}, X_{23}, X_{21}, X_{31}, X_{32}\}$ be the set of these six subgroups of $\mathrm{GL}_3(\mathbf{F})$. We can embed Σ into the lattice $\mathbf{ALG}(T_3(\mathbf{F}), \mathbf{F}^\times) = \mathbf{Z}\{e_1, e_2, e_3\}$ by the map

$$\Sigma \to \mathbf{ALG}(T_3(\mathbf{F}), \mathbf{F}^\times) = \mathbf{Z}\{e_1, e_2, e_3\} \colon X_{ij} \mapsto e_j - e_i \tag{4.10}$$

taking each subgroup $X_{ij} \in \Sigma$ to its associated reparameterization function $e_j - e_i \in \mathbf{ALG}(T_3(\mathbf{F}), \mathbf{F}^\times) = \mathbf{Z}\{e_1, e_2, e_3\}$. This embedding takes the set Σ bijectively to the root system A_2 (Example 4.6). Observe that the group $T_3(\mathbf{F})\backslash N_{\mathrm{GL}_3(\mathbf{F})}(T_3(\mathbf{F}))$ acts

$$\begin{tikzcd} \Sigma \arrow[r] \arrow[loop below, "T_3(\mathbf{F})\backslash N_{\mathrm{GL}_3(\mathbf{F})}(T_3(\mathbf{F}))"'] & \mathbf{ALG}(T_3(\mathbf{F}), \mathbf{F}^\times) \arrow[loop below, "T_3(\mathbf{F})\backslash N_{\mathrm{GL}_3(\mathbf{F})}(T_3(\mathbf{F}))"'] \end{tikzcd}$$

on the domain and codomain of the map (4.10).

The normalizer $N_{\mathrm{GL}_3(\mathbf{F})}(T_3(\mathbf{F}))$ of the maximal torus, the group of monomial matrices (with exactly one nonzero entry in each row and column), acts on the set Σ by conjugation. For example, the permutation matrix

$$n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in N_{\mathrm{GL}_3(\mathbf{F})}(T_3(\mathbf{F})) \tag{4.11}$$

corresponding to the transposition $(2,3)$ conjugates X_{12} into X_{13} because

$$x_{12}(t)^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = x_{13}(t)$$

As $T_3(\mathbf{F})$ acts trivially on $\Sigma(T)$, we in fact have an action of the quotient group $T_3(\mathbf{F})\backslash N_{\mathrm{GL}_3(\mathbf{F})}(T_3(\mathbf{F}))$ on Σ.

The normalizer $N_{\mathrm{GL}_3(\mathbf{F})}(T_3(\mathbf{F}))$ also acts on the lattice $\mathbf{ALG}(T_3(\mathbf{F}), \mathbf{F}^\times)$ by conjugation: For $\alpha \in \mathbf{ALG}(T_3(\mathbf{F}), \mathbf{F}^\times)$ and $n \in N_{\mathrm{GL}_3(\mathbf{F})}(T_3(\mathbf{F}))$, α^n is the homomorphisms $\alpha^n(s) = \alpha(s^{-n})$. For instance, with n corresponding to the transposition $(2,3)$ as in (4.11), the computation

$$(e_2 - e_1)^n(s) = (e_2 - e_1)(s^{-n}) = (e_2 - e_1)(s_1, s_3, s_2) = s_3 s_1^{-1} = (e_3 - e_1)(s)$$

shows that n takes $e_2 - e_1$ to $e_3 - e_1$. This indicates that the map (4.10) is $T_3(\mathbf{F})\backslash N_{\mathrm{GL}_3(\mathbf{F})}(T_3(\mathbf{F}))$- equivariant.

The fact that the quotient group $T_3(\mathbf{F})\backslash N_{\mathrm{GL}_3(\mathbf{F})}(T_3(\mathbf{F}))$ acts on the rank three lattice $\mathbf{ALG}(T_3(\mathbf{F}), \mathbf{F}^\times)$ means that we have a homomorphism

$$T_3(\mathbf{F})\backslash N_{\mathrm{GL}_3(\mathbf{F})}(T_3(\mathbf{F})) \to \mathbf{GRP}(\mathbf{ALG}(T_3(\mathbf{F}), \mathbf{F}^\times))^\circ \cong \mathrm{GL}_3(\mathbf{Z})$$

into the automorphism group of the lattice. This homomorphism happens to be injective and to take $T_3(\mathbf{F})\backslash N_{\mathrm{GL}_3(\mathbf{F})}(T_3(\mathbf{F}))$ isomorphically to the Weyl group $W(A_2)$.

After this concrete example, we now consider the general theory of root groups. Let K be an algebraic group and T a maximal torus of K.

Definition 4.12. *A one-parameter group X is an algebraic group isomorphic to the additive group $\mathbf{F}^+$ and a parameterization of X is an algebraic group isomorphism $x \colon \mathbf{F}^+ \to X$.*

Any two parameterizations of a one-parameter group differ by an element of the automorphism group $\mathbf{ALG}(\mathbf{F}^+)^\circ = \mathbf{F}^\times$.

Definition 4.13. *A root group of K (relative to T) is a one-parameter subgroup of K normalized by T.*

Let $\Sigma(T)$ be the set of root groups relative to T. Suppose $\Sigma(T) \ni X \leq K$ is a root group. Let $x \colon \mathbf{F}^+ \to X$ be any parameterization of X and let s be any element of the maximal torus T. Then $t \to x(t)^s$ is also a parameterization of X and therefore

$$x(t)^s = x(\alpha_X(s)t)$$

for some reparameterization factor $\alpha_X(s) \in \mathbf{ALG}(\mathbf{F}^+)^\circ = \mathbf{F}^\times$ (independent of the choice of parameterization x for X). This means that there is a map

$$\Sigma(T) \to \mathbf{ALG}(T, \mathbf{F}^\times) \colon X \mapsto \alpha_X$$

Σ	$K_u(\Sigma)$	$Z(K_u(\Sigma))$	$K_a(\Sigma)$	$\mathrm{Aut}(\Sigma)$
A_n	$\mathrm{SL}_{n+1}(\mathbf{F})$	$O_{r'}C_{n+1}$	$\mathrm{PSL}_{n+1}(\mathbf{F})$	C_2
B_n	$\mathrm{Spin}_{2n+1}(\mathbf{F})$	C_2 $r \neq 2$ 1 $r = 2$	$\mathrm{SO}_{2n+1}(\mathbf{F})$	1
C_n	$\mathrm{Sp}_{2n}(\mathbf{F})$	C_2 $r \neq 2$ 1 $r = 2$	$\mathrm{PSp}_{2n}(\mathbf{F})$	1
D_n	$\mathrm{Spin}_{2n}(\mathbf{F})$	C_2^2 $r \neq 2$, n even C_4 $r \neq 2$, n odd 1 $r = 2$	$\mathrm{PSO}_{2n}(\mathbf{F})$	Σ_3 $m = 4$ C_2 $m > 4$

TABLE 1. The simple classical algebraic groups (r denotes the characteristic of $\mathbf{F}$)

taking a root group $X \leq K$ to its reparameterization function $\alpha_X \colon T \to \mathbf{F}^\times$.

The normalizer $N_K(T)$ of the maximal torus acts on the lattice $\mathbf{ALG}(T, \mathbf{F}^\times)$. Indeed, if $n \in N_K(T)$ normalizes T and $\alpha \colon T \to \mathbf{F}^\times$ is a homomorphism define α^n to be the homomorphism that makes

$$\begin{tikzcd} T \arrow[r, "\alpha"] \arrow[d, "n"'] & \mathbf{F}^\times \arrow[d, equal] \\ T \arrow[r, "\alpha^n"'] & \mathbf{F}^\times \end{tikzcd}$$

commute, ie $\alpha^n(s^n) = \alpha(s)$ or $\alpha^n(s) = \alpha(s^{-n})$. Of course, T acts trivially, so that we in fact have an action of the quotient group $T\backslash N_K(T)$ on the lattice.

$N_K(T)$ also acts on the set $\Sigma(T)$ of root groups. Indeed, if $n \in N_K(T)$ normalizes T and $x \colon \mathbf{F}^+ \to X$ is a parameterization of a root group X then

$$x(t)^{ns} = x(t)^{nsn^{-1}n} = x(t)^{s^{-n}n} = x(\alpha_X(s^{-n}))^n = x(\alpha_X^n(s))^n$$

which says that X^n is a root group with reparameterization function $\alpha_{X^n} = \alpha_X^n$. In other words, the map $\Sigma(T) \to \mathbf{ALG}(T, \mathbf{F}^\times)$ is $N_K(T)$-equivariant. Again, T acts trivially, so that in fact $T\backslash N_K(T)$ acts on the set $\Sigma(T)$ of root groups and the map $\Sigma(T) \to \mathbf{ALG}(T, \mathbf{F}^\times)$ is $T\backslash N_K(T)$-equivariant.

The fact that $T\backslash N_K(T)$ acts on $\mathbf{ALG}(T, \mathbf{F}^\times)$ means that there is a homomorphism

$$T\backslash N_K(T) \to \mathbf{GRP}(\mathbf{ALG}(T, \mathbf{F}^\times))^\circ$$

into the automorphism group of the lattice.

Theorem 4.14. [24, Theorem 1.9.5][26, 27.1] *Assume that K is a reductive algebraic group.*

(1) *The map $\Sigma(T) \to \mathbf{ALG}(T, \mathbf{F}^\times)$ is injective and $T\backslash N_K(T)$-equivariant*
(2) *$\Sigma(T)$ is a reduced crystallographic root system in $\mathbf{ALG}(T, \mathbf{F}^\times)$ (or $\mathbf{R} \otimes_{\mathbf{Z}} \mathbf{ALG}(T, \mathbf{F}^\times)$)*
(3) *There is an isomorphism between $T\backslash N_K(T)$ and $W(\Sigma(T))$ induced by the action $T\backslash N_K(T) \to \mathbf{GRP}(\mathbf{ALG}(T, \mathbf{F}^\times))^\circ$*

The root groups in K, acted upon by $T\backslash N_K(T)$, correspond to roots in a root system Σ in $\mathbf{ALG}(T, \mathbf{F}^\times)$, acted upon by $W(\Sigma)$.

An *isogeny* is an epimorphism $K \to H$ of algebraic groups with finite kernel [24, Definition 1.10.2]. $\mathrm{SL}_{n+1}(\mathbf{F}) \to \mathrm{PSL}_{n+1}(\mathbf{F})$ and $\mathrm{SL}_{n+1}(\mathbf{F}) \times \mathbf{F}^\times \to \mathrm{GL}_{n+1}(\mathbf{F})$ are examples of isogenies.

Theorem 4.15 (Chevalley classification of semisimple algebraic groups)**.** [24, Theorem 1.10.4] *Semisimple algebraic groups are determined, up to isogeny, by their root systems. More precisely,*

(1) *If the algebraic groups K and H are connected by an isogeny then K and H have isomorphic root systems (or their root systems are B_m and C_m and $\mathbf{F}$ has characteristic two).*
(2) *Let Σ be any reduced and crystallographic root system. Then there exist unique semisimple algebraic groups, $K_u(\Sigma)$ (the universal version) and $K_a(\Sigma)$ (the adjoint version), such that for any algebraic K group with root system Σ there are isogenies $K_u(\Sigma) \to K \to K_a(\Sigma)$.*
(3) *The center of $K_u(\Sigma)$ is finite (Table 1) and the center $K_a(\Sigma)$ is trivial.*

Some of the classical algebraic groups [3, V.§23] [26, 7.2] in Table 1 are

$\mathrm{SL}_{n+1}(\mathbf{F})$: is $\{B \in \mathrm{GL}_{n+1}(\mathbf{F}) \mid \det(B) = 1\}$, the subgroup of $\mathrm{GL}_{n+1}(\mathbf{F})$ of matrices of determinant 1.

$\mathrm{SO}_{2n+1}(\mathbf{F})$: is $\{B \in \mathrm{SL}_{2n+1}(\mathbf{F}) \mid B^T P B = P\}$, the subgroup of invertible matrices preserving the nondegenerate *symmetric* bilinear form

$$((u_0, u_1, \dots, u_n, u_{-1}, \dots, u_{-n}), (v_0, v_1, \dots, v_n, v_{-1}, \dots, v_{-n})) = uPv = u_0 v_0 + \sum_{i=1}^{n} (u_i v_{-i} + u_{-i} v_i)$$

given by the nondegenerate symmetric matrix

$$P = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}$$

The associated quadratic form is $2u_0^2 + 2\sum_{i=0}^n u_i u_{-i}$. (This is under the assumption that $\mathbf{F}$ has odd characteristic r.)

$\mathrm{Sp}_{2n}(\mathbf{F})$: is $\{B \in \mathrm{GL}_{2n}(\mathbf{F}) \mid B^T P B = P\}$, the subgroup of invertible matrices preserving the nondegenerate *skew-symmetric* bilinear form

$$((u_1, \dots, u_n, u_{-1}, \dots, u_{-n}), (v_1, \dots, v_n, v_{-1}, \dots, v_{-n})) = uPv = \sum_{i=1}^{n} (u_i v_{-i} - u_{-i} v_i)$$

given by the nondegenerate skew-symmetric matrix

$$P = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

$\mathrm{SO}_{2n}(\mathbf{F})$: is $\{B \in \mathrm{SL}_{2n}(\mathbf{F}) \mid B^T P B = P\}$, the subgroup of invertible matrices preserving the nondegenerate *symmetric* bilinear form

$$((u_1, \dots, u_n, u_{-1}, \dots, u_{-n}), (v_1, \dots, v_n, v_{-1}, \dots, v_{-n})) = uPv = \sum_{i=1}^{n} (u_i v_{-i} + u_{-i} v_i)$$

given by the nondegenerate symmetric matrix

$$P = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

(This is under the assumption that $\mathbf{F}$ has odd characteristic r.)

A subgroup of an algebraic group K is unipotent if there is some embedding of K into some general linear group $\mathrm{GL}_n(\mathbf{F})$ taking the subgroup into $U_n(\mathbf{F})$. The algebraic group K is *reductive* if the trivial subgroup is the only connected normal unipotent subgroup. The algebraic group K is *semisimple* if the trivial subgroup is the only connected normal solvable subgroup. All semisimple algebraic groups are reductive.

A *simple* algebraic group is a finite central quotient of $K_u(\Sigma)$ where Σ is irreducible. (Table 1 contains a list of the classical simple algebraic groups.) A *semisimple* algebraic group is a finite central quotient of a finite product $K_1 \times \cdots \times K_h$ of simple algebraic groups. A *reductive* algebraic group is a finite central quotient of a finite product $K_1 \times \cdots \times K_h \times T$ of simple algebraic groups and a torus. (For example, $\mathrm{GL}_n(\mathbf{F})$ is reductive and not semisimple.)

Σ	$K_u(\Sigma)$	$Z(K_u(\Sigma))$	$\mathrm{Aut}(\Sigma)$
E_6	$E_6(\mathbf{F})$	C_3 $r \neq 3$ 1 $r = 3$	C_2
E_7	$E_7(\mathbf{F})$	C_2 $r \neq 2$ 1 $r = 2$	1
E_8	$E_8(\mathbf{F})$	1	1
F_4	$F_4(\mathbf{F})$	1	1
G_2	$G_2(\mathbf{F})$	1	1

TABLE 2. The simple exceptional algebraic groups (r denotes the characteristic of $\mathbf{F}$)

Example 4.16 (Root systems of $\mathrm{GL}_{n+1}(\mathbf{F})$ and $\mathrm{SL}_{n+1}(\mathbf{F})$)**.** The maximal torus of $\mathrm{GL}_{n+1}(\mathbf{F})$ is the group $T_{n+1}(\mathbf{F})$ of diagonal matrices and its dual is the lattice $\mathbf{ALG}(T_{n+1}(\mathbf{F}),\mathbf{F}^\times)=\mathbf{Z}\{e_1,\dots,e_{n+1}\}$ generated by the $n+1$ coordinates $e_i\colon T_{n+1}(\mathbf{F})\to\mathbf{F}^\times$. Generalizing Example 4.9 we learn that the root groups of $\mathrm{GL}_{n+1}(\mathbf{F})$ are the one-parameter subgroups X_{ij} $1\le i\neq j\le n+1$, with parameterization $x_{ij}(t)=I+te_{ij}$, $t\in\mathbf{F}$, and that conjugation by an element $s=(s_1,\dots,s_{n+1})$ of the maximal torus $T_{n+1}(\mathbf{F})$ results in the reparameterization

$$x_{ij}(t)^s=x_{ij}(s^{e_j-e_i}t),\qquad s\in T_{n+1}(\mathbf{F}),$$

of the root group. The set $\{X_{ij}\}$ of the $n(n+1)$ root groups corresponds to the root system

$$A_n=\{e_j-e_i\mid 1\le i\neq j\le n+1\}\subset\mathbf{Z}\{e_1,\dots,e_{n+1}\}\tag{4.17}$$

The reflection $r_{e_j-e_i}\colon\mathbf{Z}\{e_1,\dots,e_{n+1}\}\to\mathbf{Z}\{e_1,\dots,e_{n+1}\}$ for the root e_j-e_i is

$$(\beta)r_{e_j-e_i}=\beta-(\beta)(e_j^\vee-e_i^\vee)(e_j-e_i)\tag{4.18}$$

where $e_i^\vee\colon\mathbf{Z}\{e_1,\dots,e_{n+1}\}\to\mathbf{Z}$ is the usual dual, $(e_j)e_i^\vee=\delta_{ij}$, to the basis element e_i.

We now turn to the simple algebraic group $\mathrm{SL}_{n+1}(\mathbf{F})$. The n-torus

$$T_n(\mathbf{F})\to\mathrm{SL}_{n+1}(\mathbf{F})\colon(s_1,\dots,s_n)\mapsto(s_1,s_1^{-1}s_2,\dots,s_{n-1}^{-1}s_n,s_n^{-1})$$

is a maximal torus of $\mathrm{SL}_{n+1}(\mathbf{F})$. The inclusion $T_n(\mathbf{F})\hookrightarrow T_{n+1}(\mathbf{F})$ into the maximal torus $T_{n+1}(\mathbf{F})$ for $\mathrm{GL}_{n+1}(\mathbf{F})$ induces a projection $\mathbf{ALG}(T_{n+1}(\mathbf{F}),\mathbf{F}^\times)\twoheadrightarrow\mathbf{ALG}(T_n(\mathbf{F}),\mathbf{F}^\times)$ that can be identified to the projection

$$\mathbf{ALG}(T_{n+1}(\mathbf{F}),\mathbf{F}^\times)=\mathbf{Z}\{e_1,\dots,e_{n+1}\}\to\mathbf{Z}\{e_1,\dots,e_{n+1}\}/\mathbf{Z}(e_1+\cdots+e_{n+1})=\mathbf{ALG}(T_n(\mathbf{F}),\mathbf{F}^\times)\tag{4.19}$$

onto the quotient lattice by the sublattice generated by the sum $e_1+\cdots+e_{n+1}$ of the basis elements.

The root groups for $\mathrm{SL}_{n+1}(\mathbf{F})$ are the same one-parameter groups X_{ij} as for $\mathrm{GL}_{n+1}(\mathbf{F})$ and the effect of conjugation by an s element of the maximal torus $T_n(\mathbf{F})\le T_{n+1}(\mathbf{F})$

$$x_{ij}(t)^s=x_{ij}(s^{e_j-e_i}t),\qquad s\in T_n(\mathbf{F}),$$

is also as before, of course. This means that the root system for $\mathrm{SL}_{n+1}(\mathbf{F})$ is the image

$$A_n=\{e_j-e_i+\mathbf{Z}(e_1+\ldots+e_{n+1})\mid 1\le i\neq j\le n+1\}\subset\mathbf{Z}\{e_1,\dots,e_{n+1}\}/\mathbf{Z}(e_1+\ldots+e_{n+1})\tag{4.20}$$

under the quotient map (4.19) of the root system $A_n\subset\mathbf{Z}\{e_1,\dots,e_{n+1}\}$ for $\mathrm{GL}_{n+1}(\mathbf{F})$. Note that the linear forms $e_j^\vee-e_i^\vee$ factor through the quotient lattice

$$\begin{tikzcd}
\mathbf{Z}\{e_1,\dots,e_{n+1}\} \arrow[r, "e_j^\vee-e_i^\vee"] \arrow[d] & \mathbf{Z} \\
\mathbf{Z}\{e_1,\dots,e_{n+1}\}/\mathbf{Z}(e_1+\cdots e_{n+1}) \arrow[ur, dotted, "e_j^\vee-e_i^\vee"'] &
\end{tikzcd}$$

so that the reflections $r_{e_j-e_i}$ and $r_{e_j-e_i+\mathbf{Z}(e_1+\ldots+e_{n+1})}$ in the commutative diagram

$$\begin{tikzcd}
\mathbf{Z}\{e_1,\dots,e_{n+1}\} \arrow[r, "r_{e_j-e_i}"] \arrow[d] & \mathbf{Z}\{e_1,\dots,e_{n+1}\} \arrow[d] \\
\mathbf{Z}\{e_1,\dots,e_{n+1}\}/\mathbf{Z}(e_1+\ldots+e_{n+1}) \arrow[r] & \mathbf{Z}\{e_1,\dots,e_{n+1}\}/\mathbf{Z}(e_1+\ldots+e_{n+1})
\end{tikzcd}$$

both are given by formula (4.18). Lemma 4.5 shows that (4.20) indeed is a root system. Also, since the forms $\langle\cdot,\cdot\rangle$ are preserved under the bijection induced by the projection (4.19), the root systems for $\mathrm{GL}_{n+1}(\mathbf{F})$ and $\mathrm{SL}_{n+1}(\mathbf{F})$ are isomorphic.

