SEMINAR KU

SETUP prime $p \mid |G|$

$\mathcal{S}_G$ = poset of p-subgroups of G, $\mathcal{S}_G^*$ = poset of nontrivial p-subgroups of G

Example $p=2$

n	4	5	6	7	8	9	10
$\chi(A_n^*)$	1	5	-15	-175	65	5121	15105

How can we compute the Euler characteristic of a finite poset $\mathcal{S}$?

Note If $\mathcal{I} \subset \mathcal{S}$ is upward or downward closed then the Möb function for $\mathcal{S}$ restricts to a Möb function for $\mathcal{I}$ (ideal)

$$\zeta = \left(|\mathcal{S}(a,b)|\right)_{a,b\in\mathcal{S}} = \begin{pmatrix} 1 & & ? \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$\mu = \zeta^{-1} = \begin{pmatrix} 1 & & ? \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$\mu\zeta = E$: $\delta(a,c) = \sum \mu(a,b)\zeta(b,c) = \sum\{\mu(a,b) \mid b \le c\}$

$\zeta\mu = E$: $\delta(a,c) = \sum_{a \le b} \zeta(a,b)\mu(b,c) = \sum\{\mu(b,c) \mid a \le b\}$

Properties of μ

$\mu(a,a) = 1$

$\mu(a,b) = 0$ unless $a \le b$

$\sum\{\mu(a,b) \mid b \le c\}$

$\sum\{\mu(b,c) \mid a \le b\}$

Thm 1 $\chi(\mathcal{S}) = \sum_{a,b} \mu(a,b)$

Rmk 1.1 $\mathcal{S}_1 \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{S}_2$ adjoints then $\chi(\mathcal{S}_1) = \chi(\mathcal{S}_2)$ — even homotopy equivalence.

Rmk 1.2 Suppose $\mathcal{S}$ has a largest element 1 (or a smallest element 0). Then

$$\chi(\mathcal{S}) = \sum_{a,b} \mu(a,b) = \sum_a \sum_b \mu(a,b) = \sum_a \sum_{b \le 1} \mu(a,b) = \sum_a \delta(a,1) = 1.$$

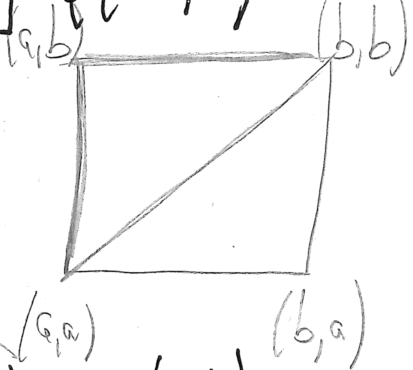
Moreover $\mathcal{S}_1 \rightleftarrows \mathcal{S}_2$ adjoints $\Rightarrow$ $\chi(\mathcal{S}_1) = \chi(\mathcal{S}_2)$

Euler character of intervals and rays

$[a,b]$ (a,b)

$[a,\infty)$ (a,∞)

$(-\infty,b] = (-\infty,b)$

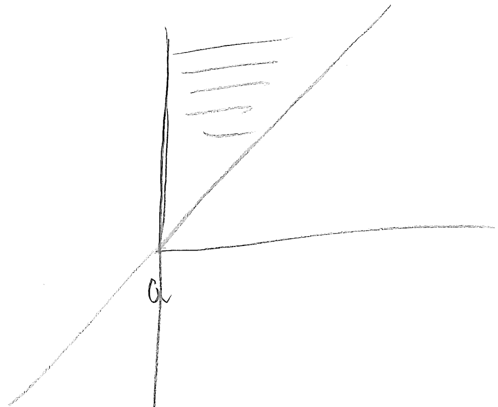


$a<b$

$$1 = \chi([a,b]) = \sum_{a\le x\le y\le b} \mu(x,y)$$

$$= \sum_{a<x\le y<b} \mu(x,y) + \overbrace{\sum_{a\le y\le b} \mu(a,y)}^{\delta(a,b)} + \overbrace{\sum_{a\le x\le b} \mu(x,b)}^{\delta(a,b)} - \mu(a,b)$$

