G finite grp

p prime $\quad p \mid |G| \quad$ In this talk p is p.

$G_p = \bigcup \mathrm{Syl}_p(G)$

$n_p = p$ p.t

$\mathcal{S}_G$ = p-subgroup poset of G $\quad \mathcal{S}_G^{p} \quad \mathcal{S}_G^{p+*}$

$\mathcal{O}_G$ = orbit cat. of G $\quad \mathcal{O}_G^{p} \quad \mathcal{O}_G^{p+*}$

$\mathcal{O}_G(H,K) = \{g \in G \mid H^g \le K\}/K \quad \mathcal{O}_G(K) = \mathcal{O}_G(K,K) = N_G(K)/K$

$K \le G \quad \mathcal{S}^{p+*}_{\mathcal{O}_G(K)} = \mathcal{S}^{p+*}_{N_G(K)/K} \quad \tilde{\chi}\left(\mathcal{S}^{p+*}_{\mathcal{O}_G(K)}\right) \neq 0 \Rightarrow K = O_p(N_G(K))$

K is p-radical

Wanted to discuss

Frobenius thm 1907 $\quad |G|_p \mid |G_p|$

Brown 1975 thm $\quad |G|_p \mid \tilde{\chi}(\mathcal{S}_G^{p+*})$

Steinberg thm 1968 $\quad K$ finite grp of Lie type (simple) $\quad |K_p| = |K|_p^2$

Rmk $\quad \sum_{n \ge 0} |(\Sigma_n)_p| \frac{x^n}{n!} = \exp\left(x + \frac{x^p}{p} + \frac{x^{p^2}}{p^2} + \dots\right)$ (Stanley)

Combinatorial formula, will give a combinatorial argument for Steinberg thm

$|G_p|$ known for all finite simple — no! Cross char case of Lie grps! Sporadic groups: Use formula on next page

Σ_n is a Weyl grp. What about the other Weyl groups?

Relation 1 (Steinberg identity)

$$\sum_{[K]\in \mathcal{S}_G^{p+\mathrm{rad}}/G} \frac{-\tilde{\chi}\left(\mathcal{S}^{p+*}_{\mathcal{O}_G(K)}\right)}{\cancel{|\mathcal{O}_G(K)|}} \frac{|G|}{|\mathcal{O}_G(K)|} = \frac{|G_p|}{\cancel{|G|}}$$

Relation 2 (Witt identity)

$$\forall H\in \mathcal{S}_G^{p+\mathrm{rad}} : \sum_{[K]\in \mathcal{S}_G^{p+\mathrm{rad}}} \frac{-\tilde{\chi}\left(\mathcal{S}^{p+*}_{\mathcal{O}_G(K)}\right)}{\cancel{|\mathcal{O}_G(K)|}} \frac{|\mathcal{O}_G(H,K)|}{|\mathcal{O}_G(K,K)|} = 1$$

(What is the intuition?)

1) Where do they come from?

2) Alternative formulation

$$\mathrm{TOM}^{p+\mathrm{rad}}_G(H,K) = |\mathcal{O}_G(H,K)| = |(K\backslash G)^H| \qquad \begin{pmatrix} * & & 0 \\ & \ddots & \\ ? & & * \end{pmatrix}$$

$$\widetilde{\mathrm{TOM}}_G(H,K) = \frac{|\mathcal{O}_G(H,K)|}{|\mathcal{O}_G(K,K)|} = |H/K^G| \qquad \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ ? & & 1 \end{pmatrix}$$

Relation 1 $\quad \dfrac{|G_p|}{|G|} = \sum \mathrm{TOM}_G^{-1}(H,K)$

Relation 2 $\quad -\tilde{\chi}\left(\mathcal{S}^{p+*}_{\mathcal{O}_G(H)}\right) = \sum_K \widetilde{\mathrm{TOM}}_G^{-1}(H,K)$

Question What happens if we take the orbit cat of a fusion system?