4.3. **Steinberg and Frobenius endomorphisms of algebraic groups.** These are the endomorphisms of algebraic groups whose fixed point groups are finite.

Definition 4.21. [24, Definition 1.15.1] *A Steinberg endomorphism of the algebraic group K is a surjective endomorphism σ whose fixed point group $C_K(\sigma)$ is finite.*

Let q be a power of r and $\varphi_q\colon \mathbf{F} \to \mathbf{F}$ the Frobenius automorphism $\varphi_q(t) = t^q$ (with fixed field the finite Galois field $\mathbf{F}_q$ with q elements). We may extend φ_q to an a bijective algebraic group endomorphism $\varphi_q\colon \mathrm{GL}_n(\mathbf{F}) \to \mathrm{GL}_n(\mathbf{F})$ by $(t_{ij})\varphi_q = (t_{ij}^q)$. (A bijective algebraic group endomorphism may not be an algebraic group automorphism as the inverse (abstract) group endomorphism may not be algebraic.) The fixed group of this φ_q is the finite group $C_{\mathrm{GL}_n(\mathbf{F})}(\varphi_q) = \mathrm{GL}_n(q)$. If K is a closed subgroup of $\mathrm{GL}_n(\mathbf{F})$ and K is normalized by φ_q then $\varphi_q|K$ is a Steinberg endomorphism of K because $C_K(\varphi_q)$ is a subgroup of the finite group $C_{\mathrm{GL}_n(\mathbf{F})}(\varphi_q)$ [24, Theorem 2.1.11].

Theorem 4.22. *Let K be a simple algebraic group and $q = r, r^2, \dots$ a positive power of r. There is a unique bijective endomorphism φ_q of K such that $(t)x_\alpha^{\varphi_q} = (t^q)x_\alpha$ for all $\alpha \in \Sigma$, $t \in \mathbf{F}$. The endomorphism φ_q is a Steinberg endomorphism.*

Frobenius endomorphisms are special Steinberg endomorphism.

Definition 4.23. [24, Definition 2.1.9] *An endomorphism σ of an algebraic group K is called a Frobenius endomorphism (of level $q^{1/m}$) if some power σ^m of σ equals $\varphi_q|K$ for some inclusion of K as a closed subgroup of some $\mathrm{GL}_n(\mathbf{F})$.*

All Frobenius endomorphisms are Steinberg endomorphisms. Indeed, the fixed point group $C_K(\sigma)$ is a subgroup of the finite group $C_K(\sigma^m) = C_K(\varphi_q)$. For *simple* algebraic groups, Steinberg endomorphism = Frobenius endomorphism [24, Theorem 2.1.11].

Example 4.24. The Dynkin diagram $\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$ for $\mathrm{SL}_3(\mathbf{F})$, with vertices $\Pi = \{e_2 - e_1, e_3 - e_2\}$, admits a symmetry which extends to an automorphism ρ of the root system $\{X_{ij} \mid 1 \leq i \neq j \leq 3\} = A_2 \subset \mathbf{Z}\{e_1, e_2, e_3\}/\mathbf{Z}(e_1 + e_2 + e_3)$ given by $(e_i)\rho = -e_{4-i}$ [13, Proposition 12.2.2].

$$\begin{array}{ccc} 1 & X_{12} & X_{13} \\ X_{21} & 1 & X_{23} \\ X_{31} & X_{32} & 1 \end{array}$$

This permutation of the root groups X_{ij} even extends to the (graph) automorphism γ_ρ of $\mathrm{SL}_3(\mathbf{F})$ given by

$$(g)\gamma_\rho = A(g^T)^{-1}A^{-1}$$

where A is the matrix whose only nonzero entries are a string of alternating ± 1 running (anti-)diagonally from the upper right to the lower left corner [24, Chp 2.7]. Observe, for instance, that

$$(t)x_{12}^{\gamma_\rho} = A(-t)x_{21}A = (t)x_{23}, \qquad t \in \mathbf{F},$$

as $((t)x_{12}^T)^{-1} = (t)x_{21}^{-1} = (-t)x_{21}$ and $A^{-1} = A$ (in this dimension 3). Actually,

$$x_\alpha^{\gamma_\rho} = x_{\alpha^\rho}$$

for all roots $\alpha \in A_2$ [24, Theorem 1.15.2]. On the maximal torus, $(s_1, s_2, s_3)\gamma_\rho = (s_3^{-1}, s_2^{-1}, s_1^{-1})$.

The composition $\sigma = \gamma_\rho\varphi_q$, for any power q of r, is a Frobenius endomorphism of $\mathrm{SL}_3(\mathbf{F})$ (of level $q^{1/2}$). Indeed, $\sigma^2 = \sigma_{q^2}$. The fixed point group $\mathrm{SL}_3^-(q) = C_{\mathrm{SL}_3(\mathbf{F})}(\sigma)$ for σ is a subgroup of the fixed point group $\mathrm{SL}_3^+(q^2) = C_{\mathrm{SL}_3(\mathbf{F})}(\sigma_{q^2})$ for $\sigma^2 = \sigma_{q^2}$.

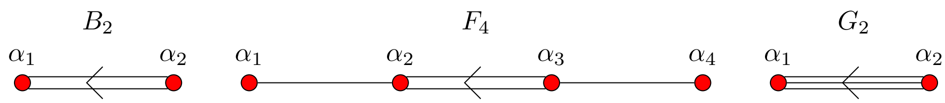
More generally, the symmetry of the Dynkin diagram for A_n extends to an automorphism ρ of order 2 of the root system A_n and even to a bijective endomorphism γ_ρ of $\mathrm{SL}_{n+1}(\mathbf{F})$ such that $x_\alpha^{\gamma_\rho} = x_{\alpha^\rho}$ for all roots $\alpha \in A_n$ [24, Theorem 1.15.2]. The composite $\sigma = \gamma_\rho\varphi_q$, where q is any power of r, is a Frobenius endomorphism with $\sigma^2 = \sigma_{q^2}$. The fixed point group $\mathrm{SL}_{m+1}^-(q) = C_{\mathrm{SL}_{m+1}(\mathbf{F})}(\sigma)$ for σ is a subgroup of the fixed point group $\mathrm{SL}_{m+1}^+(q^2) = C_{\mathrm{SL}_{m+1}(\mathbf{F})}(\sigma_{q^2})$ for $\sigma^2 = \sigma_{q^2}$.

Similarly, the root systems D_n and E_6 admit a symmetry ρ of order 2 that extends to a bijective endomorphism γ_ρ of a simple algebraic group K with these root systems. Then $\sigma = \gamma_\rho\varphi_q$ is a Frobenius endomorphism with $\sigma^2 = \varphi_{q^2}$ for any power q of r.

Also, the root system D_4 admits a symmetry ρ of order 3 that extends to a bijective endomorphism γ_ρ of a simple algebraic group K with this root system. Then $\sigma = \gamma_\rho\varphi_q$ is a Frobenius endomorphism with $\sigma^3 = \varphi_{q^3}$ for any power q of r.

These Frobenius endomorphisms exist for all characteristics r. Now follow some special constructions that only work when the field is chosen properly.

Let Σ be one of the root systems B_2, F_4, or G_2, (see Figure 2) and let Π be a basis. The Dynkin diagrams are



There exists a bijection ρ of a basis Π, taking short roots to long roots, such that the linear automorphism $\rho\colon \mathbf{R}\Sigma \to \mathbf{R}\Sigma$ preserves angles but not lengths, and ρ^2 is 2, 2, or 3. Thus ρ is not quite an automorphism of the root system since it does not preserve the form $\langle\cdot,\cdot\rangle$ or, equivalently, the inner product. Nevertheless, if K is a simple algebraic group with root system Σ and the field $\mathbf{F}$ has characteristic r equal to 2, 2, or 3, then there exists an endomorphism ψ_ρ of K such that

$$(t)x_\alpha^{\psi_\rho} = \begin{cases} (t)x_{\alpha^\rho} & \alpha \text{ is long} \\ (t^r)x_{\alpha^\rho} & \alpha \text{ is short} \end{cases}$$

and ψ_ρ^2 is φ_2, φ_2, φ_3. Thus $\sigma = \psi_\rho\varphi_{r^a}$ with square $\sigma^2 = \psi_\rho^2\varphi_{r^{2a}} = \varphi_{r^{2a+1}}$ is a Frobenius endomorphism of K for every positive exponent a.

5. Finite groups of Lie type

For any group G and any prime r, recall that $O^{r'}G$ is the subgroup of G generated by the set of r-elements of G; it is the smallest normal subgroup of G with an r'-quotient group.

Definition 5.1. [24, Definition 2.2.2] *A finite group of Lie type (in characteristic r) is a group of the form $O^{r'}C_{\bar K}(\sigma)$ where $\bar K$ is a simple algebraic group over $\bar{\mathbf{F}}_r$ and σ a Frobenius endomorphism of $\bar K$.*

In fact, we can always assume that $\sigma = \gamma_\rho\varphi_q$ for some automorphism ρ of the root system and some power q of r [24, Theorem 2.2.3]. If $\bar K$ is of universal type then $C_{\bar K}(\sigma)$ is generated by its r-elements so that $O^{r'}C_{\bar K}(\sigma) = C_{\bar K}(\sigma)$ [24, Theorem 2.2.6].

The finite group of Lie type $O^{r'}C_{\bar K}(\sigma)$ is a

- **Chevalley group:** if $\sigma = \varphi_q$ – examples are $\mathrm{SL}^+_{n+1}(q)$, $\mathrm{Spin}_{2n+1}(q)$, $\mathrm{Sp}_{2n}(q)$, $\mathrm{Spin}^+_{2n}(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$, $F_4(q)$, $G_2(q)$;
- **Steinberg group:** if $\sigma = \gamma_\rho\varphi_q$ for some *nonidentity* automorphism ρ of the root system – examples are $\mathrm{SL}^-_{n+1}(q)$ $(n \geq 2)$, $\mathrm{Spin}^-_{2n}(q)$, $(n \geq 3)$ ${}^3D_4(q)$, ${}^2E_6(q)$;
- **Suzuki–Ree group:** if $\Sigma = B_2, F_4, G_2$, $r = 2, 2, 3$, and $\sigma = \psi^{2a+1}$, where ψ is an automorphism with $\psi^2 = \varphi_r$ $(\psi = r^{\frac12})$, we get the Suzuki group ${}^2B_2(2^{a+\frac12})$, and the Ree groups ${}^2F_4(2^{a+\frac12})$, ${}^2G_2(3^{a+\frac12})$ as the fixed points for the odd powers of ψ. (The fixed points for the even powers $\psi^{2a} = \varphi_{r^a}$ are Chevalley groups $B_2(2^a)$, $F_4(2^a)$, $G_2(3^a)$.)

The Steinberg group $\mathrm{SL}^-_{n+1}(q)$ is also denoted $\mathrm{SU}_{n+1}(q)$ and it is isomorphic to the subgroup of $\mathrm{GL}_{n+1}(q^2)$ of matrices that preserve the Hermitian inner product $((u_1,\dots,u_{n+1}),(v_1,\dots,v_{n+1})) = \sum_{i=1}^{n+1} u_i v_i^q$.

6. The Frobenius category associated to a finite group

Let G be a finite group and P a Sylow p-subgroup of G.

Definition 6.1. [37, 1.8] *The Frobenius category associated with G (and P) is the category $\mathcal{F}_G$ whose objects are the subgroups of P and whose morphisms are group homomorphisms induced by conjugation in G.*

All morphisms in $\mathcal{F}_G$ are injective group homomorphisms. More explicitly, for $Q, R \leq P$

$$\mathcal{F}_G(Q,R) = C_G(Q)\backslash N_G(Q,R) \text{ and } \mathcal{F}_G(Q) = C_G(Q)\backslash N_G(Q)$$

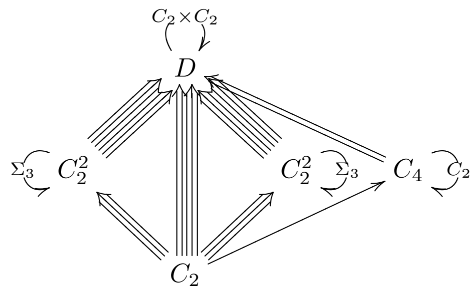
Note that $\mathcal{F}_G$ is equivalent to the category where the objects are all p-subgroups of G and the morphisms are group homomorphisms induced by conjugation in G. Therefore different choices of P will lead to equivalent (in fact, isomorphic) categories.

The category $\mathcal{F}_G$ is not a faithful representation of G. $\mathcal{F}_G$ is trivial when G is a p'-group. $\mathcal{F}_G = \mathcal{F}_P$ when G is nilpotent and thus isomorphic to the product of its Sylow p-subgroups [40, 5.2.4]. In fact, Frobenius normal p-complement theorem (Theorem 7.3) says that $\mathcal{F}_G = \mathcal{F}_P$ if and only if G is p-nilpotent. When $P \leq H \leq G$, $\mathcal{F}_H$ and $\mathcal{F}_G$ have the same objects $Q, R \leq P$ and $\mathcal{F}_H(Q,R) \subset \mathcal{F}_G(Q,R)$. When actually $\mathcal{F}_H = \mathcal{F}_G$ we say that H controls fusion in G [23, Definition 15.8]. We have

$$\mathcal{F}_H = \mathcal{F}_G \iff \forall Q \leq P \colon N_H(Q) = N_G(Q) \tag{6.2}$$

because $|\mathcal{F}_G(Q) \colon \mathcal{F}_H(Q)| = |N_G(Q) \colon N_H(Q)|$.

Example 6.3. The three simple groups $\mathrm{SL}_3(\mathbf{F}_2)$, A_6, and A_7 of order $168 = 2^3{\cdot}21$, $360 = 2^3{\cdot}45$, $2520 = 2^3{\cdot}315$ happen to have identical Frobenius categories at $p = 2$. The $p = 2$ Frobenius category of any of these three groups is



where we only show the nonidentity subgroups. In Corollary 12.2 we shall see more examples of distinct simple groups with Frobenius categories that are equivalent (as Frobenius categories).

Example 6.4. Suppose that K is a normal p'-subgroup of G. Then G and its factor group G/K have isomorphic Frobenius cateories. The group homomorphism $\pi \colon G \to G/K$ induces a (Frobenius) functor $\mathcal{F}_G \to \mathcal{F}_{G/K}$. This functor is an equivalence of categories. Let P and Q be p-subgroups of G. Since the projections $P \to P^\pi$ and $Q \to Q^\pi$ are isomorphisms the map $\mathcal{F}_G(P,Q) \to \mathcal{F}_{G^\pi}(P^\pi, Q^\pi)$ is injective. It is also surjective. Let g be an element of G so that $P^g \leq KQ$. Since Q is a Sylow p-subgroup of KQ there exists an element $x \in K$ so that $P^{gx} \leq Q$. Conjugation by gx determines an $\mathcal{F}_G$-morphism from P to Q over the $\mathcal{F}_{G^\pi}$-morphism $P^\pi \to Q^\pi$ that is conjugation by g.

For instance, consider $\mathrm{SL}_{n+1}(q)$ where q is a power of p. The center of $\mathrm{SL}_{n+1}(q)$ is cyclic of order $(n{+}1, q{-}1)$ not divisible by p. Thus $\mathrm{SL}_{n+1}(q)$ and $\mathrm{PSL}_{n+1}(q)$ have isomorphic Frobenius categories at p.

Example 6.5 ($\mathrm{GL}_2(\mathbf{F}_q)$ for $p \mid q$)**.** Let q be a positive power of p and $\mathrm{GL}_2(\mathbf{F}_q)$ the group of invertible 2×2-matrices over $\mathbf{F}_q$. The order of this group is $|\mathrm{GL}_2(\mathbf{F}_q)| = (q^2-1)(q^2-q) = q(q^2-1)(q-1)$ and the p-part of the order is $|\mathrm{GL}_2(\mathbf{F}_q)|_p = q$. Consider the subgroups

$$H = \left\{ \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \mid s_1, s_2 \in \mathbf{F}_q^\times \right\} \cong \mathbf{F}_q^\times \times \mathbf{F}_q^\times, \qquad U = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbf{F}_q \right\} \cong \mathbf{F}_q^+,$$

$$B = HU = \left\{ \begin{pmatrix} s_1 & t \\ 0 & s_2 \end{pmatrix} \mid s_1, s_2 \in \mathbf{F}_q^\times,\ t \in \mathbf{F}_q \right\}, \qquad B \rhd U, \quad U \cap H = 1, \quad B/U = H$$

of diagonal, (upper) unitriangular matrices, and triangular matrices. H normalizes U, ie $H \leq N_{\mathrm{GL}_2(\mathbf{F}_q)}(U)$, because

$$\begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} = \begin{pmatrix} 1 & s_2 s_1^{-1} t \\ 0 & 1 \end{pmatrix}$$

and therefore B is a subgroup of $N_{\mathrm{GL}_2(\mathbf{F}_q)}(U)$. In fact, $B = N_{\mathrm{GL}_2(\mathbf{F}_q)}(U)$ *is* the normalizer of U. To see this we (refer to theory of parabolic subgroups or) compute the effect of conjugation by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{F}_q), \qquad D = ad - bc,$$

on U:

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}^g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = D^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a+tc & b+td \\ c & d \end{pmatrix} = D^{-1} \begin{pmatrix} ? & tbd \\ -tc^2 & ? \end{pmatrix}$$

This shows that

$$U^g = U \iff c = 0 \iff U \cap U^g \neq 1 \iff g \in B$$

U is a Sylow p-subgroup of $\mathrm{GL}_2(\mathbf{F}_q)$ because $|U| = q$. The above discussion shows that B controls fusion in $\mathrm{GL}_2(\mathbf{F}_q)$, ie $\mathcal{F}_{\mathrm{GL}_2(\mathbf{F}_q)} = \mathcal{F}_B$. The objects of $\mathcal{F}_B$ are the subgroups of the elementary abelian p-group $U \cong \mathbf{F}_q^+ = \mathbf{F}_p^+ \times \cdots \times \mathbf{F}_p^+$. The morphisms in $\mathcal{F}_B$ are restrictions of the group automorphisms in the image of $C_{q-1} = \mathbf{F}_q^\times \to \mathbf{GRP}(\mathbf{F}_q^+)^\circ$.

We conclude that the Frobenius category $\mathcal{F}_{\mathrm{GL}_2(\mathbf{F}_q)}$ associated to $\mathrm{GL}_2(\mathbf{F}_q)$ is the category of subgroups of U with maps induced from $B = HU$, ie $\mathcal{F}_{\mathrm{GL}_2(\mathbf{F}_q)} = \langle \mathcal{F}_B(U) \rangle = \langle\, \mathbf{F}_q^+ \circlearrowleft \mathbf{F}_q^\times \,\rangle$.