$$= \chi(a,b) - \mu(a,b)$$



$$1 = \chi([a,\infty)) = \sum_{a\le x\le y} \mu(x,y)$$

$$= \sum_{a\le y} \mu(a,y) + \sum_{a<x\le y} \mu(x,y)$$

$$= \sum_{y} \mu(a,y) + \tilde{\chi}(a/\!/\mathcal{P})$$

$\mathcal{P}/\!/b = \{a \mid a<b\}$

$a/\!/\mathcal{P} = \{b \mid a<b\}$

CONCLUSION:

Not really needed

$$\mu(a,b) = \tilde{\chi}(a,b)$$

Mb function are Euler char of an interval

$$-\tilde{\chi}(a/\!/\mathcal{P}) = \sum_b \mu(a,b)$$

Euler char of rays are sums of Mb function

$$-\tilde{\chi}(\mathcal{P}/\!/b) = \sum_a \mu(a,b)$$

COR: $\displaystyle\sum_a -\tilde{\chi}(a/\!/\mathcal{P}) = \chi(\mathcal{P}) = \sum_b -\tilde{\chi}(\mathcal{P}/\!/b)$

Euler characteristics detect p-radical and elementary abelian subgroups

SETUP prime $p \mid |G|$

$\mathcal{S}_G$ = poset of p-subgroups of G

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Def $H \leq G$ is p-radical if $H = O_p N_G(H)$ $\left(\Leftrightarrow O_p\left(N_G(H)/H\right) = 1\right)$

$K \leq G$ is p-elementary abelian if $K \cong C_p \times \cdots \times C_p$.

For any $H \leq G$:

$$H/\!/\mathcal{S}_G^* = \{K \mid H < K\} = (H, G]$$

$$\boxed{\begin{array}{c} H/\!/\mathcal{S}_G^* \rightleftarrows \mathcal{S}^*_{N_G(H)/H} \\ K \longrightarrow N_K(H)/H \\ \bar{K} \leftarrow \bar{K}/H \end{array}}$$

For any $K \leq G$

$$\mathcal{S}_G^*/\!/K = (1, K)$$

Consider functions $\quad Ob(\mathcal{S}_G^*) \longrightarrow \mathbb{Z}$

$$H \to \tilde{\chi}(H/\!/\mathcal{S}_G^*) = \tilde{\chi}(\mathcal{S}^*_{N_G(H)/H})$$

$\mathcal{S}_G^*/\!/H$ is not a functor

$$K \to \tilde{\chi}(\mathcal{S}_G^*/\!/K) = \tilde{\chi}(1, K)$$

FACT

$\tilde{\chi}(H/\!/\mathcal{S}_G^*) \neq 0 \Rightarrow H$ is p-radical

$\tilde{\chi}(\mathcal{S}_G^*/\!/K) \neq 0 \Leftrightarrow K$ is p-elementary abelian

STRONG QUILLEN CONJECTURE

$\tilde{\chi}(H/\!/\mathcal{S}_G^*) \neq 0 \Leftrightarrow H$ is p-radical

The poset $\mathcal{S}_G^*$ knows about p-radical and elementary abelian subgroups

Defining property

$$\sum_{H \le K} -\tilde{\chi}\left(\mathcal{S}_G^{*} /\!/ H\right) = 1$$

for all fixed K

$$\left(-\tilde{\chi}(\mathcal{S}_G^{*} /\!/ H)\right) \begin{array}{c} \begin{pmatrix} \quad & \vert & \quad \end{pmatrix} \\ K \end{array} = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}$$

(weightings)

$$\sum_{H \le K} -\tilde{\chi}\left(K /\!/ \mathcal{S}_G^{*}\right) = 1$$

for all fixed H

$$\begin{array}{cc} H & \begin{pmatrix} \xi_G(H,K) \end{pmatrix} \end{array} \begin{pmatrix} \vdots \\ -\tilde{\chi}(K /\!/ \mathcal{S}_G^{*}) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