We note five general properties of $\mathcal{F}_G$:

1. $\mathcal{F}_P$ is a subcategory of $\mathcal{F}_G$
2. $\mathcal{F}_P(P) = C_G(P) \cap P \backslash P = C_G(P) \backslash PC_G(P)$ is a normal Sylow p-subgroup of $\mathcal{F}_G(P) = C_G(P) \backslash N_G(P)$ because P is Sylow p-subgroup of $N_G(P)$ and $C_G(P)$ and $PC_G(P)$ are normal in $N_G(P)$ (Proposition 3.1, Corollary 3.5).
3. If for some group homomorphism $\varphi\colon R \to S$ and some $\mathcal{F}_G$-morphism $y \to y^g$ from S to T, the composition $R \xrightarrow{\varphi} S \xrightarrow{y \to y^g} T$ is an $\mathcal{F}_G$-morphism, $(x^\varphi)^g = x^h$, then $x^\varphi = x^{hg^{-1}}$, so φ is an $\mathcal{F}_G$-morphism too:
$$\begin{tikzcd} R \arrow[r, "\varphi \in \mathbf{GRP}(R,S)"] \arrow[rr, bend right, "\varphi\psi \in \mathcal{F}_G(R,T)"'] & S \arrow[r, "\psi \in \mathcal{F}_G(S,T)"] & T \end{tikzcd} \implies \varphi \in \mathcal{F}_G(R,S)$$
4. For a subgroup Q of P and $K \leq \mathbf{GRP}(Q)^\circ$, the K-normalizer of Q in P is a p-subgroup of the K-normalizer of Q in G but $N_P^K(Q)$ need *not* be a Sylow p-subgroup of $N_G^K(Q)$, as it may be too small. When is $N_P^K(Q)$ a Sylow p-subgroup of $N_G^K(Q)$? See subsection 6.1.
5. Some homomorphisms $Q \to P$ in $\mathcal{F}_G$ may extend to a larger subgroup $R \geq Q$ of P. How far can you extend a given homomorphism in $\mathcal{F}_G$? See subsection 6.3.

6.1. **Fully K-normalized subgroups.** We shall now discuss in more detail Sylow p-subgroups of centralizers or normalizers of p-subgroups. Let $Q \leq P$.

Lemma 6.6. *The following two conditions are equivalent:*

1. *$N_P(Q)$ is a Sylow p-subgroup of $N_G(Q)$*
2. *$C_P(Q)$ is a Sylow p-subgroup of $C_G(Q)$ and $\mathcal{F}_P(Q)$ is a Sylow p-subgroup of $\mathcal{F}_G(Q)$*

Proof. This is clear as $|N_G(Q)\colon N_P(Q)| = |C_G(Q)\colon C_P(Q)||\mathcal{F}_G(Q)\colon \mathcal{F}_P(Q)|$ simply because $|N_G(Q)| = |C_G(Q)||\mathcal{F}_G(Q)|$ and $|N_P(Q)| = |C_P(Q)||\mathcal{F}_P(Q)|$. □

Lemma 6.7. *There is $g \in N_G(Q,P)$ so that $N_P(Q^g)$ is a Sylow p-subgroup of $N_G(Q^g)$ (and $C_P(Q^g)$ is a Sylow p-subgroup of $C_G(Q^g)$).*

Proof. Let N be a Sylow p-subgroup of $N_G(Q)$. We can choose N so that N contains Q. Also, we can find $g \in G$ so that $N^g \leq P$. Now

$$N_G(Q)^g = N_G(Q^g) \geq N_P(Q^g) = P \cap N_G(Q^g) = P \cap N_G(Q)^g \geq P \cap N^g = N^g$$

But N^g is a maximal p-subgroup of $N_G(Q)^g$ and therefore $N_P(Q^g) = N^g$ is a Sylow p-subgroup of $N_G(Q)^g = N_G(Q^g)$. □

Lemma 6.8. *The following conditions are equivalent:*

1. *$C_P(Q)$ is a Sylow p-subgroup of $C_G(Q)$*
2. *$|C_P(Q)| \geq |C_P(Q^g)|$ for all $g \in N_G(Q,P)$*
3. *$C_P(Q)^g = C_P(Q^g)$ for all $g \in N_G(Q \cdot C_P(Q), P)$*

Proof. (1) $\implies$ (2) $C_P(Q)^g$ is a Sylow p-subgroup of $C_G(Q)^g = C_G(Q^q)$ and $C_P(Q^g)$ is a p-subgroup of $C_G(Q)^g = C_G(Q)^g$.
(2) $\implies$ (3) $C_P(Q)^g = P \cap C_P(Q)^g = P \cap C_{P^g}(Q^g) = C_{P\cap P^g}(Q^q) \leq C_P(Q^g)$ (as in Proposition 3.8), and from (2), $|C_P(Q)^g| = |C_P(Q)| \geq |C_P(Q^g)|$. Therefore, in fact, $C_P(Q)^g = C_P(Q^g)$.
(3) $\implies$ (1) Choose $g \in N_G(Q, P)$ so that $C_P(Q^g)$ is a Sylow p-subgroup of $C_G(Q^g)$ (Lemma 6.7). Now $C_P(Q)^g$ is a p-subgroup of $C_G(Q)^g = C_G(Q^g)$ so $C_P(Q)^{gh} \leq C_P(Q^g)$ for some $h \in C_G(Q^g)$ (Theorem 3.4). Then $Q^{gh} = Q^g$ as h centralizes Q^g and $(Q \cdot C_P(Q))^{gh} = Q^g \cdot C_P(Q)^{gh} \leq Q^g \cdot C_P(Q^g) \leq P \cdot P = P$ so that $gh \in N_G(Q \cdot C_P(Q), P)$ and $C_P(Q)^{gh} = C_P(Q^{gh}) = C_P(Q^g)$ where the first equality comes from assumption (3). Thus $C_P(Q^g) = C_P(Q)^{gh}$ is a Sylow p-subgroup of $C_G(Q^g) = C_G(Q^{gh}) = C_G(Q)^{gh}$ and $C_P(Q)$ is a Sylow p-subgroup of $C_G(Q)$. □

Lemma 6.9. *The following conditions are equivalent:*

(1) $N_P(Q)$ *is a Sylow p-subgroup of* $N_G(Q)$
(2) $C_P(Q)$ *is a Sylow p-subgroup of* $C_G(Q)$ *and* $\mathcal{F}_P(Q)$ *is a Sylow p-subgroup of* $\mathcal{F}_G(Q)$
(3) $|N_P(Q)| \geq |N_P(Q^g)|$ *for all* $g \in N_G(Q, P)$
(4) $N_P(Q)^g = N_P(Q^g)$ *for all* $g \in N_G(N_P(Q), P)$

Proof. (1) $\iff$ (2) Lemma 6.6.
(1) $\implies$ (3) $N_P(Q^g)$ is a p-subgroup and $N_P(Q)^g$ a Sylow p-subgroup of $N_G(Q)^g = N_G(Q^g)$.
(3) $\implies$ (4) Suppose that $N_P(Q)^g \leq P$. Then $N_P(Q)^g \leq N_P(Q^g)$ by Proposition 3.8.(2). As also $Q^g \leq P$, we know from (3) that $|N_P(Q)^g| = |N_P(Q)| \geq |N_P(Q^g)|$. Therefore, in fact, $N_P(Q)^g = N_P(Q^g)$.
(4) $\implies$ (1) Choose $g \in N_G(Q, P)$ so that $N_P(Q^g)$ is a Sylow p-subgroup of $N_G(Q^g)$ (Lemma 6.7). Now $N_P(Q)^g$ is a p-subgroup of $N_G(Q)^g = N_G(Q^g)$ so $N_P(Q)^{gh} \leq N_P(Q^g) \leq P$ for some $h \in N_G(Q^g)$. Then $Q^{gh} = Q^g$ as h normalizes Q^g and $(Q \cdot N_P(Q))^{gh} = Q^g \cdot N_P(Q)^{gh} \leq P \cdot P = P$. This means that $gh \in N_G(Q \cdot N_G(H), P)$ and therefore $N_P(Q)^{gh} = N_P(Q^{gh}) = N_P(Q^g)$ by assumption (4). Thus $N_P(Q)^{gh}$ is a Sylow p-subgroup of $N_G(Q^g) = N_G(Q^{gh}) = N_G(Q)^{gh}$ and, equivalently, $N_P(Q)$ is a Sylow p-subgroup of $N_G(Q)$. □

Definition 6.10. *Q is fully centralized in $\mathcal{F}_G$ if it satisfies any one of the conditions of Lemma* 6.8 *and fully normalized in $\mathcal{F}_G$ if it satisfies any one of the conditions of Lemma* 6.9.

By Lemma 6.9.(2): Q is fully normalized $\implies$ Q is fully centralized.

Central subgroups of P are fully centralized in $\mathcal{F}_G$ as $C_P(Q) = P$ (Lemma 6.8.(2)) and normal subgroups of P are fully normalized in $\mathcal{F}_G$ as $N_P(Q) = P$ (Lemma 6.9.(3)).

Observe that the preceding results about centralizers and normalizer have a great deal in common. In fact, they both are extreme special cases of a general statement using K-normalizers. In this framework, Lemma 6.6–6.9 are special cases (with $K = \{1\}$ the trivial group or $K = \mathbf{GRP}(Q)^\circ$ the full automorphism group) of the following three statements.

Let $Q \leq P$, $K \leq \mathbf{GRP}(Q)^\circ$, and let $N_P^K(Q)$ and $N_G^K(Q)$ be the K-normalizers of Q in P and G (Definition 3.7).

Lemma 6.11. *The following conditions are equivalent:*

(1) $N_P^K(Q)$ *is a Sylow p-subgroup of* $N_G^K(Q)$
(2) $C_P(Q)$ *is a Sylow p-subgroup of* $C_G(Q)$ *and* $K \cap \mathcal{F}_P(Q)$ *is a Sylow p-subgroup of* $K \cap \mathcal{F}_G(Q)$

Proof. By definition, there is an exact sequence $1 \to C_G(Q) \to N_G^K(Q) \to K \cap \mathcal{F}_G(Q) \to 1$ so that $|N_G(Q)| = |C_G(Q)||K \cap \mathcal{F}_G(Q)|$. Similarly, $|N_P(Q)| = |C_P(Q)||K \cap \mathcal{F}_P(Q)|$. These two identities prove the lemma. □

Lemma 6.12. *There is $g \in N_G(Q, P)$ so that $N_P^{K^g}(Q^g)$ is a Sylow p-subgroup of $N_G^{K^g}(Q^g)$.*

Proof. We may assume that Q is already fully normalized in $\mathcal{F}_G$ (Lemma 6.7) so that $C_P(Q)$ is a Sylow p-subgroup of $C_G(Q)$ and $\mathcal{F}_P(Q)$ is a Sylow p-subgroup of $\mathcal{F}_G(Q)$ (Lemma 6.6). Choose $g \in N_G(Q, Q) = N_G(Q)$ so that (Corollary 3.5.(1)) $(K \cap \mathcal{F}_G(Q))^g \cap \mathcal{F}_P(Q) = K^g \cap \mathcal{F}_G(Q^g)$ is a Sylow p-subgroup of $(K \cap \mathcal{F}_G(Q))^g = K^g \cap \mathcal{F}_G(Q^g)$ (and $Q^g = Q$). Then $N_P^{K^g}(Q^g)$ is a Sylow p-subgroup of $N_G^{K^g}(Q^g)$ (Lemma 6.11). □

Lemma 6.13. *The following conditions are equivalent:*

(1) $N_P^K(Q)$ *is a Sylow p-subgroup of* $N_G^K(Q)$
(2) $C_P(Q)$ *is a Sylow p-subgroup of* $C_G(Q)$ *and* $K \cap \mathcal{F}_P(Q)$ *is a Sylow p-subgroup of* $K \cap \mathcal{F}_G(Q)$

(3) $|N_P^K(Q)| \geq |N_P^{K^g}(Q^g)|$ *for all* $g \in N_G(Q,P)$
(4) $N_P^K(Q)^g = N_P^{K^g}(Q^g)$ *for all* $g \in N_G(Q \cdot N_P^K(Q), P)$

Proof. (1) $\iff$ (2) Lemma 6.11.
(1) $\implies$ (3) $N_P^{K^g}(Q^g)$ is a p-subgroup and $N_P^K(Q)^g$ a Sylow p-subgroup of $N_G^K(Q)^g = N_G^{K^g}(Q^g)$.
(3) $\implies$ (4) Proceed exactly as in the proof of Lemma 6.9.
(4) $\implies$ (1) Proceed exactly as in the proof of Lemma 6.9 (replacing Lemma 6.7 by Lemma 6.12). □

Definition 6.14. *Q is fully K-normalized in $\mathcal{F}_G$ if it satisfies any one of the conditions of Lemma* 6.13*.*

By Lemma 6.13.(2):

$$Q \text{ is fully } K\text{-normalized in } \mathcal{F}_G \implies Q \text{ is fully centralized in } \mathcal{F}_G \iff Q \text{ is fully } \mathcal{F}_P(G)\text{-normalized in } \mathcal{F}_G$$

Lemma 6.15. *If Q is fully K-normalized in $\mathcal{F}_G$ and $H \lhd K$ then Q is also fully H-normalized in $\mathcal{F}_G$.*

Proof. As $N_G^H(Q) \lhd N_G^K(Q)$, and $N_P^K(Q)$ is a Sylow p-subgroup of $N_G^K(Q)$, the intersection $N_P^K(Q) \cap N_G^H(Q) = N_P^H(Q)$ is a Sylow p-subgroup of $N_G^H(Q)$ (Corollary 3.5). □

Remark 6.16 (K-normalizer Frobenius categories)**.** Assume that Q is fully K-normalized in $\mathcal{F}_G$ so that $N_P^K(Q)$ is a Sylow p-subgroup of $N_G^K(Q)$. The Frobenius category $\mathcal{F}_{N_G^K(Q)}$ is a subcategory of $\mathcal{F}_G$. The objects of $\mathcal{F}_{N_G^K(Q)}$ are subgroups R, T, of $N_P^K(Q)$ and the morphisms sets are

$$\mathcal{F}_{N_G^K(Q)}(R,T) = C_G(R)\backslash(N_G^K(Q) \cap N_G(R,T))$$

An element $g \in G$ belongs to $N_G^K(Q) \cap N_G(R,T)$ if and only if conjugation with g defines a homomorphism $Q \cdot R \xrightarrow{g} Q \cdot T$ that restricts to a morphism $R \xrightarrow{g} T$ and an isomorphism $Q \xrightarrow[\cong]{g} Q$ in K. Observe that the following two conditions are equivalent for $N \leq P$:

(1) N is K-normal in P and $N_G^K(N)$ controls fusion in G
(2) N is fully K-normalized in $\mathcal{F}_G$ and $\mathcal{F}_{N_G^K(N)} = \mathcal{F}_G$.

If we assume (1), the objects of $\mathcal{F}_{N_G^K(N)} = \mathcal{F}_G$ are all subgroups, R, T, of $N_P^K(N) = P$ and the morphisms are $\mathcal{F}_{N_G^K(N)}(R,T) = \mathcal{F}_G(R,T)$ since $N_G^K(N)$ controls fusion in G. Conversely, if we assume (2), then $N_P^K(N)$ is a Sylow p-subgroup of $N_G^K(N)$. But P is also a subgroup of $N_G^K(N)$ because P is an object of $\mathcal{F}_G = \mathcal{F}_{N_G^K(N)}$. Thus $N_P^K(N) \leq P \leq N_G^K(N)$ so that $N_P^K(N) = P$ by maximality. The equation $\mathcal{F}_{N_G^K(N)} = \mathcal{F}_G$ means that $N_G^K(N)$ controls fusion in G.

This shows that the Frobenius categories of the local subgroups of G are subcategories of the Frobenius category of G.

6.2. **Selfcentralizing subgroups.** In this section we introduce the p-selfcentralizing p-subgroups of G.

Definition 6.17. (1) *A subgroup H of G is selfcentralizing if the center of H equals the centralizer of H: $Z(H) = C_H(H) = C_G(H)$*
(2) *A p-subgroup Q of G is p-selfcentralizing if the center of Q is a Sylow p-subgroup of the centralizer of Q: $Z(Q) = C_Q(Q) \in \mathrm{Syl}_p(C_G(Q))$*

Lemma 6.18. *The following conditions are equivalent:*

(1) *$Z(Q)$ is a Sylow p-subgroup of $C_G(Q)$*
(2) *Q is a Sylow p-subgroup of $QC_G(Q)$*
(3) *$C_P(Q^g) \leq Q^g$ for all $g \in N_G(Q,P)$*
(4) *$C_P(Q^g) = Z(Q^g)$ for all $g \in N_G(Q,P)$*
(5) *$C_G(Q) \cong Z(Q) \times O_{p'}(C_G(Q))$*

Proof. (1) $\iff$ (2) Q is normal in $QC_G(Q)$ and the isomorphism $QC_G(Q)/Q \cong C_G(Q)/(Q \cap C_G(Q)) = C_G(Q)/Z(Q)$ shows that Q has index prime to p in $C_G(Q)$ if and only if $Z(Q)$ has index prime to p in $C_G(Q)$.
(1) $\implies$ (3) Under assumption (1), $Z(Q)$ is the *unique* Sylow p-subgroup of $C_G(Q)$ because it is evidently central. $C_P(Q^g)$ is a p-subgroup of $C_G(Q^g) = C_G(Q)^g$, so it is contained in the unique Sylow p-subgroup $Z(Q)^g$, which of course is contained in Q^g.

(3) $\implies$ (1) Choose $g \in N_G(Q,P)$ so that Q^g is fully centralized (Definition 6.10), ie $C_P(Q^g)$ is a Sylow p-subgroup of $C_G(Q^g) = C_G(Q)^g$. By assumption (3) we also have that $C_P(Q^g)$ is contained in Q^g so that $C_P(Q^g) = C_P(Q^g) \cap Q^g = C_{Q^g}(Q^g) = C_Q(Q)^g = Z(Q)^g$. We have now shown (1).
(3) $\iff$ (4) This is clear.
(5) $\iff$ (1) If $Z(Q)$ is a Sylow p-subgroup of $C_G(Q)$ then it is a central Sylow p-subgroup and there is a commutative diagram with exact row

$$\begin{tikzcd} 1 \arrow[r] & Z(Q) \arrow[r] & C_G(Q) \arrow[r] & C_G(Q)/Z(Q) \arrow[r] & 1 \\ & & O_{p'}(C_G(Q)) \arrow[u, hook] \arrow[ur, "\cong"'] & & \end{tikzcd}$$

which splits by the Schur–Zassenhaus theorem [40, 9.1.2]. Conversely, if there is a splitting of $C_G(Q)$ as in (5) then clearly $Z(Q)$ is a Sylow p-subgroup. $\square$

Definition 6.19. *A subgroup $Q \le P$ is $\mathcal{F}_G$-selfcentralizing if it satisfies any of the conditions of Lemma* 6.18*.*

Proposition 6.20. *Assume that $Q \le P$ is $\mathcal{F}_G$-selfcentralizing.*
1. *Q is fully centralized in $\mathcal{F}_G$.*
2. *If $N_G(Q,R) \neq \emptyset$, then $R \le P$ is $\mathcal{F}_G$-selfcentralizing.*

Proof. (1) Note that the order of $C_P(Q^g)$ is constant for all $g \in N_G(Q,P)$ and apply Lemma 6.8.
(2) For $g \in N_G(Q,R)$ and $h \in N_G(R,P)$ we have $Q^g \le R$ and $Q^{gh} \le R^h$ so that $C_P(R^h) \le C_P(Q^{gh}) \le Q^{gh} \le R^h$. $\square$

Some easy observations:
- P itself is $\mathcal{F}_G$-selfcentralizing and the trivial subgroup is not $\mathcal{F}_G$-selfcentralizing
- Proposition 6.20.(2) says that supergroups of $\mathcal{F}_G$-selfcentralizing subgroups are $\mathcal{F}_G$-selfcentralizing.

Let $Q \le P$. We say that Q is *selfcentralizing in* P if Q contains its centralizer in P, $Q \ge C_P(Q)$. Then $Z(Q) = C_Q(Q) = Q \cap C_P(Q) = C_P(Q)$ (cf Lemma 6.18.(4)) but still Q may not be $\mathcal{F}_G$-selfcentralizing.

Example 6.21. Let $p = 2$ and let $G = \mathrm{GL}_2(\mathbf{F}_3)$ (of order 48). Let P be a Sylow 2-subgroup of G and B its derived group. Then B is cyclic of order 4 and B has two other G-conjugates that we call A. The two groups A happen to lie in P. The centralizer in P of B is a Singer cycle (a cyclic group of order 8) and the centralizer of A is A itself. Thus A is selfcentralizing in P but not $\mathcal{F}_G$-selfcentralizing. (I heard of this example from Bernd Stellmacher and Andy Chermak.)

In Example 6.21, the group action $\mathcal{F}_G(A) \times \mathcal{F}_G(A,S) \times \mathcal{F}_G(S) \to \mathcal{F}_G(A,S)$, of $\mathcal{F}_G(A) \times \mathcal{F}_G(S)$ on the set of $\mathcal{F}_G$-morphisms from A to S, is not transitive. (Any automorphism of S must preserve the set of subgroups that are selfcentralizing in P.)

6.3. **The extension property.** We shall now discuss the problem of extending homomorphisms in $\mathcal{F}_G$ [37, Chapter 2][9, Appendix A].

Lemma 6.22. *Suppose that Q, R and S are subgroups of G and that Q is contained in both R and S. Then*
$$C_G(Q) \cdot N_R^{\mathcal{F}_S(Q)}(Q) = C_G(Q) \cdot N_S^{\mathcal{F}_R(Q)}(Q)$$

Proof. $N_R^{\mathcal{F}_S(Q)}(Q)/C_R(Q) = N_R^{\mathcal{F}_R(Q) \cap \mathcal{F}_S(Q)}(Q)/C_R(Q) = \mathcal{F}_R(Q) \cap \mathcal{F}_S(Q) = N_S^{\mathcal{F}_R(Q)}(Q)/C_S(Q)$. $\square$

Lemma 6.23 (Extension Lemma)**.** *Let $Q \le P$ be subgroup of the Sylow p-subgroup P and let $g \in N_G(Q,P)$. Assume that Q^g is fully centralized in $\mathcal{F}_G$ (and hence fully $\mathcal{F}_P(Q^g)$-normalized). Then the morphism $Q \xrightarrow{g} P$ in $\mathcal{F}_G$ extends to a morphism $N_P^{\mathcal{F}_P(Q^g)^{g^{-1}}}(Q) \to P$ in $\mathcal{F}_G$.*

Proof. [9, Proposition 1.3] The element g defines an isomorphism $Q \to Q^g$ and these isomorphisms

$$\begin{tikzcd} Q \arrow[r, "g"] \arrow[loop below, "\mathcal{F}_P(Q^g)^{g^{-1}}"] & Q^g \arrow[loop below, "\mathcal{F}_P(Q^g)"] \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[r, "g"] \arrow[loop below, "\mathcal{F}_P(Q)"] & Q^g \arrow[loop below, "\mathcal{F}_{P^g}(Q^g)"] \end{tikzcd}$$

between groups of automorphisms of Q and Q^g.

Put $N_Q = C_G(Q) \cdot N_P^{\mathcal{F}_P(Q^g)^{g^{-1}}}(Q)$ and $N_{Q^g} = C_G(Q) \cdot N_P^{\mathcal{F}_P(Q)^g}(Q^g)$. The image of N_Q under conjugation by g is

$$N_Q^g = N_{P^g}^{\mathcal{F}_P(Q^g)}(Q^g) \cdot C_G(Q^g) = N_P^{\mathcal{F}_{P^g}(Q^g)}(Q^g) \cdot C_G(Q^g) = N_P^{\mathcal{F}_P(Q)^g}(Q^g) \cdot C_G(Q^g) = N_{Q^g}$$

where we use Lemma 6.22. $N_P^{\mathcal{F}_P(Q)^g}(Q^g)$ is a p-subgroup of N_{Q^p} and since the index $[N_{Q^g} : N_P^{\mathcal{F}_P(Q)^g}(Q^g)] = [C_G(Q^g) : C_P(Q^g)]$ is prime to p, as $C_P(Q^g)$ is a Sylow p-subgroup of $C_G(Q^g)$, it is in fact a Sylow p-subgroup. Note also that $C_G(Q^g)$ is a normal subgroup of N_{Q^g} and its index

$$|N_{Q^g} : C_G(Q^g)| = |N_P^{\mathcal{F}_P(Q)^g}(Q^g) : N_P^{\mathcal{F}_P(Q)^g}(Q^g) \cap C_G(Q)| = |N_P^{\mathcal{F}_P(Q)^g}(Q^g) : C_P(Q)|$$

is a p-power.

Conjugation by g takes the p-subgroup $N_P^{\mathcal{F}_P(Q^g)^{g^{-1}}}(Q)$ of N_Q into a p-subgroup of $N_Q^g = N_{Q^g}$, so by Lemma 3.6, there is some $h \in C_G(Q^g)$ such that

$$\big(N_P^{\mathcal{F}_P(Q^g)^{g^{-1}}}(Q)\big)^{gh} \leq N_P^{\mathcal{F}_P(Q)^g}(Q^g) \leq P$$

Thus the group element gh defines an $\mathcal{F}_G$-morphism $N_P^{\mathcal{F}_P(Q^g)^{g^{-1}}}(Q) \to P$.

The group $N_P^{\mathcal{F}_P(Q^g)^{g^{-1}}}(Q)$ contains Q because $\mathcal{F}_P(Q^g)^{g^{-1}}$ contains the group $\mathcal{F}_{Q^g}(Q^g)^{g^{-1}} = \mathcal{F}_Q(Q)$ of inner automorphisms of Q. The morphism gh restricted to Q is the composite $Q \xrightarrow{g} Q^g \xrightarrow{h} Q^{gh} \leq P$. However, since h centralizes Q^g, $Q^g = Q^{gh}$ and the morphism $Q^g \xrightarrow{h} Q^{gh} = Q^g$ is the identity. $\square$

Here is a perhaps more accessible version of Lemma 6.23.

Corollary 6.24. *Assume that Q^g is fully centralized in $\mathcal{F}_G$ for some $g \in N_G(Q,P)$. If $K \leq \mathbf{GRP}(Q)^\circ$ and $K^g \leq \mathcal{F}_P(Q^g)$, then the $\mathcal{F}_G$-morphism $Q \xrightarrow{g} P$ extends to a $\mathcal{F}_G$-morphism $Q \cdot N_P^K(Q) \to P$.*

Proof. It is immediate from Lemma 6.23 that we may extend $Q \xrightarrow{g} P$ to $N_P^K(Q)$ because $N_P^K(Q) \leq N_P^{\mathcal{F}_P(Q^g)^{g^{-1}}}(Q)$ as $K^g \leq \mathcal{F}_P(Q^g)$ in $\mathbf{GRP}(Q^g)^\circ$ or, equivalently, $K \leq \mathcal{F}_P(Q^g)^{g^{-1}}$ in $\mathbf{GRP}(Q)^\circ$.

Of course, $\mathcal{F}_Q(Q)^g = \mathcal{F}_{Q^g}(Q^g) \leq \mathcal{F}_P(Q^g)$ so that, after replacing K by $K \cdot \mathcal{F}_Q(Q)$ if necessary, we can assume that $\mathcal{F}_Q(Q) \leq K$. This assures that $Q \leq N_P^K(Q)$. $\square$

Corollary 6.25. *Let $Q \leq P$, $K \leq \mathbf{GRP}(Q)^\circ$, and $g \in N_G(Q,P)$. Assume that Q^g is fully K^g-normalized in $\mathcal{F}_G$. Then, for some $\chi \in K \cap \mathcal{F}_G(Q)$, the morphism $Q \xrightarrow{\chi} Q \xrightarrow{g} P$ in $\mathcal{F}_G$ extends to a morphism $Q \cdot N_P^K(Q) \to P$ in $\mathcal{F}_G$.*

Proof. By assumption, Q^g is fully centralized in $\mathcal{F}_G$ and $K^g \cap \mathcal{F}_P(Q^g)$ is a Sylow p-subgroup of $K^g \cap \mathcal{F}_G(Q^g)$. Now

$$\begin{aligned}
&K^g \cap \mathcal{F}_P(Q^g) \text{ is a Sylow } p\text{-subgroup of } K^g \cap \mathcal{F}_G(Q^g) = K^g \cap \mathcal{F}_G(Q)^g \\
\Longrightarrow\ &K \cap \mathcal{F}_P(Q^g)^{g^{-1}} \text{ is a Sylow } p\text{-subgroup of } K \cap \mathcal{F}_G(Q) \\
\Longrightarrow\ &\exists \chi \in K \cap \mathcal{F}_G(Q)\colon (K \cap \mathcal{F}_P(Q))^\chi \leq K \cap \mathcal{F}_P(Q^g)^{g^{-1}} \\
\Longrightarrow\ &\exists \chi \in K \cap \mathcal{F}_G(Q)\colon (K \cap \mathcal{F}_P(Q))^{\chi g} \leq \mathcal{F}_P(Q^g) \\
\overset{6.24}{\Longrightarrow}\ &Q \xrightarrow{\chi g} Q \text{ extends to } Q \cdot N_P^{K \cap \mathcal{F}_P(Q)}(Q) \to P
\end{aligned}$$

where $N_P^{K \cap \mathcal{F}_P(Q)}(Q) = N_P^K(Q)$ as a general fact. $\square$

Here is a perhaps more accessible version of Corollary 6.25 (preceded by a lemma needed in the proof).

Lemma 6.26. *Suppose that Q is fully centralized in $\mathcal{F}_G$.*

(1) *If $Q \leq R \leq N_P(Q)$, then Q is fully $\mathcal{F}_R(Q)$-normalized in $\mathcal{F}_G$.*
(2) *If $\mathcal{F}_Q(Q) \leq K \leq \mathcal{F}_P(Q)$, then Q is fully K-normalized in $\mathcal{F}_G$.*

Proof. (1) We check condition Lemma 6.13.(4). As all elements of R normalize Q, $R \leq N_R(Q)$, and thus $R = N_R(Q)$. Now $N_P^{\mathcal{F}_R(Q)}(Q) = N_P(Q) \cap N_R(Q) \cdot C_P(Q) = N_P(Q) \cap R \cdot C_P(Q) = R \cdot C_P(Q)$ as $R \cdot C_P(Q) \leq N_P(Q)$. Let $g \in G$ be any group element so that $N_P^{\mathcal{F}_R(Q)}(Q)^g = (R \cdot C_P(Q))^g \leq P$. Here, $C_P(Q)^g = C_P(Q^g)$ because $(Q \cdot C_P(Q))^g \leq P$ and Q is assumed fully centralized in $\mathcal{F}_G$ (Lemma 6.8.(3)). Also $R^g \leq P$ and this implies that $Q^g \leq R^g \leq P \cap N_P(Q)^g = P \cap N_{P^g}(Q^g) \leq N_P(Q^g)$. We have just seen that $N_P^{\mathcal{F}_{R^g}(Q^g)}(Q^g) = R^g \cdot C_P(Q^g)$ in this situation. We conclude that

$$N_P^{\mathcal{F}_R(Q)}(Q)^g = (R \cdot C_P(Q))^g = R^g \cdot C_P(Q^g) = N_P^{\mathcal{F}_{R^g}(Q^g)}(Q^g) = N_P^{\mathcal{F}_R(Q)^g}(Q^g)$$

which shows that Q is fully $\mathcal{F}_G(Q)$-normalized in $\mathcal{F}_G$.
(2) Since $Q \leq N_P^K(Q) \leq N_P(Q)$, Q is fully $\mathcal{F}_{N_P^K(Q)}(Q)$-normalized in $\mathcal{F}_G$ by (1). We already observed that $\mathcal{F}_{N_P^K(Q)}(Q) = K$. □

$$\begin{tikzcd} Q \arrow[r, "g"] \arrow[d, hook] & P \\ R \arrow[ur, dotted, "?"'] & \end{tikzcd} \qquad Q \leq R \leq N_P(Q) \qquad \begin{tikzcd} Q \arrow[r, "g"] \arrow[loop below, "\mathcal{F}_R(Q)"] & Q^g \arrow[loop above, "\mathcal{F}_P(Q^g)"] \arrow[loop below, "\mathcal{F}_R(Q)^g"] \end{tikzcd}$$

Corollary 6.27. [37, 2.10.1] *If Q^g is fully centralized in $\mathcal{F}_G$ for some $g \in N_G(Q, P)$, $Q \leq R \leq N_P(Q)$, and $\mathcal{F}_R(Q)^g \leq \mathcal{F}_P(Q^g)$, then the $\mathcal{F}_G$-morphism $Q \xrightarrow{g} P$ extends to a $\mathcal{F}_G$-morphism $R \to P$.*

Proof. Observe that

$$Q \leq R \implies Q^g \leq R^g \implies \mathcal{F}_{Q^g}(Q^g) \leq \mathcal{F}_{R^g}(Q^g) = \mathcal{F}_R(Q)^g$$

and that $\mathcal{F}_R(Q)^g \leq \mathcal{F}_P(Q^g)$ by assumption. Hence Q^g is fully $\mathcal{F}_R(Q)^g$-normalized in $\mathcal{F}_G$ by Lemma 6.26.(2). According to Corollary 6.25 there exists a morphism $\chi \in \mathcal{F}_R(Q)$ such that $Q \xrightarrow{\chi g} Q$ extends to an $\mathcal{F}_G$-morphism $Q \cdot N_P^{\mathcal{F}_R(Q)}(Q) \xrightarrow{\psi} P$. Here, $N_P^{\mathcal{F}_R(Q)}(Q) = R \cdot C_P(Q)$ by the proof of Lemma 6.26.(1). In particular, $Q \xrightarrow{\chi g} Q$ extends to $R \xrightarrow{\psi} P$. The $\mathcal{F}_G$-homomorphism $\chi \in \mathcal{F}_R(Q)$ extends to an inner morphism $\chi \in \mathcal{F}_R(R)$. Composing with the inverse $\chi^{-1} \in \mathcal{F}_R(R)$ as in the commutative diagram

$$\begin{tikzcd} Q \arrow[r, "\chi^{-1}"] \arrow[d, hook] & Q \arrow[r, "\chi"] \arrow[d, hook] & Q \arrow[r, "g"] & P \\ R \arrow[r, "\chi^{-1}"'] & R \arrow[urr, "\psi"'] & & \end{tikzcd}$$

we see that $R \xrightarrow{\chi^{-1}\psi} P$ extends $Q \xrightarrow{g} P$. □

7. What does $\mathcal{F}_G$ know about G?

A general question is how much $\mathcal{F}_G$ knows about G. We have already seen that $\mathcal{F}_G = \mathcal{F}_{G/N}$ where N is any normal p'-subgroup of G (Example 6.4). Also, the Frobenius p-complement Theorem 7.3 says that $\mathcal{F}_S = \mathcal{F}_G$ when the Sylow p-subgroup S of G has a normal complement. On the other hand, Proposition 10.8 shows that $\mathcal{F}_G$ is strong enough to guarantee that G is simple up to a normal p'-subgroup. We shall here see two other examples of information about the group that is retained by the Frobenius category.

7.1. **Group cohomology.** Let $Q \leq P$ and $g \in N_G(Q, P)$. The morphism $Q \xrightarrow{g} P$ factors as $Q \xrightarrow{g} Q^g \hookrightarrow P$. Since conjugation by g induces the identity map on $H^*(G)$ there is a commutative diagram

$$\begin{tikzcd} H^*(G) \arrow[r, equal] \arrow[d, hook, "\mathrm{res}^G_P"'] & H^*(G) \arrow[r, equal, "g^*=\mathrm{id}"] \arrow[d] & H^*(G) \arrow[r, equal] \arrow[d] & H^*(G) \arrow[d, hook, "\mathrm{res}^G_P"] \\ H^*(P) \arrow[r, "\mathrm{res}^P_{Q^g}"'] & H^*(Q^g) \arrow[r, "g^*"'] & H^*(Q) & H^*(P) \arrow[l, "\mathrm{res}^P_Q"] \end{tikzcd}$$

where the vertical arrows are restriction homomorphisms. This means that the image of $H^*(G)$ in $H^*(P)$ is contained in the subring

$$H^*(P)^{\mathcal{F}_G} = \{x \in H^*(P) \mid \forall Q \leq P \; \forall \varphi_1, \varphi_2 \in \mathcal{F}_G(Q,P) \colon \varphi_1^*(x) = \varphi_2^*(x)\}$$

of $\mathcal{F}_G$-*stable* elements. An old theorem of Cartan and Eilenberg tells us that even more is true. (Here, $H^*(K)$ is short for $H^*(K; \mathbf{F}_p)$.)

Theorem 7.1. [12, Theorem 10.1] [1, Theorem II.6.6] *The restriction map $H^*(G) \to H^*(P)^{\mathcal{F}_G}$ is an isomorphism.*

7.2. **Frobenius normal p-complement theorem.** We consider here finite groups that are p-groups up to normal p'-subgroups.

Definition 7.2. *A group G is p-nilpotent if a Sylow p-subgroup of G is a retract of G.*

In other words, G is said to be p-nilpotent [40, p 270] (or to have a normal p-complement) if $G = O_{p'}(G)P$ where P is a Sylow p-subgroup of G.

The following theorem is the motivation to name $\mathcal{F}_G$ after Frobenius.

Theorem 7.3. [24, Proposition 16.10] [21] *Let G be a finite group and P a Sylow p-subgroup of G. The following conditions are equivalent:*

(1) *G is p-nilpotent*
(2) *$N_G(Q)$ is p-nilpotent for every nontrivial $Q \leq P$*
(3) *$\mathcal{F}_G(Q)$ is a p-group for every nontrivial $Q \leq P$*
(4) *$\mathcal{F}_G = \mathcal{F}_P$ (P controls fusion in G)*
(5) *$H^*(G; \mathbf{F}_p) \to H^*(P; \mathbf{F}_p)$ is an isomorphism*

Proof. (1) $\iff$ (2) $\iff$ (3): This is Frobenius' original theorem [22, Theorem 7.4.4] [40, 10.3.2].
(4) $\implies$ (3): Clear.
(1) $\implies$ (4): If $G = N \rtimes P$ then there is a retraction (an idempotent) $\sigma \colon G \to P$ of G onto P given by $(kh)^\sigma = h$. But then any two elements of P that are conjugate in G are already conjugate in P: If $h \in P$, $g \in G$ and $h^g \in P$ then $h^g = (h^g)^\sigma = h^{g^\sigma}$.
(1) $\iff$ (5): [38] □

This means that the Frobenius category of a p-nilpotent group depends only on its Sylow p-subgroup. It is now easy to find many examples of distinct groups with identical Frobenius categories.

Example 7.4. The short exact sequence $1 \to C_3 \to D_{24} \to D_8 \to 1$, involving the dihedral groups of order 24 and 8, shows that $D_{24} = C_3 \rtimes D_8$ is 2-nilpotent and therefore $\mathcal{F}_{D_{24}} = \mathcal{F}_{D_8}$.

The alternating group $G = A_4$ of order 12 has Sylow 2-subgroup $C_2 \times C_2 = \langle (12)(34), (13)(24) \rangle$ generated by the double transpositions. G is not 2-nilpotent and the Frobenius categories $\mathcal{F}_P$ and $\mathcal{F}_G$ are not isomorphic.

For odd primes it is possible to strengthen Frobenius's theorem. The Thompson subgroup $J(P)$ of a finite p-group P is the subgroup generated by the set of abelian subgroups of maximal order.

Theorem 7.5 (Glauberman–Thompson normal p-complement theorem)**.** [24, Theorem 16.12]. *Let G be a finite group and P a Sylow p-subgroup of G and assume that p is an odd prime. Then G has a normal p-complement if and only if $N_G(Z(J(P)))$ has a normal p-complement.*

See also Glauberman's fusion theorem 9.5 which in particular applies when P is abelian.

8. Abstract Frobenius categories

Let P be a finite p-group.

Definition 8.1. [37, Chapter 2] *A P-category is a category $\mathcal{F}$ where*

- *the objects of $\mathcal{F}$ are the subgroups of P*
- *the morphisms of $\mathcal{F}$ are injective group homomorphisms*
- *$\mathcal{F}_P$ is a subcategory of $\mathcal{F}$*

A P-category $\mathcal{F}$ that satisfies the

Divisibility axiom: *If $R, S, T \le P$ and $R \xrightarrow{\psi} S \xrightarrow{\varphi} T$ are group homomorphisms then*

$$\varphi \in \mathcal{F}(S,T),\ \psi\varphi \in \mathcal{F}(R,T) \implies \psi \in \mathcal{F}(R,S)$$

is called a divisible *P-category.*

If $\mathcal{F}$ is a divisible P-category and $\varphi \in \mathcal{F}(Q,P)$ a morphism then $\varphi \in \mathcal{F}(Q,Q^\varphi)$ and $\varphi^{-1} \in \mathcal{F}(Q^\varphi, Q)$.

Let $\mathcal{F}$ be a divisible P-category and $Q \le P$, $K \le \mathbf{GRP}(Q)^\circ$.