(coweightings)

$$\chi(\mathcal{S}_G^*) = \sum \mu(H,K)$$

$$= \sum_{H \text{ p-radical}} -\tilde{\chi}(\mathcal{S}^*_{N_G(H)/H})$$

$$= \sum_{K \text{ elementary}} -\tilde{\chi}(1,K)$$

<u>Lemma</u> $\mu(H,K)=0$ unless $H \trianglelefteq K$ and ~~$K/H \cong p^n$~~ $H\backslash K = p^n$ ~~and~~ - and then $\mu(H,K)=(-1)^n p^{\binom{n}{2}}$

Why is ~~χ~~ $|G|_p \mid \tilde{\chi}(\mathcal{S}_G^*)$?

~~We see that~~

<u>COR</u> $\chi(\mathcal{S}_G^{*+\text{rad}}) = \chi(\mathcal{S}_G^*) = \chi(\mathcal{S}_G^{*+\text{eab}})$

$$\sum_{\substack{H \text{ p-radical} \\ 1 \lneq H \leq G}} -\tilde{\chi}(\mathcal{S}^*_{N_G(H)/H}) = \sum_{\substack{K \cong p^n \\ K \leq G}} (-1)^n p^{\binom{n}{2}}$$

Are $\mathcal{S}_G^{*+\text{rad}} \hookrightarrow \mathcal{S}_G^* \hookleftarrow \mathcal{S}_G^{*+\text{eab}}$ htpy equivalences?

CRITERIA for HOMOTOPY EQUIVALENCE*

Now we want to prove that $\int_G^{*} \hookrightarrow \int_G$ and $\int_G^{*+\mathrm{triv}} \hookrightarrow \int_G^{*}$ are homotopy equivalences.

We formulate this for categories, not just posets.

<u>Def</u>: A functor $\mathcal{C} \to \mathcal{D}$ is a htpy equivalence if $B\mathcal{C} \to B\mathcal{D}$ is a homotopy equivalence.

A category $\mathcal{C}$ is contractible if $1 \to \mathcal{C}$ is a homotopy equiv

What criteria do we have?

<u>Example</u> If $\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$ is an adjunction then L and R are homotopy equivalences.

isomorphism $\Rightarrow$ equivalence $\Rightarrow$ adjunction $\Rightarrow$ homotopy equivalence

<u>Slices and coslices</u>

Suppose $\mathcal{A} \hookrightarrow \mathcal{C}$ is an inclusion of a full subcat $\mathcal{A}$ into $\mathcal{C}$. Let x, y be objects of $\mathcal{C}$: $x, y \in Ob(\mathcal{C})$

<u>Slice category $\mathcal{A}/y$</u> obj: $Ob(\mathcal{A}/y) = \bigcup_{a \in Ob(\mathcal{A})} \mathcal{C}(a, y)$

<u>Also</u> $\mathcal{A}/\!/y$
Double slice

<u>morphisms</u>

$$\begin{tikzcd} a_1 \arrow[rr, "\mathcal{A}"] \arrow[dr, "\varphi_1"'] & & a_2 \arrow[dl, "\varphi_2"] \\ & y & \end{tikzcd}$$

~~If $y_1 \to y_2$ then functor $\mathcal{A}/y_1 \to \mathcal{A}/y_2$ - so~~

~~$\mathcal{A}/\circ,\ \mathcal{A}/\!/\circ : \mathcal{C} \to CAT$~~

Let $supp(\mathcal{A}/\!/\circ) = \{ y \in Ob(\mathcal{C}) \mid \mathcal{A}/\!/y$ is noncontractible $\}$

Coslice category: $X/\mathcal{A}$ obj $\mathrm{Ob}(X/\mathcal{A}) = \bigcup_{a\in\mathrm{Ob}(\mathcal{A})} \mathcal{C}(X,a)$

$X/\!/\mathcal{A}$

morphisms

$$\begin{tikzcd} & X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ a_1 \arrow[rr, dashed] & & a_2 \end{tikzcd}$$