Definition 8.2. *Q is fully K-normalized in $\mathcal{F}$ if $N_P^K(Q)^\varphi = N_P^{K^\varphi}(Q^\varphi)$ for all $\varphi \in \mathcal{F}(Q \cdot N_P^K(Q), P)$.*

In particular, Q is fully centralized in $\mathcal{F}$ if $C_P(Q)^\varphi = C_P(Q^\varphi)$ for all $\varphi \in \mathcal{F}(Q \cdot C_P(Q), P)$, and Q is fully normalized in $\mathcal{F}$ if $N_P(Q)^\varphi = N_P(Q^\varphi)$ for all $\varphi \in \mathcal{F}(N_P(Q), P)$. Obviously, P itself is fully normalized.

Definition 8.3. [37, Chapter 2] *A Frobenius P-category is a divisible P-category $\mathcal{F}$ satisfying these two additional axioms:*

Sylow axiom: *The group $\mathcal{F}_P(P)$ of inner automorphisms of P is a Sylow p-subgroup of the group $\mathcal{F}(P)$ of $\mathcal{F}$-automorphisms of P*

Extension axiom I: *For any subgroup Q of P, any $K \le \mathbf{GRP}(Q)^\circ$, and any $\mathcal{F}$-morphism $Q \xrightarrow{\varphi} P$ such that Q^φ is fully K^φ-normalized in $\mathcal{F}$, there exists $\chi \in K \cap \mathcal{F}(Q)$ such that $Q \xrightarrow{\chi} Q \xrightarrow{\varphi} P$ extends to an $\mathcal{F}$-morphism $Q \cdot N_P^K(Q) \to P$.*

Extension axiom I, corresponding to Corollary 6.25, can be replaced by this variant [37, Proposition 2.11] corresponding to Corollary 6.27:

Extension axiom II: For any subgroup Q of P and any $\mathcal{F}$-morphism $Q \xrightarrow{\varphi} P$ such that Q^φ is fully centralized and fully normalized in $\mathcal{F}$ and any subgroup R such that $Q \le R \le N_P(Q)$ and $\mathcal{F}_R(Q)^\varphi \le \mathcal{F}_P(Q^\varphi)$ there exists an $\mathcal{F}$-morphism $R \to P$ extending $Q \xrightarrow{\varphi} P$

Let $\mathcal{F}$ be a Frobenius P-category and $Q \le P$, $K \le \mathbf{GRP}(Q)^\circ$.

Proposition 8.4. [37, Proposition 2.7] *There exists a morphism $\varphi \in \mathcal{F}(Q,P)$ such that Q^φ is fully K^φ-normalized in $\mathcal{F}$.*

Proposition 8.5. [37, Proposition 2.11] *The following conditions are equivalent:*

(1) *Q is fully centralized in $\mathcal{F}$ and $K \cap \mathcal{F}_P(Q)$ is a Sylow p-subgroup of $K \cap \mathcal{F}(Q)$*
(2) *$|N_P^K(Q)| \ge |N_P^{K^\varphi}(Q^\varphi)|$ for all $\varphi \in \mathcal{F}(Q,P)$*
(3) *Q is fully K-normalized in $\mathcal{F}$*

By Lemma 8.5.(1):

Q is fully K-normalized in $\mathcal{F}$ $\implies$ Q is fully centralized in $\mathcal{F}$ $\iff$ Q is fully $\mathcal{F}_P(Q)$-normalized in $\mathcal{F}$

If Q is K-normal in P then Q is fully K-normalized because $N_P^K(Q) = P$ of course implies that the order $|N_P^K(Q)|$ is maximal among the subgroups $\mathcal{F}$-isomorphic to Q.

Definition 8.6. [37, 2.14] *Assume that Q is fully K-normalized in $\mathcal{F}$. The K-normalizer of Q in $\mathcal{F}$ is the divisible $N_P^K(Q)$-category $N_\mathcal{F}^K(Q)$ where the morphism sets are*

$$N_\mathcal{F}^K(Q)(R,T) = \left\{ \varphi \in \mathcal{F}(R,T) \mid \exists \psi \in \mathcal{F}(Q\cdot R, Q \cdot T),\ \chi \in K \cap \mathcal{F}(Q) : \begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, hook] & T \arrow[d, hook] \\ Q \cdot R \arrow[r, "\psi"] & Q \cdot T \\ Q \arrow[u, hook] \arrow[r, "\chi"'] & Q \arrow[u, hook] \end{tikzcd} \right\}$$

for all $R, T \le N_P^K(Q)$.

Proposition 8.7. [37, Proposition 2.16] *If Q is fully K-normalized in $\mathcal{F}$, then $N_\mathcal{F}^K(Q)$ is a Frobenius $N_P^K(Q)$-category.*

Example 8.8. If G is a finite group with Sylow p-subgroup P, then the category $\mathcal{F}_G$ (Definition 6.1) is a Frobenius P-category. If $Q \le P$ is fully K-normalized for some subgroup $K \le \mathbf{GRP}(Q)^\circ$ then $N_{\mathcal{F}_G}^K(Q) = \mathcal{F}_{N_G^K(Q)}$ (Remark 6.16). The Sylow p-subgroup P is always fully K-normalized in $\mathcal{F}_G$ for any group K of automorphisms of P. If P is K-normal in G, ie $N_G^K(P) = G$, then $N_{\mathcal{F}_G}^K(P) = \mathcal{F}_{N_G^K(P)} = \mathcal{F}_G$ and $\mathcal{F}_G = \langle K \cap \mathcal{F}_G(P) \rangle$ (see below).

In particular, if Q is fully normalized or fully centralized in $\mathcal{F}$, then the morphisms sets in the categories $N_{\mathcal{F}}(Q)$ and $C_{\mathcal{F}}(Q)$ are

$$N_{\mathcal{F}}(Q)(R,T) = \left\{ \varphi \in \mathcal{F}(R,T) \;\middle|\; \begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, hook] & T \arrow[d, hook] \\ Q\cdot R \arrow[r, "\mathcal{F}(QR,QT)"] & Q\cdot T \\ Q \arrow[u, hook] \arrow[r, "\mathcal{F}(Q)"'] & Q \arrow[u, hook] \end{tikzcd} \right\} \qquad R,T \leq N_P(Q)$$

$$C_{\mathcal{F}}(Q)(R,T) = \left\{ \varphi \in \mathcal{F}_G(R,T) \;\middle|\; \begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, hook] & T \arrow[d, hook] \\ Q\cdot R \arrow[r, "\mathcal{F}_G(QR,QT)"] & Q\cdot T \\ Q \arrow[u, hook] \arrow[r, equal] & Q \arrow[u, hook] \end{tikzcd} \right\} \qquad R,T \leq C_P(Q)$$

We shall write $\mathcal{F} = \langle \mathcal{F}(Q_1), \dots, \mathcal{F}(Q_k) \rangle$ if there are subgroups $Q_1, \dots, Q_k \leq P$ such that any morphism $R_1 \to R_t$ in $\mathcal{F}$ is a composition

$$\begin{tikzcd} & Q_1 \arrow[r, "\varphi_1"] & Q_1 & & Q_2 \arrow[r, "\varphi_2"] & Q_2 & & \cdots & & Q_{t-1} \arrow[r, "\varphi_{t-1}"] & Q_{t-1} & \\ R_1 \arrow[ur, hook] \arrow[rrr] & & & R_2 \arrow[ul, hook] \arrow[ur, hook] \arrow[rrr] & & & R_3 \arrow[ul, hook] \arrow[r] & \cdots \arrow[r] & R_{t-1} \arrow[ur, hook] \arrow[rrr] & & & R_t \arrow[ul, hook] \end{tikzcd}$$

of restrictions of automorphisms $\varphi_1 \in \mathcal{F}(Q_1), \dots, \varphi_{t-1} \in \mathcal{F}(Q_{t-1})$.

The group P is always fully K-normalized in $\mathcal{F}$ for any $K \leq \mathbf{GRP}(P)^\circ$ so that the category $N_{\mathcal{F}}^K(P)$ is defined. It is immediate from Definition 8.6 that

$$N_{\mathcal{F}}^K(P) = \mathcal{F} \iff \mathcal{F} = \langle K \cap \mathcal{F}(P) \rangle \tag{8.9}$$

so that, in particular, $N_{\mathcal{F}}(P) = \mathcal{F}$ if and only if $\mathcal{F} = \langle \mathcal{F}(P) \rangle$ and $C_{\mathcal{F}}(P) = \mathcal{F}$ if and only if $\mathcal{F}$ is the category of posets of P.

Theorem 8.10 (Burnside fusion theorem)**.** [23, Lemma 16.2] *$\mathcal{F} = N_{\mathcal{F}}(P)$ when P is abelian.*

Proof. Let $Q \leq P$ be any subgroup of P. Since P is abelian, Q is fully centralized in $\mathcal{F}$. The claim of the theorem is that every morphism $\varphi \in \mathcal{F}(Q,P)$ extends to an $\mathcal{F}$-morphism defined on P. This follows immediately from Extension axiom I. $\square$

The concept of selfcentralizing subgroups (Definition 6.19) translates immediately to this abstract setting.

Definition 8.11. [37, 4.8.1] *Q is $\mathcal{F}$-selfcentralizing if $C_P(Q^\varphi) \leq Q^\varphi$ for all $\varphi \in \mathcal{F}(Q,P)$*

If Q is $\mathcal{F}$-selfcentralizing and and $\mathcal{F}(Q,R) \neq \emptyset$ then also R is $\mathcal{F}$-selfcentralizing because $C_P(R^\varphi) \leq C_P(Q^{\psi\varphi}) \leq Q^{\psi\varphi} \leq R^\varphi$ for any $\psi \in \mathcal{F}(Q,R)$ and any $\varphi \in \mathcal{F}(R,P)$. (Anything receiving a morphism from a selfcentralizing subgroup is selfcentralizing [37, Proposition 4.5].)

If Q is fully centralized in $\mathcal{F}$ then $QC_P(Q)$ is $\mathcal{F}$-selfcentralizing because

$$C_P((QC_P(Q))^\varphi) = C_P(Q^\varphi C_P(Q)^\varphi) = C_P(Q^\varphi C_P(Q^\varphi)) \leq C_P(Q^\varphi) \overset{\text{Defn 8.2}}{=} C_P(Q)^\varphi \leq (QC_P(Q))^\varphi$$

for any $\varphi \in \mathcal{F}(QC_P(Q),P)$. This observation implies that

$$\begin{aligned} Q \text{ is } \mathcal{F}\text{-selfcentralizing} &\iff Q \text{ is fully centralized in } \mathcal{F} \text{ and } Z(Q) = C_P(Q) \\ &\iff Q \text{ is fully centralized in } \mathcal{F} \text{ and selfcentralizing in } P \end{aligned}$$

for any $Q \leq P$ (as in Proposition 6.20). To see this, assume first that Q is $\mathcal{F}$-selfcentralizing. Then $C_P(Q^\varphi) = Q^\varphi \cap C_P(Q^\varphi) = C_{Q^\varphi}(Q^\varphi) = Z(Q^\varphi) = Z(Q)^\varphi$ is independent of $\varphi \in \mathcal{F}(Q,P)$, and therefore Q is fully centralized in $\mathcal{F}$. Conversely, if Q is fully centralized in $\mathcal{F}$ and $Z(Q) = C_P(Q)$ then $QC_P(Q) = Q$ is $\mathcal{F}$-selfcentralizing.

Proposition 8.12. [9, Proposition A8] [37, Proposition 4.6] *Suppose that $R \leq Q \leq P$ where R is $\mathcal{F}$-selfcentralizing. If $\varphi \in \mathcal{F}(Q,P)$ is the identity on R, then $x^\varphi = x^u$ for all $x \in Q$ for some $u \in C_P(R) = Z(R)$.*

Proof. Assume first that R is normal in Q. Then $Q^\varphi = Q$. To see this, note that for all $r \in R$, $q \in Q$, $r^{(q)\varphi} = (r^\varphi)^{(q)^\varphi} = (r^q)^\varphi = r^q$, means that $(q)\varphi q^{-1} \in C_P(R) \leq R$ so that $(q)\varphi \in R^q \leq Q$. Thus $\varphi \in \mathcal{F}(Q)$. Let $K = \{\alpha \in \mathbf{GRP}(Q)^\circ \mid \alpha|R = \mathrm{id}|R\}$ be the group of automorphisms of Q extending the identity on R. We can, by replacing Q by an $\mathcal{F}$-isomorphic subgroup, if necessary, assume that Q is fully K-normalized in $\mathcal{F}$ (Proposition 8.4). Because $C_Q(R) = Q \cap C_P(R) \leq Q \cap R = R$, it is easily seen that any $\alpha \in K$ will induce the identity on Q/R. (As above, $q^\alpha \equiv q \bmod R$ for all $q \in Q$.) This implies that K is a p-group [22, Corollary 5.3.3, p 179]. Now $K \cap \mathcal{F}_P(Q)$ is a Sylow p-subgroup of $K \cap \mathcal{F}(Q)$ (Proposition 8.5). As the latter is a p-group, this, in fact, means that $K \cap \mathcal{F}_P(Q) = K \cap \mathcal{F}(Q)$. Because φ lies in $\mathcal{F}_P(Q)$, $x^\varphi = x^u$ for some $u \in N_P(R)$. Because $x^r = x^\varphi = x$ when $x \in R$, this $u \in C_P(R)$.

In general, R is not normal but it is subnormal in Q: Taking successive normalizers we obtain a normal chain $R = R_0 \lhd R_1 \lhd \cdots \lhd R_t = Q$ because normalizers grow in a nilpotent group [40, 5.2.4] [27, III.2.3]. Here, $C_P(R) = C_P(R_0) \geq C_P(R_1) \geq \cdots \geq C_P(R_{t-1})$. By applying the above result finitely many times, we get the proposition. □

Definition 8.13 (Exterior quotient of $\mathcal{F}$)**.** *$\widetilde{\mathcal{F}}$ is the left exterior quotient of $\mathcal{F}$ by the left interior structure $\mathcal{I}(Q) = \mathcal{F}_Q(Q)$ for all $Q \leq P$.*

Because $q \circ \varphi = \varphi \circ q^\varphi$ for any $q \in Q$, $\varphi \in \mathcal{F}(Q,R)$, it is clear that $Q \to \mathcal{F}_Q(Q)$ is a left interior structure in $\mathcal{F}$. The morphisms and automorphisms in the exterior quotient $\widetilde{\mathcal{F}}$ (Definition 2.2)

$$\widetilde{\mathcal{F}}(Q,R) = \mathcal{F}(Q,R)/\mathcal{F}_R(R), \qquad \widetilde{\mathcal{F}}(Q) = \mathcal{F}(Q)/\mathcal{F}_Q(Q) = \widetilde{\mathcal{F}}(Q)^\circ$$

are conjugacy classes of $\mathcal{F}$-morphisms and $\mathcal{F}$-automorphisms (outer $\mathcal{F}$-automorphisms). $\widetilde{\mathcal{F}}(P) = \mathcal{F}(P)/\mathcal{F}_P(P)$ is a p'-group because $\mathcal{F}_P(P)$ is a Sylow p-subgroup of $\mathcal{F}(P)$ by the Sylow axiom (Definition 8.3).

8.1. **Alperin fusion.** A subgroup M of a group X is strongly p-embedded if $1 < M < X$, $|M|_p > 0$, and $|M \cap M^x|_p = 0$ for all $x \in X - M$ [24, Definition 17.1, Lemma 17.11].

Lemma 8.14. *A group X with a strongly p-embedded subgroup has no nontrivial normal p-subgroups: $O_p(X) = 1$.*

Proof. Suppose that N is a normal p-subgroup in a group X containing a strongly p-embedded subgroup M. Let S be a Sylow p-subgroup of X. Then $M \geq N_G(S) \geq S \geq N$ [24, Lemma 17.11] and $M^x \geq N^x = N$ for all $x \in X$. If we choose $x \notin M$, we see that $M \cap M^x \geq N$ and thus N must be trivial. □

Definition 8.15. [37, Theorem 5.11] *$Q \leq P$ is $\mathcal{F}$-essential if Q is $\mathcal{F}$-selfcentralizing and $\widetilde{\mathcal{F}}(Q)$ contains a strongly p-embedded subgroup.*

By Lemma 8.14 and some easy manipulations with the functor O_p,

$$Q \text{ is } \mathcal{F}\text{-essential} \implies O_p\widetilde{\mathcal{F}}(Q) = 1 \iff O_p\mathcal{F}(Q) = \mathcal{F}_Q(Q)$$

P itself is not $\mathcal{F}$-essential as the p'-group $\widetilde{\mathcal{F}}(P)$ can not contain a strongly p-embedded subgroup.

Theorem 8.16 (Alperin Fusion Theorem)**.** [37, 5.11][30, 5.2] $\mathcal{F} = \langle \mathcal{F}(Q) \mid Q$ *is $\mathcal{F}$-essential or* $Q = P\rangle$

For instance, if P is abelian, there are no $\mathcal{F}$-essential subgroups because the only $\mathcal{F}$-selfcentralizing subgroup is P itself. (If $Q \leq P$ is $\mathcal{F}$-selfcentralizing then $P = C_P(Q) \leq Q$ by Lemma 6.18.(3).) Therefore $\mathcal{F} = \langle \mathcal{F}(P) \rangle = \mathcal{F}_{P \rtimes \mathcal{F}(P)}$ by Alperin's Fusion Theorem (reproving Theorem 8.10) so that P and $\mathcal{F}(P)$ completely determine $\mathcal{F}$.

Example 8.17. Let $G = \mathrm{GL}_2(\mathbf{F}_q)$ where q is a power of the prime r. The order is $|G| = (q^2-1)(q^2-q) = q(q-1)^2(q+1)$. We shall determine $\mathcal{F}_G$ at the (odd) primes p where it is nontrivial.
$q \equiv 0 \bmod p$: The subgroup $U \cong \mathbf{F}_q^+$ of upper unitriangular matrices from Example 6.5 is a Sylow p-subgroup of G. Its normalizer is $N_G(U) = B$. Since U is abelian, $\mathcal{F}_G = \langle \mathcal{F}_G(U) \rangle$, where $\mathcal{F}_G(U) \cong \mathbf{F}_q^\times$ (Theorem 8.10). Note that the computations of Example 6.5 show that B is a strongly p-embedded subgroup of $\mathrm{GL}_2(\mathbf{F}_q)$.
$q \equiv 1 \bmod p$, $p > 2$: The maximal torus $T_2(\mathbf{F}_q) \cong \mathbf{F}_q^\times \times \mathbf{F}_q^\times$ contains a Sylow p-subgroup $P = O_pT_2(\mathbf{F}_q)$ for G. As P is abelian, $\mathcal{F}_G = \langle \mathcal{F}_G(P) \rangle$. Elementary matrix calculations show that $N_G(P) = \langle T_2(\mathbf{F}_q), \Sigma_2 \rangle$ (the monomial matrices) where Σ_2 are the permutation matrices, and so $\mathcal{F}_G(P) = \Sigma_2$. Observe that $\mathcal{F}_G$ only depends on the p-adic valuation $\nu_p(q-1)$ of $q-1$ in this case.

$\underline{q \equiv -1 \bmod p,\ p > 2}$: We follow the description given in [24, Chp 4.10] to determine a Sylow p-subgroup of $G = \mathrm{GL}_2(q)$. Let r be the characteristic of the field $\mathbf{F}_q$ and view G as $C_{\bar{K}}(\sigma)$ where $\bar{K} = \mathrm{GL}_2(\bar{\mathbf{F}}_r)$ and $\sigma = \varphi_q$ is the Frobenius map.

Let n be the permutation matrix for the transposition $(1,2)$. Choose a matrix $g \in \bar{K}$, such that $gg^{-\sigma} = n$. (Use elementary matrix calculations or Lang's Theorem [24, Theorem 2.1.1] for this.) The commutative diagram

$$\begin{tikzcd} N_{\bar{K}}(\bar{T}) \arrow[r, "g"] \arrow[d, "\sigma n^{-1}"'] & N_{\bar{K}}(\bar{T})^g \arrow[d, "\sigma"] \\ N_{\bar{K}}(\bar{T}) \arrow[r, "g"'] & N_{\bar{K}}(\bar{T}) \end{tikzcd}$$

expresses that the action σn^{-1} on $N_{\bar{K}}(\bar{T})$ corresponds to the action of σ on $N_{\bar{K}}(\bar{T})$. This follows from the computation

$$(x^g)^\sigma = x^{\sigma g^\sigma} = x^{\sigma n^{-1} g} = (x^{\sigma n^{-1}})^g$$

where x is any element of $N_{\bar{K}}(\bar{T})$. Since $(t_1, t_2)^{\sigma n^{-1}} = (t_2^q, t_1^q)$, elementary calculations give that

$$C_{N_{\bar{K}}(\bar{T})}(\sigma n^{-1}) = \langle \begin{pmatrix} u & 0 \\ 0 & u^q \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$$

where u is a primitive element of $\mathbf{F}_{q^2}$. This group contains a normal cyclic subgroup of order $q+1$,

$$C_{q+1} = \langle \begin{pmatrix} v & 0 \\ 0 & v^q \end{pmatrix} \rangle, \qquad v = u^{q-1},$$

and, by the above commutative diagram, $C_{N_{\bar{K}}(\bar{T})}(\sigma n^{-1})^g$ is a subgroup of $C_{N_{\bar{K}}(\bar{T})}(\sigma) \leq C_{\bar{K}}(\sigma) = G$. We have now shown that P^g, with $P = O_p C_{q+1}$, is a Sylow p-subgroup in G. The normalizer of P^g in G is

$$N_G(P^g) = N_{C_{\bar{K}}(\sigma)}(P^g) = C_{N_{\bar{K}}(P^g)}(\sigma) = C_{N_{\bar{K}}(P)^g}(\sigma) = C_{N_{\bar{K}}(\bar{T})^g}(\sigma) = C_{N_{\bar{K}}(\bar{T})}(\sigma n^{-1})^g$$

where we used that $N_{\bar{K}}(P) = N_{\bar{K}}(\bar{T})$. This shows that $\mathcal{F}_G(P^g)$ has order 2. Again, Alperin's Fusion Theorem says that $\mathcal{F}_G = \langle \mathcal{F}_G(P^g) \rangle$.

At $p = 2$, $2 \nmid q$, it is convenient to distinguish between the cases $q \equiv \pm 1 \bmod 4$ because

$$\nu_2(q^2-1) = \begin{cases} 1 + \nu_2(q-1) & q \equiv +1 \bmod 4 \\ 1 + \nu_2(q+1) & q \equiv -1 \bmod 4 \end{cases}, \qquad \nu_2(|\mathrm{GL}_2(q)|) = \begin{cases} 1 + 2\nu_2(q-1) & q \equiv +1 \bmod 4 \\ 2 + \nu_2(q+1) & q \equiv -1 \bmod 4 \end{cases}$$

We shall not go further into this here.

8.2. **Frobenius functors.** We introduce the category **FRB** of Frobenius categories. Let $\mathcal{F}$ be a P-category, $\mathcal{F}'$ a P'-category, and $P \xrightarrow{\alpha} P'$ a group homomorphism.

Definition 8.18. [37, 12.1] *The homomorphism α is $(\mathcal{F}, \mathcal{F}')$-functorial if $\mathcal{F}(Q,R) \circ \alpha \subset \alpha \circ \mathcal{F}'(Q^\alpha, R^\alpha)$, for every $Q, R \leq P$.*

Equivalently, α is $(\mathcal{F}, \mathcal{F}')$-functorial if for every $\varphi \in \mathcal{F}(Q,R)$, there exists $\varphi^\alpha \in \mathcal{F}'(Q^\alpha, R^\alpha)$ such that

$$\begin{tikzcd} Q \arrow[r, two heads, "\alpha"] \arrow[d, "\varphi"'] & Q^\alpha \arrow[d, "\varphi^\alpha"] \\ R \arrow[r, two heads, "\alpha"'] & R^\alpha \end{tikzcd}$$

commutes. In particular, $(\ker(\alpha) \cap Q)^\varphi \leq \ker(\alpha) \cap R$ for all morphisms $\varphi \in \mathcal{F}(Q,R)$ in $\mathcal{F}$. If φ^α exists, it is unique. The functor $\mathfrak{f}_\alpha \colon (\mathcal{F}, \mathcal{F}_P) \to (\mathcal{F}', \mathcal{F}_{P'})$, $\varphi \in \mathcal{F}(Q,R) \to \varphi^\alpha \in \mathcal{F}(Q^\alpha, R^\alpha)$, induced by an $(\mathcal{F}, \mathcal{F}')$-functorial $\alpha \colon P \to P'$ is called a *Frobenius functor* [37, 12.1.2].

Let **FRB** be the category of Frobenius categories $\mathcal{F}$ and $\mathcal{F}'$ with $(\mathcal{F}, \mathcal{F}')$-functorial *epimorphisms* $\alpha \colon P \twoheadrightarrow P'$. The automorphism group (equal to the endomorphism monoid) of a Frobenius P-category $\mathcal{F}$

$$\mathbf{FRB}(\mathcal{F})^\circ = \{\alpha \in \mathbf{GRP}(P)^\circ \mid \alpha \text{ is } (\mathcal{F}, \mathcal{F})\text{-functorial}\} = \mathbf{FRB}(\mathcal{F})$$

consists of the $(\mathcal{F},\mathcal{F})$-functorial automorphisms of P. The automorphisms of $\mathcal{F}$ are the group automorphisms α of P that normalize $\mathcal{F}$ in the sense that $\mathcal{F}(Q^\alpha, R^\alpha) = \mathcal{F}(Q,R)^\alpha$ for all $Q,R \leq P$. In symbols, the automorphism group of the Frobenius cateogry $\mathcal{F}$ is the group

$$\mathbf{FRB}(\mathcal{F}) = \{\alpha \in \mathbf{GRP}(P)^\circ \mid \forall Q,R \leq P \colon \mathcal{F}(Q,R)^\alpha = \mathcal{F}(Q^\alpha, R^\alpha)\}$$

Always, $\mathbf{FRB}(\mathcal{F}_P) = \mathbf{GRP}(P)^\circ$ and $\mathbf{FRB}(\mathcal{F}) \leq N_{\mathbf{GRP}(P)^\circ}(\mathcal{F}(P))$. If $P \leq H \leq G$, then $\mathbf{FRB}(\mathcal{F}_H) \geq \mathbf{FRB}(\mathcal{F}_G)$. Different choices of Sylow p-subgroup in G lead to isomorphic objects $\mathcal{F}_G$ in the category $\mathbf{FRB}$.

Proposition 8.19. *Suppose that $Q \leq P$ is fully K-normalized in $\mathcal{F}$. For any $(\mathcal{F},\mathcal{F})$-functorial automorphism $\alpha \in \mathbf{FRB}(\mathcal{F})$, Q^α is fully K^α-normalized and $\alpha|N_P^K(Q) \colon N_P^K(Q) \to N_P^{K^\alpha}(Q^\alpha)$ is an isomorphism of Frobenius categories, $\alpha|N_P^K(Q) \in \mathbf{FRB}(N_{\mathcal{F}}^K(Q), N_{\mathcal{F}}^{K^\alpha}(Q^\alpha))$. If, in addition, Q and K are normalized by $\mathbf{FRB}(\mathcal{F})$, then there is a restriction homomorphism $\mathbf{FRB}(\mathcal{F}) \to \mathbf{FRB}(N_{\mathcal{F}}^K(Q))$.*

The $\mathcal{F}$-automorphism groups of P, $\mathcal{F}(P) \lhd \mathbf{FRB}(\mathcal{F})$, define an interior structure (Definition 2.2) in the category of Frobenius categories. In the exterior quotient $\widetilde{\mathbf{FRB}}$ the morphisms sets and the automorphism groups are

$$\widetilde{\mathbf{FRB}}(\mathcal{F},\mathcal{F}') = \mathbf{FRB}(\mathcal{F},\mathcal{F}')/\mathcal{F}(P'), \qquad \widetilde{\mathbf{FRB}}(\mathcal{F}) = \mathbf{FRB}(\mathcal{F})/\mathcal{F}(P)$$

Observe here that any $\varphi \in \mathcal{F}(P)$ is $(\mathcal{F},\mathcal{F})$-functorial as there are commutative diagrams

$$\begin{tikzcd} Q \arrow[r, "\psi"] \arrow[d, "\varphi"', "\cong"] & R \arrow[d, "\varphi", "\cong"'] \\ Q^\varphi \arrow[r, "\psi^\varphi"'] & R^\varphi \end{tikzcd}$$

within the category $\mathcal{F}$. This shows that $\mathcal{F}(P) \leq \mathbf{FRB}(\mathcal{F})$. Also, $\mathcal{F}(P) \circ \alpha \subset \mathcal{F}'(P') \circ \alpha$ for any $\alpha \in \mathbf{FRB}(\mathcal{F},\mathcal{F}')$ as α is $(\mathcal{F},\mathcal{F}')$-functorial and maps P *onto* P'. Always, $\widetilde{\mathbf{FRB}}(\mathcal{F}) \leq N_{\mathbf{GRP}(P)^\circ}(\mathcal{F}(P))/\mathcal{F}(P)$. There are group homomorphisms

$$\kappa_G \colon N_{\mathbf{GRP}(G)^\circ}(P) \to \mathbf{FRB}(\mathcal{F}_G), \qquad \widetilde{\kappa}_G \colon \widetilde{\mathbf{GRP}}(G)^\circ \to \widetilde{\mathbf{FRB}}(\mathcal{F}_G) \tag{8.20}$$

for any finite group G. The homomorphism $\widetilde{\kappa}_G$ is surjective if any $(\mathcal{F}_G,\mathcal{F}_G)$-functorial automorphism of P extends to an automorphism of G.

Example 8.21. If $\mathcal{F} = \langle \mathcal{F}(P) \rangle$ (for instance if P is abelian) then the automorphism group and the outer automorphism groups are

$$\mathbf{FRB}(\mathcal{F}) = N_{\mathbf{GRP}(P)^\circ}(\mathcal{F}(P)) \qquad \widetilde{\mathbf{FRB}}(\mathcal{F}) = N_{\mathbf{GRP}(P)^\circ}(\mathcal{F}(P))/\mathcal{F}(P)$$

In Example 6.5 we determined the Frobenius category $\mathcal{F}_{\mathrm{GL}_2(\mathbf{F}_q)}$ associated to $\mathrm{GL}_2(\mathbf{F}_q)$ for q a power of p. The automorphism and outer automorphism groups are

$$\mathbf{FRB}(\mathcal{F}_{\mathrm{GL}_2(\mathbf{F}_q)}) = N_{\mathbf{GRP}(\mathbf{F}_q^+)^\circ}(\mathbf{F}_q^\times) = \mathbf{F}_q^\times \operatorname{Gal}(\mathbf{F}_q/\mathbf{F}_p),\ \widetilde{\mathbf{FRB}}(\mathcal{F}_{\mathrm{GL}_2(\mathbf{F}_q)}) = N_{\mathbf{GRP}(\mathbf{F}_q^+)^\circ}(\mathbf{F}_q^\times)/\mathbf{F}_q^\times = \operatorname{Gal}(\mathbf{F}_q/\mathbf{F}_p)$$

where $\operatorname{Gal}(\mathbf{F}_q/\mathbf{F}_p)$ is the Galois group (acting on $\mathbf{F}_q^+ = \mathbf{F}_p^+ \times \cdots \times \mathbf{F}_p^+$ by the cyclic permutation). We have shown that $\widetilde{\kappa}_{\mathrm{GL}_2(\mathbf{F}_q)} \colon \widetilde{\mathbf{GRP}}(\mathrm{GL}_2(\mathbf{F}_q))^\circ \to \widetilde{\mathbf{FRB}}(\mathcal{F}_{\mathrm{GL}_2(\mathbf{F}_q)})$ is surjective.

Let G be a p-nilpotent group (Theorem 7.3). Since $\mathcal{F}_G(P) = \mathcal{F}_P(P) = \mathrm{Inn}(P)$, the outer automorphism group of the Frobenius category

$$\widetilde{\mathbf{FRB}}(\mathcal{F}_G)^\circ = N_{\mathbf{GRP}(P)^\circ}(\mathrm{Inn}(P))/\mathrm{Inn}(P) = \mathbf{GRP}(P)^\circ/\mathrm{Inn}(P) = \widetilde{\mathbf{GRP}}(P)^\circ$$

is the outer automorphism group of P and the homomorphism $\widetilde{\kappa}_G$ is the standard group homomorphism

$$\widetilde{\kappa}_G \colon \widetilde{\mathbf{GRP}}(G)^\circ \to \widetilde{\mathbf{GPR}}(P)^\circ$$

from the outer automorphism group of G to the outer automorphism group of its Sylow p-subgroup. In case of $G = C_5 \rtimes C_4 = {}^2B_2(2)$, $\widetilde{\kappa}_G$ is not onto for the outer automorphism group of G is trivial but the (outer) automorphism group of C_4 has order 2. In case of $G = C_3^2 \rtimes Q_8 = {}^2A_2(2)$, $\widetilde{\kappa}_G$ is an isomorphism and both outer automorphism groups are isomorphic to the symmetric group Σ_3.

9. $\mathcal{F}$-normal and $\mathcal{F}$-stable subgroups of P

Assume that $N \leq P$ and $K \leq \mathbf{GRP}(N)^\circ$.

Definition 9.1. [37, 12.6] *N is $\mathcal{F}$-K-normal if N is fully K-normalized in $\mathcal{F}$ and $N_{\mathcal{F}}^K(N) = \mathcal{F}$.*

Equivalently, N is $\mathcal{F}$-K-normal if N is K-normal in P (hence fully K-normalized in $\mathcal{F}$) and every $\mathcal{F}$-morphism $Q \to R$ extends to an $\mathcal{F}$-morphism $QN \to RN$ taking N to N by an automorphism in K. To see this, start by recalling that $N_{\mathcal{F}}^K$ is a Frobenius $N_P^K(N)$-category (Definition 8.6) and that the equality $N_{\mathcal{F}}^K(N) = \mathcal{F}$ implies $N_P^K(N) = P$. In particular, we may speak of $\mathcal{F}$-central subgroups ($K = 1$) and $\mathcal{F}$-normal subgroups ($K = \mathbf{GRP}(N)^\circ$).

Definition 9.2. [37, 12.2] [23, Definition 16.18] *$U \leq P$ is $\mathcal{F}$-stable if $(U \cap Q)^\varphi \leq U$ for all $Q \leq P$ and all morphisms $\varphi \in \mathcal{F}(Q, P)$.*

In words, U is $\mathcal{F}$-stable if all elements of U stay in U under $\mathcal{F}$-morphisms. The trivial subgroup and P itself are always $\mathcal{F}$-stable.

For any $N \leq P$ we have

$$N \text{ is } \mathcal{F}\text{-normal} \implies N \text{ is } \mathcal{F}\text{-stable} \implies N \text{ is normal in } P \text{ and fully normalized in } \mathcal{F}$$

If N is $\mathcal{F}$-normal then any $\varphi \in \mathcal{F}(Q, P)$ extends to a morphism $\psi \in \mathcal{F}(QN, P)$ taking N to N and therefore $(N \cap Q)^\varphi = (N \cap Q)^\psi \leq N^\psi \leq N$. If U is $\mathcal{F}$-stable then $U^\varphi = U$ all $\varphi \in \mathcal{F}_P(P)$, so that U is normal in P and therefore fully normalized in $\mathcal{F}$.

Proposition 9.3. [37, Proposition 19.5] *Let $N \leq P$ be a subgroup of P.*

$$N \textit{ is } \mathcal{F}\textit{-normal} \iff N \textit{ is contained in } Q \textit{ and stabilized by } \mathcal{F}(Q) \textit{ for any } \mathcal{F}\textit{-essential } Q \leq P \textit{ and } Q = P$$
$$N \textit{ is } \mathcal{F}\textit{-stable} \iff N \cap Q \textit{ is stabilized by } \mathcal{F}(Q) \textit{ for any } \mathcal{F}\textit{-essential } Q \leq P \textit{ and } Q = P$$

Proof. This is a consequence of Alperin's Fusion Theorem (Theorem 8.16). □

Proposition 9.3 reveals that subgroups generated by $\mathcal{F}$-normal subgroups are $\mathcal{F}$-normal.

Definition 9.4. *$O_p(\mathcal{F})$ is the biggest $\mathcal{F}$-normal subgroup of P.*

In particular,

$$O_p(\mathcal{F}) = 1 \iff \text{The trivial subgroup is the only } \mathcal{F}\text{-normal subgroup of } P$$

$$O_p(\mathcal{F}) = P \iff P \text{ is } \mathcal{F}\text{-normal} \iff N_{\mathcal{F}}(P) = \mathcal{F} \overset{(8.9)}{\iff} \mathcal{F} = \langle \mathcal{F}(P) \rangle$$

If $\mathcal{F} = \mathcal{F}_G$ for some finite group G and $N \leq P$ then

$$N \text{ is } \mathcal{F}_G\text{-}K\text{-normal} \iff N \text{ is } K\text{-normal in } P \text{ and } N_G^K(N) \text{ controls fusion in } G$$

according to Remark 6.16. Special cases are

$$N \text{ is } \mathcal{F}_G\text{-normal} \iff N \text{ is normal in } P \text{ and } \mathcal{F}_G = \mathcal{F}_{N_G(N)}$$
$$N \text{ is } \mathcal{F}_G\text{-central} \iff N \text{ is central in } P \text{ and } \mathcal{F}_G = \mathcal{F}_{C_G(N)}$$

We have that $O_p(G) \leq O_p(\mathcal{F}_G)$ because $O_p(G)$ is $\mathcal{F}_G$-normal. In case $N = P$, the above says that

$$O_p(\mathcal{F}_G) = P \iff P \text{ is } \mathcal{F}_G\text{-normal} \iff \mathcal{F}_G = \langle \mathcal{F}_G(P) \rangle \iff \mathcal{F}_G = \mathcal{F}_{N_G(P)}$$

In fact, a similar but much stronger statement is true.

Theorem 9.5 (Glauberman fusion theorem)**.** [24, Proposition 16.20] *Any abelian $\mathcal{F}_G$-stable subgroup $A \leq P$ is $\mathcal{F}_G$-normal.*

For every $\mathcal{F}$-stable $U \leq P$ there is a Frobenius P/U-category $\mathcal{F}/U$, the U-quotient of $\mathcal{F}$ [37, Proposition 12.3]. For $U \leq Q, R \leq P$ the $\mathcal{F}/U$ morphisms $Q/U \to R/U$ are the group homomorphisms induced by the $\mathcal{F}$-morphisms $Q \to R$.

Proposition 9.6. [37, Remark 12.4] *If U is $\mathcal{F}$-stable and $\mathcal{F}$-selfcentralizing, then $\mathcal{F}/U$ is the Frobenius category associated to the group $\widetilde{\mathcal{F}}(U)$.*

Proof. As U is fully normalized, $\mathcal{F}_P(U)$ is a Sylow p-subgroup of $\mathcal{F}(U)$ (Proposition 8.5) and $N_P(U)/UC_P(U) = P/UC_P(U)$ a Sylow p-subgroup of $\widetilde{\mathcal{F}}(U)$ (Definition 8.13). Here, $N_P(U) = P$ as U is normal in P. If U is also $\mathcal{F}$-selfcentralizing (Definition 8.11), $UC_P(U) = U$ and $P/UC_P(U) = P/U$. It is now easy to check that $\mathcal{F}/U = \mathcal{F}_{\widetilde{\mathcal{F}}(U)}$. □

10. Normal subcategories of Frobenius categories

Normal subcategories are supposed to mimic $\mathcal{F}_N$ inside $\mathcal{F}_G$ where N is normal in G (see Example 10.2.(1)).

Definition 10.1. [37, 12.6] *Let $P' \leq P$ and let $\mathcal{F}'$ be a divisible P'-subcategory of $\mathcal{F}$. Then $\mathcal{F}'$ is normal in $\mathcal{F}$ if*

- *P' is $\mathcal{F}$-stable*
- *$\mathcal{F}(P') \leq \mathbf{FRB}(\mathcal{F}')$ ($\mathcal{F}(P')$ normalizes $\mathcal{F}'$)*
- *$\mathcal{F}'(Q)^{\varphi} = \mathcal{F}'(Q^{\varphi})$ for all $Q \leq P'$ and $\varphi \in \mathcal{F}(Q, P)$ ($\mathcal{F}'(Q) \lhd \mathcal{F}(Q)$, in particular)*

Observe that $\mathcal{F}'$ is just required to be a divisible P'-category, not necessarily a Frobenius category. It is easily seen that if $\mathcal{F}'_1$ and $\mathcal{F}'_2$ are two normal divisible P'-categories in $\mathcal{F}$ then their intersection $\mathcal{F}'_1 \cap \mathcal{F}'_2$ (defined in the obvious way) is also a normal divisible P'-subcategory in $\mathcal{F}$.

Example 10.2. Here are three examples, with increasing levels of abstraction, of normal subcategories of Frobenius categories.