$\cdot/\!/\mathcal{A}$, $\cdot/\mathcal{A}$

Coslice is a cofunctor $\mathcal{C}^{op} \longrightarrow CAT$

$\mathrm{supp}(\cdot/\!/\mathcal{A}) = \{X \in \mathrm{Ob}(\mathcal{C}) \mid X/\mathcal{A} \text{ noncontractible}\}$

<u>Quillen's Thm A</u> $\mathcal{A} \hookrightarrow \mathcal{C}$ is a homotopy equivalence if $\mathrm{supp}(\cdot/\!/\mathcal{A}) = \emptyset$ or $\mathrm{supp}(\mathcal{A}/\!/\cdot) = \emptyset$ (i.e. if all slices or coslices are contractible)

Do not use supp - just says $c/\!/\mathcal{A} \simeq *$ or $\mathcal{A}/\!/c \simeq *$

<u>Bouc's thm for finite EI-categories</u> $\mathcal{A} \hookrightarrow \mathcal{C}$ is a homotopy equivalence if $\mathrm{supp}(\cdot/\!/\mathcal{C}) \subseteq \mathrm{Ob}(\mathcal{A})$ or $\mathrm{supp}(\mathcal{C}/\!/\cdot) \subseteq \mathrm{Ob}(\mathcal{A})$

$\mathcal{S}^*_G > H$

<u>Thm</u> $\mathcal{S}^{*+\mathrm{rab}}_G \hookrightarrow \mathcal{S}^*_G$, $\mathcal{S}^{*+\mathrm{rad}}_G \hookrightarrow \mathcal{S}^*_G$, are homotopy equivalences

<u>Proof</u> $\mathrm{supp}(\cdot/\!/\mathcal{S}^*_G) = \{H \le G \mid H/\!/\mathcal{S}^*_G \text{ noncontractible}\}$

$\subseteq \{H \le G \mid \tilde{\chi}(\mathcal{S}^*_{N_G(H)/H}) \neq 0\}$

$\subseteq \{H \le G \mid H \text{ is } p\text{-radical}\}$

$\mathrm{supp}(\mathcal{S}^*_G/\!/\cdot) = \{K \le G \mid \mathcal{S}^*_G/\!/K \text{ noncontractible}\}$

$\subseteq \{K \le G \mid \tilde{\chi}(1,K) \neq 0\}$

$\subseteq \{K \le G \mid K \text{ is elementary abelian}\}$

Bouc's thm for posets: If $\mathcal{P}_{a<} \not\simeq *$ only if a in $\mathcal{A}$, then $\mathcal{A} \hookrightarrow \mathcal{P}$ is a homotopy equivalence

Quillen's conjecture: $O_p G = 1 \Rightarrow$
$\begin{cases} \mathcal{S}^*_G \text{ noncontractible} \\ \mathcal{S}^{*+\mathrm{rab}}_G \text{ —"—} \\ \mathcal{S}^{*+\mathrm{rad}}_G \text{ —"—} \end{cases}$

SUBGROUP CATEGORIES $\quad p$ prime $p \mid |G|$

$\mathcal{S}_G$, $\qquad \mathcal{S}_G(H,K) \neq \emptyset \Leftrightarrow H \leq K$

$\mathcal{T}_G \qquad \mathcal{T}_G(H,K) = N_G(H,K)$

$\mathcal{F}_G \qquad \mathcal{F}_G(H,K) = C_G(H) \backslash N_G(H,K)$

$\mathcal{O}_G \qquad \mathcal{O}_G(H,K) = N_G(H,K)/K$

$\mathcal{L}_G$

$\tilde{\mathcal{F}}_G \qquad \tilde{\mathcal{F}}_G(H,K) = C_G(H) \backslash N_G(H,K)/K$

$$
\begin{tikzcd}
 & & \mathcal{L}_G \arrow[r] & \mathcal{F}_G \arrow[dr] & \\
\mathcal{S}_G \arrow[r] & \mathcal{T}_G \arrow[ur] \arrow[dr] & & & \tilde{\mathcal{F}}_G \\
 & & \mathcal{O}_G \arrow[urr] & &
\end{tikzcd}
$$

Also categories have Euler characteristics (they are rational numbers). Play the same game with computing Euler characteristics.