1. Let N be a normal subgroup of G. Then $\mathcal{F}_N$ is a normal $(N \cap P)$-subcategory of $\mathcal{F}_G$:
   - Let $Q \leq P$ and $g \in N_G(Q, P)$. Conjugation by g takes $N \cap P \cap Q = N \cap Q$ into $N \cap P$. Thus $N \cap P$ is $\mathcal{F}_G$-stable.
   - Let $Q, R \leq N \cap P$ and $g \in N_G(N \cap P)$. Then $Q^g, R^g \leq (N \cap P)^g = N \cap P$ and conjugation by g takes $N_N(Q, R)$ to $N_N(Q^g, R^g)$. Thus $\mathcal{F}_G(N \cap P)$ normalizes $\mathcal{F}_N(Q, R)$.
   - Let $Q \leq N \cap P$ and $g \in N_G(Q, P)$. Conjugation by g takes $N_N(Q)$ to $N_N(Q^g)$. This means that $\mathcal{F}_N(Q)^{\varphi} = \mathcal{F}_N(Q^{\varphi})$ for all $\varphi \in \mathcal{F}_G(Q, P)$.
2. Suppose that $Q \leq P$ is fully K-normalized in $\mathcal{F}_G$ for some $K \leq \mathbf{GRP}(Q)^{\times}$. Then $\mathcal{F}_{C_G(Q)}$ is a Frobenius $C_P(Q)$-subcategory of the Frobenius $N_P(Q)$-category $\mathcal{F}_{N_G^K(Q)}$ (Remark 6.16). Since $C_G(Q)$ is normal in $N_G^K(Q)$, the $C_P(Q)$-subcategory $\mathcal{F}_{C_G(Q)} = C_{\mathcal{F}_G}(Q)$ is normal in the $N_P(Q)$-category $\mathcal{F}_{N_G^K(Q)} = N_{\mathcal{F}_G}^K(Q)$.
3. Let $Q \leq P$, $K \leq \mathbf{GRP}(Q)^{\circ}$, and assume that Q is fully K-normalized in $\mathcal{F}$. Then Q is fully centralized in $\mathcal{F}$ by Proposition 8.5.(1). Therefore $C_{\mathcal{F}}(Q)$ is Frobenius $C_P(Q)$-category and $N_{\mathcal{F}}^K(Q)$ a $N_P^K(Q)$-category (Proposition 8.7). It is easy to check that $C_{\mathcal{F}}(Q)$ is normal in $N_{\mathcal{F}}^K(Q)$ [37, 18.2].

Proposition 10.3 (Frattini argument)**.** [37, Proposition 12.8] *Assume that $P' \leq P$ and that $\mathcal{F}'$ is a divisible P'-subcategory of $\mathcal{F}$. Then $\mathcal{F}'$ is normal in $\mathcal{F}$ if and only if*

- *P' is $\mathcal{F}$-stable*
- *$\mathcal{F}(P') \leq \mathbf{FRB}(\mathcal{F}')$*
- *$\mathcal{F}(Q, P') = \mathcal{F}'(Q, P') \circ \mathcal{F}(P')$ for all $Q \leq P'$.*

$$\begin{tikzcd} Q \arrow[r, "\mathcal{F}'(Q,P')"'] \arrow[rr, bend left, "\mathcal{F}(Q,P')"] & P' \arrow[r, "\mathcal{F}(P')"'] & P' \end{tikzcd}$$

The *adjoin subcategory* $\mathcal{F}^a$ of $\mathcal{F}$ is the smallest Frobenius P-category $\mathcal{F}'$ in $\mathcal{F}$ satisfying a certain condition [37, 12.7, Corollary 12.17] [7, Definition 3.1]. $\mathcal{F}^a$ is a Frobenius P-category, $\mathcal{F}^a$ is normal in $\mathcal{F}$, and $(\mathcal{F}^a)^a = \mathcal{F}^a$. By Proposition 10.4, $\mathcal{F}^a$ can also be characterized as the smallest normal Frobenius P-category in $\mathcal{F}$.

Proposition 10.4. *Let $\mathcal{F}$ be a Frobenius P-category.*

1. *If $\mathcal{F}'$ is a normal divisible P-category in $\mathcal{F}$ then $\mathcal{F}'(Q)$ has index prime to p in $\mathcal{F}(Q)$ for every $Q \leq P$.*
2. *$\mathcal{F}^a$ is the smallest normal Frobenius P-category in $\mathcal{F}$.*
3. *The intersection of finitely many normal Frobenius categories in $\mathcal{F}$ is a normal Frobenius category in $\mathcal{F}$.*

Proof. (1) [30] Let $Q \leq P$. Choose $\varphi \in \mathcal{F}(Q, P)$ so that Q^{φ} is fully $\mathcal{F}$-normalized (Proposition 8.4). Now

$$\mathcal{F}_P(Q^{\varphi}) \leq \mathcal{F}'(Q^{\varphi}) \lhd \mathcal{F}(Q^{\varphi})$$

where $\mathcal{F}_P(Q^\varphi)$ is Sylow p-subgroup of the last group (Proposition 8.5.(1)), $\mathcal{F}(Q^\varphi)$, and hence also of the middle group, $\mathcal{F}'(Q^\varphi)$. Thus $\mathcal{F}'(Q^\varphi) = \mathcal{F}'(Q)^\varphi$ has index prime to p in $\mathcal{F}(Q^\varphi) = \mathcal{F}(Q)^\varphi$ and $\mathcal{F}'(Q)$ has index prime to p in $\mathcal{F}(Q)$.
(2) Let $\mathcal{F}'$ be a normal Frobenius P-category in $\mathcal{F}$. In order to show that $\mathcal{F}'$ contains $\mathcal{F}^a$ it suffices to show that $X_{\mathcal{F}}(Q) \leq \mathcal{F}'(Q)$ for every $\mathcal{F}$-essential $Q \leq P$ [37, Corollary 12.17] where $X_{\mathcal{F}}(Q)$ is the subgroup defined in [37, Corollary 5.13]. The group $\mathcal{F}_Q(Q)$ is contained in $\mathcal{F}_P(Q)$ and in $\mathcal{F}'(Q)$ because $\mathcal{F}_P$ is a subcategory of $\mathcal{F}'$. Since we know that $\mathcal{F}'(Q)$ has index prime to p in $\mathcal{F}(Q)$ also $\mathcal{F}'(Q)/\mathcal{F}_Q(Q)$ has index prime to p in $\mathcal{F}(Q)/\mathcal{F}_Q(Q) = \widetilde{\mathcal{F}}(Q)$ (Definition 8.13). Thus every p-element of $\widetilde{\mathcal{F}}(Q)$ lies in $\mathcal{F}'(Q)/\mathcal{F}_Q(Q)$. Let X be a normal subgroup of $\widetilde{\mathcal{F}}(Q)$ whose order is divisible by p. Then the same is true of $X \cap \mathcal{F}'(Q)/\mathcal{F}_Q(Q)$. This shows that the smallest such subgroup, $X_{\widetilde{\mathcal{F}}}(Q)$, is contained in $\mathcal{F}'(Q)/\mathcal{F}_Q(Q)$. Therefore, its preimage, $X_{\mathcal{F}}(Q)$, is contained in $\mathcal{F}'(Q)$.
(3) Let $\mathcal{F}'_1$ and $\mathcal{F}'_2$ be normal Frobenius P-categories in $\mathcal{F}$. Their intersection, $\mathcal{F}'_1 \cap \mathcal{F}'_2$, is a divisible P-category in $\mathcal{F}$ containing $\mathcal{F}^a$. By [37, Corollary 12.17], $\mathcal{F}'_1 \cap \mathcal{F}'_2$ is a Frobenius category. □

The extreme cases are that $\mathcal{F}^a$ equals $\mathcal{F}$ or $\mathcal{F}_P$. We have

$$\mathcal{F} = \mathcal{F}^a \iff \mathcal{F} \text{ is the only normal Frobenius } P\text{-category in } \mathcal{F}$$

according to Proposition 10.4.

Corollary 10.5. *The following conditions are equivalent:*

(1) $\mathcal{F}_P = \mathcal{F}^a$
(2) $\mathcal{F}_P$ *is normal in* $\mathcal{F}$
(3) $\mathcal{F} = N_{\mathcal{F}}(P)$
(4) $\mathcal{F} = \langle \mathcal{F}(P) \rangle$

Proof. (1) and (2) are equivalent because $\mathcal{F}^a$ is normal in $\mathcal{F}$ and $\mathcal{F}_P$ is the smallest Frobenius P-category in $\mathcal{F}$. We previously observed that (3) and (4) are equivalent. That (2) implies (4) follows from Proposition 10.3. The converse is clear. □

Definition 10.6. [37, 12.20] *The Frobenius P-category $\mathcal{F}$ is simple if there are no nontrivial proper $\mathcal{F}$-stable subgroups of P and $\mathcal{F} = \mathcal{F}^a$.*

Example 10.7. [7, p 3839] Let $G = A_5$ of order 60 and $p = 2$. The Sylow p-subgroup $P = C_2 \times C_2$ is abelian so $\mathcal{F}_G = \langle \mathcal{F}_G(P) \rangle$ where $\mathcal{F}_G(P) = C_3$ (Theorem 8.10). It is easy to verify directly that the discrete category $\mathcal{F}_P$ is a normal P-category in $\mathcal{F}_G$ [30, Proposition 7.3]. Thus $\mathcal{F}_G^a = \mathcal{F}_P$ and $\mathcal{F}_G$ is not simple even though G is simple. Also, $O^{p'}(G) = G$ because G is simple so that $\mathcal{F}^a$ is not $\mathcal{F}_{O^{p'}(G)}$.

Since the Frobenius category is unable to distinguish between a group G and any factor group G/N by a normal p'-subgroup N (Example 6.4) the following result is about the best we could hope for.

Proposition 10.8. [37, 12.20.1] *If $\mathcal{F}_G$ is simple and $O_{p'}(G) = 1$, then G is simple.*

Proof. Let N be a nontrivial normal subgroup of G. N is not a p'-group for $O_{p'}(G) = 1$. Let P be a Sylow p-subgroup of G. Then $N \cap P$, a Sylow p-subgroup of N (Corollary 3.5.(2)), is nontrivial. $N \cap P$ is also $\mathcal{F}_G$-stable (Example 10.2), and therefore $N \cap P = P$. This shows that N contains P. We have now shown that N contains all Sylow p-subgroups of G. This means that N contains $O^{p'}(G)$, the normal subgroup generated by all Sylow p-subgroups of G.

We know that $\mathcal{F}_N$ is a normal P-subcategory of $\mathcal{F}_G$ (Example 10.2). For any $Q \leq P$, $N_N(Q)$ is normal in $N_G(Q)$ and $N_G(Q)/N_N(Q)$ is a p'-group because $N_N(Q)$ contains all p-elements of $N_G(Q)$ (Proposition 3.2). The quotient group $\mathcal{F}_G(Q)/\mathcal{F}_N(Q)$ is therefore also a p'-group. This means that $\mathcal{F}_N(Q)$ contains $O^{p'}\mathcal{F}_G(Q)$. But the adjoin subcategory $\mathcal{F}_G^a$ is the smallest normal P-subcategory with this property and therefore $\mathcal{F}_G^a$ is a P-subcategory of $\mathcal{F}_N$. By assumption, $\mathcal{F}_G^a$ equals $\mathcal{F}_G$, so we conclude that $\mathcal{F}_N = \mathcal{F}_G$ and that $N_G(P) = N_N(P)$ (6.2). Now $G = N_G(P)N = N_N(P)N = N$ where the first equality is the Frattini argument (Corollary 3.5.(3)). □

The *hyperfocal subcategory* $\mathcal{F}^h$ of $\mathcal{F}$ [37, 13.8] is the smallest normal $H_{\mathcal{F}}$-subcategory of $\mathcal{F}$ such that $\mathcal{F}^h(R) \geq O^p(\mathcal{F}(R))$ for any $R \leq H_{\mathcal{F}}$ fully normalized in $\mathcal{F}$ [37, Theorem 13.6]; $\mathcal{F}(R)/\mathcal{F}^h(R)$ is p-group. Here,

$$H_{\mathcal{F}} = \langle u^{-1}u^\sigma \mid u \in Q \leq P, \sigma \in O^p(\mathcal{F}(Q)) \rangle$$

is the $\mathcal{F}$*-hyperfocal subgroup* of P [37, 13.2]. (In the special case of the Frobenius category of a finite group G, $H_{\mathcal{F}_G} = P \cap O^p(G)$ [37, 13.2.1] [7, Lemma 2.2].) [2]

Definition 10.9. [2] *The group G is wild if (assuming p odd)*

- $O_p(\mathcal{F}) = 1$
- $\mathcal{F}^a_G = \mathcal{F}_G = \mathcal{F}^h_G$
- $\widetilde{\kappa}_G$ (8.20) *is not split surjective*

It is not known if wild groups exist. If they do then they will produce exotic fusion categories [2].

Albert showed that there are simple groups such that the fusion system contains a normal subfusion system.

11. Frobenius categories and classifying spaces

Let G be a finite group, BG the classifying space, and $C_p(BG)$ the Bousfield–Kan p-completion of BG (Subsection 11.2). Whereas the classifying space BG is a $K(\pi, 1)$, the p-completed classifying space may have higher homotopy groups. However, whatever happens to BG under p-completion is invisible seen from a finite p-group.

Theorem 11.1. *For any finite p-group Q and any finite group G, there are $H\mathbf{Z}/p$-equivalences*

$$\coprod_{\rho \in \widetilde{\mathbf{GRP}}(Q,G)} BC_G(Q^\rho) \simeq \mathrm{map}(BQ, BG) \simeq \mathrm{map}(BQ, C_p(BG))$$

Equivalently,

1. *The natural maps $\widetilde{\mathbf{GRP}}(Q,G) \to [BQ, BG] \to [BQ, C_p(BG)]$ are bijections*
2. *The natural maps $BC_G(Q^\rho) \to \mathrm{map}(BQ, BG)_\rho \to \mathrm{map}(BQ, C_p(BG))_\rho$ are $H\mathbf{Z}/p$-equivalences for any group homomorphism $\rho \colon Q \to G$.*

Proof. (1) Dwyer and Zabrodsky [14, Theorem 1.1] show that $\widetilde{\mathbf{GRP}}(Q,G) = [BQ, BG]$ and Mislin [34, Equation (4)] shows that $[BQ, BG] = [BQ, C_p(BG)]$.
(2) The group homomorphism $C_G(Q^\rho) \times Q \to G$ induces a map $BC_G(Q^\rho) \times BQ \to BG$ between classifying spaces whose adjoint is the map $BC_G(Q^\rho) \to \mathrm{map}(BQ, BG)_\rho$. Dwyer and Zabrodsky [14, Theorem 1.1] show that $BC_G(Q^\rho) \to \mathrm{map}(BQ, BG)_\rho$ is an $H\mathbf{Z}/p$-equivalence, and Mislin [34] or Broto and Levi [8, Proposition 2.1] that also $\mathrm{map}(BQ, BG)_\rho \to \mathrm{map}(BQ, C_p(BG))_\rho$ is an $H\mathbf{Z}/p$-equivalence. (These results would not have been possible without the Sullivan conjecture [33].) $\square$

The finite group G determines a presheaf

$$\widetilde{\mathbf{GRP}}(-, G) \colon p\mathbf{GRP}^{\mathrm{op}} \to \mathbf{SET} \tag{11.2}$$

on the category $p\mathbf{GRP}$ of p-groups. Recall that elements of $\widetilde{\mathbf{GRP}}(Q,G)$ have kernels (Example 2.3.(1)). The constant epimorphism $Q \to 1$ induces an injection $\{1\} = \widetilde{\mathbf{GRP}}(1,G) \to \widetilde{\mathbf{GRP}}(Q,G)$ that picks out the constant homomorphism $Q \xrightarrow{1} G$. If G_1 and G_2 are two finite groups and $A \colon \widetilde{\mathbf{GRP}}(-, G_1) \Rightarrow \widetilde{\mathbf{GRP}}(-, G_2)$ a natural transformation of the associated presheaves, then the component $A(Q) \colon \widetilde{\mathbf{GRP}}(Q, G_1) \Rightarrow \widetilde{\mathbf{GRP}}(Q, G_2)$, for any finite p-group Q, takes 1 to 1 and it increases kernels in the sense that $\ker \varphi \leq \ker A(Q)(\varphi)$ for any $\varphi \in \widetilde{\mathbf{GRP}}(Q, G_1)$ by naturality.

Proposition 11.3. *The following are equivalent for two given finite groups G_1 and G_2:*

1. *$\mathcal{F}_{G_1}$ and $\mathcal{F}_{G_2}$ are isomorphic Frobenius categories*
2. *$\widetilde{\mathbf{GRP}}(-, G_1)$ and $\widetilde{\mathbf{GRP}}(-, G_2)$ are isomorphic presheaves on $p\mathbf{GRP}$*

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[2] The adjoin subcategory $\mathcal{F}^a$ is denoted $O^{p'}(\mathcal{F})$ in [7, p 3838]. The $\mathcal{F}$-hyperfocal subgroup $H_{\mathcal{F}}$ is denoted $O^p_{\mathcal{F}}(P)$ [7, Definition 2.1] and the hyperfocal subcategory $\mathcal{F}^h$ is $O^p(\mathcal{F})$ in [7, p 3823].

Proof. Suppose that $\mathcal{F}_{G_1}$ and $\mathcal{F}_{G_2}$ are isomorphic Frobenius categories. This means that there exists an $(\mathcal{F}_{G_1}, \mathcal{F}_{G_2})$-functorial isomorphism $\alpha\colon P_1 \to P_2$ of Sylow p-subgroups (Definition 8.18). There is an induced bijection, natural in Q, as in the diagram

$$\begin{tikzcd} \mathbf{GRP}(Q,P_1) \arrow[r, "\alpha", "\cong"'] \arrow[d] & \mathbf{GRP}(Q,P_2) \arrow[d] \\ \widetilde{\mathbf{GRP}}(Q,G_1) \arrow[r, dotted, "A"] & \widetilde{\mathbf{GRP}}(Q,G_2) \end{tikzcd} \tag{11.4}$$

The reason that this bijection is well-defined is that if $\varphi_1, \varphi_2\colon Q \to P_1$ are two lifts up to conjugacy in G_1 of the same $\varphi\colon Q \to G_1$, then $\varphi_1^g = \varphi_2$ for some g in G_1. Because α is functorial there is some g^α in G_2 such that the diagram

$$\begin{tikzcd} & Q^{\varphi_1} \arrow[r, "\alpha"] \arrow[d, "g"] & Q^{\varphi_1\alpha} \arrow[r] \arrow[d, "g^\alpha"] & G_2 \arrow[d, "g^\alpha"] \\ Q \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & & & \\ & Q^{\varphi_2} \arrow[r, "\alpha"] & Q^{\varphi_2\alpha} \arrow[r] & G_2 \end{tikzcd}$$

commutes. Thus $\varphi_1\alpha$ and $\varphi_2\alpha$ are G_2-conjugate.

Conversely, assume that $A\colon \widetilde{\mathbf{GRP}}(-,G_1) \Rightarrow \widetilde{\mathbf{GRP}}(-,G_2)$ is a natural isomorphism of presheaves on $p\mathbf{GRP}$. Then A preserves kernels. Let $P_1 \xhookrightarrow{\iota_1} G_1$ be the inclusion of a Sylow p-subgroup. Consider the component $A(P_1)\colon \widetilde{\mathbf{GRP}}(P_1,G_1) \to \widetilde{\mathbf{GRP}}(P_1,G_2)$ of A. Now

$$A(P_1)(P_1 \xhookrightarrow{\iota_1} G_1)^{G_1} = (P_1 \xhookrightarrow{\alpha} G_2)^{G_2}$$

for some monomorphism $\alpha\colon P_1 \hookrightarrow G_2$. Put $P_2 = \alpha(P_1) \le G_2$. Then P_2 is a Sylow p-subgroup of G_2 because it is a p-subgroup of maximal order. Moreover, the isomorphism $\alpha\colon P_1 \xrightarrow{\cong} P_2$ realizes the natural isomorphism A as in the above commutative diagram (11.4). An easy inspection reveals that this precisely means that α is $(\mathcal{F}_{G_1}, \mathcal{F}_{G_2})$-functorial. □

As the presheaf $\widetilde{\mathbf{GRP}}(-,G) = [B-, C_p(BG)]$ is determined by the p-completed classifying space by Theorem 11.1, we arrive at the following corollary which says that we can construct isomorphisms of Frobenius categories by constructing homotopy equivalences between localized classifying spaces.

Corollary 11.5. [32] *If $C_p(BG_1)$ and $C_p(BG_2)$ are homotopy equivalent topological spaces then $\mathcal{F}_{G_1}$ and $\mathcal{F}_{G_2}$ are isomorphic Frobenius categories.*

The converse of Corollary 11.5, called the Martino–Priddy conjecture, has been proved by Bob Oliver [35, 36] using the classification of the finite simple groups. The conclusion is

$$C_p(BG_1) \simeq C_p(BG_2) \iff \mathcal{F}_{G_1} \cong \mathcal{F}_{G_2}$$

for any two finite groups G_1 and G_2. This means that two finite groups have the same p-local structure if and only if their classifying spaces have the same p-local structure (Remark 11.9).

11.1. **An algebraization of $C_p(BG)$.** There is a functor

$$L\colon \mathcal{F}_G^{\mathrm{op}} \to G\mathbf{SET}\colon (Q \xrightarrow{g} R) \mapsto (C_G(Q)\backslash G \xleftarrow{C_G(Q)gh \leftarrow C_G(R)h} C_G(R)\backslash G)$$

taking values in the category of transitive right G-sets. Observe that the the morphism $L(g)$ is well-defined because $gC_G(R) \subset C_G(Q)g$, or $C_G(R) \le C_G(Q)^g = C_G(Q^g)$, whenever $g \in N_G(Q,R)$. Define

$$L_p\colon \mathcal{F}_G^{\mathrm{op}} \to \mathbf{GRPOID} \tag{11.6}$$

be the composite functor $\mathcal{F}_G^{\mathrm{op}} \xrightarrow{L} G\mathbf{SET} \to \mathbf{GRPOID} \xrightarrow{M \mapsto O^pM\backslash M} \mathbf{GRPOID}$. More explicitly, $L_p(Q)$, for any $Q \le P$, is the connected groupoid on the object set $C_G(Q)\backslash G$ with morphism sets and automorphism groups

$$L_p(Q)(C_G(Q)g_1, C_G(Q)g_2) = O^pC_G(Q)^{g_1}\backslash g_1^{-1}C_G(Q)g_2, \qquad L_p(Q)(C_G(Q)g) = O^pC_G(Q)^g\backslash C_G(Q)^g$$

$$\begin{tikzcd}
C_p(BG) \arrow[rr, shift left] \arrow[dd] & & L_p \rtimes \mathcal{F}_G^{\mathrm{op}} \arrow[ll, shift left] \arrow[dd, shift left] \\
& G \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \\
H^*(G;\mathbf{F}_p) & & \mathcal{F}_G \arrow[ll] \arrow[uu, shift left]
\end{tikzcd}$$

FIGURE 3. p-local information attached to the finite group G

In particular, when Q is selfcentralizing, $L_p(Q)$ is a connected groupoid with automorphism groups isomorphic to

$$L_p(Q)(C_G(Q)) = O^pC_G(Q)\backslash C_G(Q) \cong Z(Q)$$

The isomorphism here comes from the splitting $C_G(Q) \cong Z(Q) \times O_{p'}C_G(Q)$ from Lemma 6.18. In this case the classifying space $BL_p(Q)$ of the groupoid $L_p(Q)$ is homotopy equivalent to $BZ(Q)$.

Define $\mathcal{F}_G^{\mathrm{sc}}$ to be the full subcategory of the Frobenius category $\mathcal{F}_G$ generated by all its selfcentralizing objects. Let L_p^{sc} denote the restriction to $\mathcal{F}_G^{\mathrm{sc}}$ of the sheaf L_p on $\mathcal{F}_G$.

Theorem 11.7. [9, 15] *The spaces $C_p(\operatorname{hocolim} L_p^{\mathrm{sc}})$ and $C_p(BG)$ are homotopy equivalent.*

According to this theorem we can build the p-completed classifying space $C_p(BG)$ from the very simple building blocks $BZ(Q)$ where Q is a $\mathcal{F}_G$-selfcentralizing object. Thus the functor (11.6), or its restriction to the selfcentralizing subcategory, is an algebraization of $C_p(BG)$. The homotopy colimit of the functor L_p^{sc} is homotopy equivalent to the classifying space of the Grothendieck construction $\mathcal{L}_G^{\mathrm{sc}} = L_p^{\mathrm{sc}} \rtimes \mathcal{F}_G^{\mathrm{op}}$ on the functor L_p^{sc} [41]. Therefore we have

$$C_p(BG_1) \simeq C_p(BG_2) \iff \mathcal{L}_{G_1}^{\mathrm{sc}} \cong \mathcal{L}_{G_2}^{\mathrm{sc}}$$

The hard part in proving the Martino–Priddy conjecture is to show that $\mathcal{F}_{G_1} \cong \mathcal{F}_{G_2} \implies \mathcal{L}_{G_1}^{\mathrm{sc}} \cong \mathcal{L}_{G_2}^{\mathrm{sc}}$. (The category $\mathcal{L}_G^{\mathrm{sc}}$ is the 'centric linking system' of [9] and an example of an '$\mathcal{F}_G$-locality' of [37, 17.3]. The existence or uniqueness of centric linking systems associated to abstract Frobenius categories is an important unsolved problem.)

11.2. **The Bousfield–Kan p-completion functor C_p.** Let us say that a space C is p-local if $[A, C] \leftarrow [B, C]$ is bijective whenever $A \to B$ is an $H\mathbf{Z}/p$-equivalence. (C is p-local if it believes that all $H\mathbf{Z}/p$-equivalence are homotopy equivalences.) Ideally, we would like to have a functorial construction of an $H\mathbf{Z}/p$-equivalence $X \to CX$ into a p-local space CX (a p-local space not far from X).

Bousfield and Kan partially succeed in this. They functorially associate to any space X a map $X \to C_p(X)$. The Bousfield–Kan p-completion, $C_p(X)$, is indeed p-local but X and $C_p(X)$ do not always have the same $H\mathbf{Z}/p$-homology, only when X is '$\mathbf{Z}/p$-good' [6, I.5.1]. The main properties of the Bousfield–Kan p-completion are:

- $C_p(X)$ is p-local [6, II.2.8]
- $X \to C_p(X)$ is an $H\mathbf{Z}/p$-equivalence when X is connected and nilpotent or $\pi_1(X)$ is finite [6, VI.5.3, VII.5.1] (and $C_p(X)$ agrees with the Bousfield $H\mathbf{Z}/p$-localization of X] [5, §4])
- $X \to Y$ is an $H\mathbf{Z}/p$-equivalence $\iff$ $C_p(X) \to C_p(Y)$ is a homotopy equivalence [6, I.5.5]
- X is k-connected $\implies$ $C_p(X)$ is k-connected [6, I.6.1]
- C_p preserves $H\mathbf{Z}/p$-nilpotent fibrations of connected spaces [6, II.5.1]
- $BQ \to C_p(BQ)$ is a homotopy equivalence for any finite p-group Q [6, VI.3]

It follows for instance that $C_p(X) \simeq * \iff X \to *$ is an $H\mathbf{Z}/p$-equivalence $\iff \overline{H}_*(X; \mathbf{Z}/p) = 0$.

By definition, X is $\mathbf{Z}/p$-complete if $X \to C_p(X)$ is a homotopy equivalence [6, I.5.1]. BQ, Q a finite p-group, is $\mathbf{Z}/p$-complete. If X is connected and nilpotent or has a finite fundamental group, then $X \to C_p(X)$ is a $H\mathbf{Z}/p$-equivalence and $C_p(X)$ is $\mathbf{Z}/p$-complete [6, I.5.2]. In that case, the $H\mathbf{Z}/p$-equivalence $X \to C_p(X)$ is terminal among $H\mathbf{Z}/p$-equivalences from X to some space and initial among maps from X to $\mathbf{Z}/p$-complete spaces [6, VII.2.1]:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, "\simeq"] \\ C_p(X) \arrow[r] \arrow[ur, dotted] & C_p(Y) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \arrow[dl, dotted] \\ C_p(X) \arrow[r, "\simeq"] & C_p(Y) \end{tikzcd}$$

Proposition 11.8. *Let G be a finite group.*

(1) $\pi_1(C_p(BG)) = O^pG\backslash G$
(2) $C_p(BO^p(G))$ *is the universal covering space of* $C_p(BG)$
(3) $C_p(BG)$ *is aspherical* $\iff$ O^pG *contains no elements of order* p

Proof. The normal subgroup O^pG, generated by all p'-elements is $\mathbf{Z}/p$-perfect, $H_1(O^pG;\mathbf{Z}/p) = 0$, so that $C_p(BO^pG)$ is simply connected [6, VII.3.2]. The fibration $B(O^pG) \to BG \to B(G/O^pG)$ induces a fibration

$$C_p(BO^pG) \to C_p(BG) \to B(G/O^pG)$$

of p-completed spaces because G/O^pG is a finite p-group. Thus $\pi_1 C_p(BG) = G/O^pG$ and $C_p(BO^pG)$ is the universal covering space of $C_p(BG)$. Therefore

$$C_p(BG) \text{ is a } K(\pi,1) \iff C_pB(O^pG) \simeq * \iff \overline{H}_*(O^pG;\mathbf{Z}/p) = 0 \iff O^pG \text{ has order prime to } p$$

For the last step we use that if the finite group K contains an element of order p then $H^*(K;\mathbf{Z}/p)$ will contain a polynomial subring by Quillen's theorem that the Krull dimension of $H^*(K;\mathbf{F}_p)$ equals the p-rank of K [39, 19]. $\square$

Remark 11.9. Let X and Y be topological spaces. Let us say that X_1 and X_2 have the same p-local structure, $X_1 \sim_p X_2$, if there there exist a topological space Y and $H\mathbf{Z}/p$-equivalences $X_1 \to Y \leftarrow X_2$. This is an equivalence relation. To prove transitivity, consider the diagram

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] & Y_1 \arrow[dr] & X_2 \arrow[l, "g_2"'] \arrow[r, "f_2"] & Y_2 \arrow[dl] & X_3 \arrow[l, "g_3"'] \\ & & C(f_2,g_2) & & \end{tikzcd}$$

where $C(f_2,g_2) = Y_1 \cup_{g_2} \cup (I \times X_2) \cup_{f_2}$ is the double mapping cylinder. The homology of the double mapping cylinder is easily computed from the Mayer–Vietoris sequence and we see that the maps $Y_1 \to C(f_2,g_2) \leftarrow Y_2$ are $H\mathbf{Z}/p$-equivalences when f_2 and g_2 are $H\mathbf{Z}/p$-equivalences. It is immediate that $X_1 \sim_p X_2$ if and only if $C_pX_1 \sim C_pX_2$ when X_1 and X_2 are $H\mathbf{Z}/p$-good spaces (eg X_1 and X_2 are connected and have finite fundamental groups). In particular, when G_1 and G_2 are finite groups, $C_p(BG_1) \sim C_p(BG_2)$ if and only if there are $H\mathbf{Z}/p$-equivalences $BG_1 \to B \leftarrow BG_2$.

12. Friedlander description of finite groups of Lie type

Friedlander [20] describes the p-completed classifying spaces of finite groups of Lie type as homotopy fixed point spaces for homotopy equivalences of p-completed classifying spaces of Lie groups. I shall now explain what these words mean.

Let X be a topological space and $\alpha\colon X \to X$ a map of X into itself. The *fixed-point space* of α is the space $X^{\langle\alpha\rangle} = \{x \in X \mid x^\alpha = x\}$ and the *homotopy fixed-point space* of α is the space $X^{h\langle\alpha\rangle} = \{(x,u) \in X \times \mathrm{map}(I,X) \mid 0^u = x, 1^u = x^\alpha\}$ defined by the pull-back diagrams

$$\begin{tikzcd} X^{\langle\alpha\rangle} \arrow[r] \arrow[d] & X \arrow[d, "(\mathrm{id},\mathrm{id})"] \\ X \arrow[r, "(\mathrm{id},\alpha)"] & X \times X \end{tikzcd} \qquad \begin{tikzcd} X^{h\langle\alpha\rangle} \arrow[r] \arrow[d] & \mathrm{map}(I,X) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ X \arrow[r, "(\mathrm{id},\alpha)"] & X \times X \end{tikzcd}$$

of topological spaces.

Let $K(\mathbf{Z}_p,2)^n = C_pK(\mathbf{Z},2)^n$ be the Eilenberg–MacLane space with second homotopy group isomorphic to the group $\mathbf{Z}_p^{\oplus n}$ where $\mathbf{Z}_p$ is the ring of p-adic integers. For each p-adic unit u, let ψ^u be the homotopy self-equivalence of $K(\mathbf{Z}_p,2)^n$ that is multiplication by u on the second homotopy group.

The space $K(\mathbf{Z}_p,2)^n$ is the p-completed classifying space $C_pBT_n(\mathbf{C})$ of the maximal torus $T_n(\mathbf{C}) \to \mathrm{SL}_{n+1}(\mathbf{C})$ of the special linear group. According to Jackowski–McClure–Oliver [28] the map ψ^u on the maximal torus extends to a unique self-homotopy equivalence ψ^u

$$\begin{tikzcd} C_pBT_n(\mathbf{C}) \arrow[r, "\psi^u"] \arrow[d] & C_pBT_n(\mathbf{C}) \arrow[d] \\ C_pB\,\mathrm{SL}_{n+1}(\mathbf{C}) \arrow[r, "\psi^u"] & C_pB\,\mathrm{SL}_{n+1}(\mathbf{C}) \end{tikzcd}$$

of the p-completed classifying space of $\mathrm{SL}_{n+1}(\mathbf{C})$. Moreover, these maps account for the entire group of self-homotopy equivalences,

$$\mathrm{Out}(C_pB\,\mathrm{SL}_{n+1}(\mathbf{C})) \cong \mathbf{Z}_p^\times$$

for $n>1$. (When $n=1$, the maps ψ^u and ψ^{-u} are homotopic.)

Let q be a prime power not divisible by p so that q is a unit in the ring $\mathbf{Z}_p$ of p-adic integers and $\psi^{\pm q}$ is a self-homotopy equivalence of $C_pB\,\mathrm{SL}_{n+1}(\mathbf{C})$. Friedlander's theorem, for this special case,

$$C_pB\,\mathrm{SL}^{\pm}_{n+1}(\mathbf{F}_q) \simeq C_pB\,\mathrm{SL}_{n+1}(\mathbf{C})^{h\langle\psi^{\pm q}\rangle}$$

describes the p-completed classifying space of $\mathrm{SL}^{\pm}_{n+1}(\mathbf{F}_q)$ as the homotopy fixed point space for the homotopy self-equivalence $\psi^{\pm q}$ of $C_pB\,\mathrm{SL}_{n+1}(\mathbf{C})$. We now see that it would be good to be able to compare homotopy fixed point spaces.

Let B be a topological space. Let $U_k \le \mathrm{Out}(B)$ be the subgroup of self-homotopy equivalences inducing the identity on $H^*(B;\mathbf{Z}/p^k)$. Define the p-adic topology on the group $\mathrm{Out}(B)$ to be the topology whose basis at the identity consists of the U_k. (If $H^i(B;\mathbf{Z}/p)$ is finite for all $i\ge 0$, $H^i(B;\mathbf{Z}_p)$ is the inverse limit of the groups $H^i(B;\mathbf{Z}/p^k)$ (by a change of functors spectral sequence), and $\mathrm{Out}(B)$ is a Hausdorff topological group if and only if $\mathrm{Out}(B)$ is detected on $H^*(B;\mathbf{Z}_p)$. Here, $\mathbf{Z}_p$ stands for the ring of p-adic integers.)

Theorem 12.1. [11, Theorem 2.4] *Let B be a connected p-complete space such that $H^i(B;\mathbf{Z}/p)$ is finite for all $i\ge 0$, $H^*(B;\mathbf{Z}/p)$ is a finitely generated $\mathbf{Z}/p$-algebra, and* $\mathrm{Out}(B)$ *is detected on $H^*(B;\mathbf{Z}_p)$. Let $\alpha,\beta\in\mathrm{Out}(B)$ be two self-homotopy equivalences of B. Then*

$$B^{h\langle\alpha\rangle} \simeq B^{h\langle\beta\rangle}$$

if $\overline{\langle\alpha\rangle} = \overline{\langle\beta\rangle}$ in the topological group $\mathrm{Out}(B)$.

As the conditions of this theorem are satisfied for $B = C_pB\,\mathrm{SL}_{n+1}(\mathbf{C})$, we get this immediate consequence:

Corollary 12.2. *Let q_1 and q_2 be two prime powers not divisible by p. If $\overline{\langle q_1\rangle} = \overline{\langle \pm q_2\rangle}$ in the topological group $\mathbf{Z}_p^\times$, then $C_pB\,\mathrm{SL}^{+}_{n+1}(q_1) \simeq C_pB\,\mathrm{SL}^{\pm}_{n+1}(q_2)$.*

It is not hard to see that

$$\overline{\langle q_1\rangle} = \overline{\langle q_2\rangle} \iff \begin{cases} q_1 \equiv q_2 \bmod p \text{ and } \nu_p(q_1^e-1)=\nu_p(q_2^e-1) & p \text{ odd} \\ q_1 \equiv q_2 \bmod 4 \text{ and } \nu_p(q_1^2-1)=\nu_p(q_2^2-1) & p=2 \end{cases}$$

where e is the multiplicative order of q_1 modulo p.

According to Corollary 11.5, these topological statements about p-completed classifying spaces translate to the purely group theoretical statements

$$\overline{\langle q_1\rangle} = \overline{\langle \pm q_2\rangle} \implies \mathcal{F}_{\mathrm{SL}^{+}_{n+1}(q_1)} \cong \mathcal{F}_{\mathrm{SL}^{\pm}_{n+1}(q_2)}$$

about Frobenius categories of finite groups of Lie type. (This is only one instance of a more general statement [11, Theorem A] comparing Frobenius categories for finite groups of Lie type.)

In [10] we replace the p-completed classifying spaces of Lie groups by the more general p-compact groups and show that the homotopy fixed point spaces define Frobenius categories that in many cases are exotic in the sense that they are not Frobenius categories of any finite group.

Theorem 12.3. [10] *Let BX be a connected p-compact group, $u\in\mathbf{Z}_p^\times$ a p-adic unit, and ψ^u the corresponding unstable Adams operation on BX. Then $BX^{h\langle\psi^u\rangle}$ is the classifying space of a Frobenius category.*

At odd primes, this provides many examples of exotic Frobenius categories. At the prime $p = 2$ with $BX = B\mathrm{DI}(4)$ the Dwyer–Wilkerson space [16], this construction provides the only known exotic Frobenius system [29].

13. Open problems

- Is it possible to classify Frobenius categories? Would that be helpful eg in the classification of the finite simple groups?
- Does every abstract Frobenius category $\mathcal{F}$ admit a (unique) functor like (11.6) (and hence a classifying space)?
- Every Frobenius category of a finite group is the Frobenius category of a block – is the converse true?
- Describe the exotic Frobenius categories, the Frobenius categories not associated to any finite group.
- Is it true that there is essentially just one exotic Frobenius category at $p = 2$?

14. Student activities

Exercises:

(1) Determine $\mathcal{F}_G$ for your favorite groups G.
(2) Find examples of nonisomorphic groups with isomorphic Frobenius categories.
(3) Show that the Frobenius categories associated to Σ_m are the same when $np \le m < (n+1)p$. (Here, Σ_m is the symmetric group on m letters.)
(4) Find all Frobenius categories on $C_2 \times C_2$ and $C_2 \times C_2 \times C_2$. (This exercise was suggested by Martin.)
(5) Complete the proof of Proposition 9.6.
(6) Do Example 8.17 for $p = 2$.
(7) Show that $U \le P$ is $\mathcal{F}$-stable if and only if U is the kernel of some $(\mathcal{F}, \mathcal{F}')$-functorial group homomorphism $\alpha\colon P \to P'$.
(8) Identify the group $O_p(\mathcal{F}_G)$ (Definition 9.4) for a finite group G.
(9) Imagine that you were to make an exam for this course. What questions would you put in the exam?

Short (30–60 minutes) presentations:

(1) What is an affine group group scheme? [42, 18]
(2) Classification of root systems [4]
(3) Homotopy colimits [17, 25]
(4) The Bousfield–Kan p-completion functor [6]
(5) Sharp homology decompositions of classifying spaces of finite groups
(6) Euler characteristic of Frobenius categories (Martin Wedel Jacobsen)

Links:

- Algebraic groups
- Classical groups
- Normal p-complement

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Department of Mathematical Sciences, Universitetsparken 5, DK–2100 København
E-mail address: moller@math.ku.dk
URL: htpp://www.math.ku.dk/~moller