<u>Thm 1.2</u> from MG + JMM

<u>Corollary</u>

$$
\begin{tikzcd}
\mathcal{F}^{*} \arrow[r] & \tilde{\mathcal{F}}^{*} \\
\mathcal{F}^{*+eab} \arrow[u, "\simeq"] \arrow[r, Rightarrow] & \tilde{\mathcal{F}}^{*+eab} \arrow[u, "\simeq"']
\end{tikzcd}
$$

is a homotopy equivalence

An exampl of a htpy equivalence that is not an adjoint nor satisfies Quillen Thm A (?)

Cor 4.2 from MWJ-JTM:

$$1+(p-1)\sum_C |C| = p|G| \sum_{[H]} \frac{-\tilde{\chi}(\mathcal{S}^*_{N_G(H)/H})}{|O_G(H)|}$$

We have

$$|G|\sum_{[H]} \frac{-\tilde{\chi}(\mathcal{S}^*_{N_G(H)/H})}{|O_G(H)|} = |G|\sum_{[H]} \frac{-\tilde{\chi}(\quad)|H|}{|N_G(H)|}$$

$$= \cancel{|G|} \sum_H \frac{-\tilde{\chi}(\quad)|H|}{|N_G(H)|} \frac{1}{|\cancel{G}:N_G(H)|}$$

$$= \sum_H -|H|\,\tilde{\chi}(\mathcal{S}^*_{N_G(H)/H})$$

From $\mathcal{S}^*_G$:

$$\sum_H +\tilde{\chi}(\mathcal{S}^*_{N_G(H)/H}) = \sum_K \mu(1,K)$$

$$= \sum_{1<p^n\leq G} (-1)^n p^{\binom{n}{2}}$$

The formula of Cor 4.2.(3) reads

Correspondence between cyclic and poset diagrams

$$\boxed{1+(p-1)\sum_{1\leq C\leq G}|C| + p\sum_{1\leq H\leq G}|H|\,\tilde{\chi}(\mathcal{S}^*_{N_G(H)/H}) = 0}$$

but if $H=G$ $\mathcal{S}^*_{G/G} = \mathcal{S}^*_1 = \emptyset$

Corollary $\quad (1-p)\sum|C| \equiv 1 \mod p|G|_p$

The function $H \to \tilde{\chi}(\mathcal{S}^*_{N_G(H)/H})$ connects the radical, cyclic, and elementary abelian subgroups of G

radical = upward closed?

Burnside counting lemma $\Rightarrow$ double cosets

$$\cancel{\chi(\mathcal{O}^*_G) = \frac{1}{|G|}}$$

$$|G|\,\chi(\mathcal{O}^*_G) = \left|\{g \in G \mid |g| \mid |G|_p\}\right|$$

$$= \left|\bigcup \{S \mid S \in \mathrm{Syl}_p(G)\}\right|$$

$$(1-p)\sum_{1\lneq C\leq G}|C| \equiv 1 \ (|G|_p \cdot p)$$

$$p \cdot \#\mathrm{gen}(C) = |C|(p-1) \qquad |C| = p^n$$

$$\#\mathrm{gen} = p^n - p^{n-1}$$

$$\#\text{generators in } C = \varphi(|C|) = \frac{|C|}{p}(p-1) = |C| - \frac{|C|}{p}$$

$$\equiv -p \sum_{1\lneq C\leq G} \#\text{generators in } C + (1-p)$$

$$\cancel{\equiv -p\ \#\text{elms. of } p\text{-power order}}$$

$$= (1-p) - p \cdot \#\text{elms. of } p\text{-power} > 1 \text{ order}$$

$$= 1 - p(\#\text{ elms. of } p\text{-power order})$$

Shown

$$\boxed{\#\text{elms. of } p\text{-power order} \equiv 0 \ (|G|_p)}$$

$$\left|\bigcup_{S\in \mathrm{Syl}_p(G)} S\right| \equiv 0 \ (|G|_p)$$