Steinberg identity

Relation 1 shows that Frobenius $\Leftrightarrow$ Brown

Application 1

Maybe skip this

$$\frac{|G_p|}{\cancel{|G|}} = \sum_{K \in \mathcal{S}_G^{p\text{-rad}}} \frac{-\tilde{\chi}(\mathcal{S}^{p+*}_{O_G(K)})}{|O_G(K)|} |G|$$

$$= \underbrace{-\tilde{\chi}(\mathcal{S}^{p+*}_{O_G(K)})}_{K=1} + \sum_{K>1} \frac{-\tilde{\chi}(\mathcal{S}^{p+*}_{O_G(K)})}{|O_G(K)|_p} \frac{|G|}{|O_G(K)|_{p'}}$$

(If $K=1$ is radical, this is OK. If $K=1$ is not radical, $O_p(G)>1$ and this is still true as $\tilde{\chi}(\mathcal{S}_G^{p+*})=0$)

1) Assume $|G|_p \mid |G_p|$. Then $|G|_p$ divides LHS. By induction we can assume that $|O_G(K)|_p \mid \tilde{\chi}(\mathcal{S}^{p+*}_{O_G(K)})$. Then $|G|_p$ divides the sum on the RHS. Then $|G|_p$ divides $-\tilde{\chi}(\mathcal{S}^{p+*}_{O_G(K)})$.

2) Assume $|G|_p \mid -\tilde{\chi}(\mathcal{S}_G^{p+*})$ for all G. Then $|G|_p$ divides the RHS. So $|G|_p$ divides $|G_p|$.

Application 2

Steinbergs thm on $|K_p|$

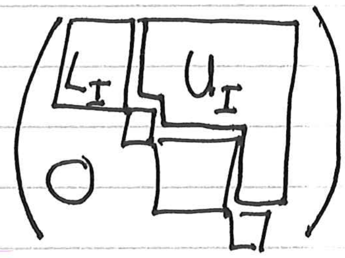
Σ root system simple $\Pi \subseteq \Sigma$ fundamental roots

$\overline{K}(\Sigma)$ simple $\overline{\mathbb{F}}_p$-algebraic group with root system Σ

$\sigma : \overline{K}(\Sigma) \to \overline{K}(\Sigma)$ Steinberg endo

$K = O^{p'} C_{\overline{K}(\Sigma)}(\sigma)$

$$\Sigma \to \tilde{\Sigma} \to \hat{\Sigma}$$
$$\Pi \to \tilde{\Pi} \to \hat{\Pi}$$



Parabolic subgroups and unipotent radicals

$\hat{\Pi} \supseteq I \supseteq J$: $P_J \supseteq P_I \supseteq U_I \supseteq U_J$ $P_I = L_I \ltimes U_I$ $U_I = O_p P_I$

$P_I = N_K(U_I)$

$L_I = C_K(U_I)$

$I = \hat{\Pi}$: $K = P_{\hat{\Pi}} = L_{\hat{\Pi}} \ltimes U_{\hat{\Pi}}$ $L_{\hat{\Pi}} = K$, $U_{\hat{\Pi}} = 1$ (maximal parabolic)

$I = \emptyset$: $B = P_{\emptyset} = L_{\emptyset} \ltimes U_{\emptyset}$ $L_{\emptyset}$ = max torus $U_{\emptyset} \in Syl_p(K)$ (minimal parabolic)

What do the relations say:

$$\text{①} \quad \cancel{|(P_I)_p|} = |G_p| = \sum_{[K]} -\tilde{\chi}(\mathcal{S}^{p+*}_{Q_G(K)}) \frac{|G||K|}{|N_G(K)|}$$

$$= \sum_{[K]} -\tilde{\chi}(\mathcal{S}^{p+*}_{Q_G(K)}) |G:N_G(K)||K|$$

Applied to $G = \{P_I\}_p$: (not to K)

$$|(P_I)_p| = \sum_{I \supseteq J} -\tilde{\chi}(\mathcal{S}^{p+*}_{L_J}) |P_I : P_J||U_J|$$

can we ign... Do we need to know $\tilde{\chi}(\mathcal{S}^{p+*}_{L_J})$

$$\text{②} \quad \sum_{I \supseteq J} -\tilde{\chi}(\mathcal{S}^{p+*}_{L_J}) \frac{|O_K(U_I, U_J)|}{|O_K(U_J, U_J)|} = 1$$

Lemma 1 (Solomon-Tits thm) $-\tilde{\chi}(\mathcal{S}^{p+*}_{L_J}) = (-1)^{|J|}|U_\emptyset : U_J|$

Lemma 2 $$\frac{|O_K(U_I, U_J)|}{|O_K(U_J, U_J)|} = \begin{cases} |P_I : P_J| & I \supseteq J \\ 0 & \text{otherwise} \end{cases}$$

Relation 2 says

$$\sum_{I \supseteq J} (-1)^J |U_\emptyset : U_J| |P_I : P_J| = 1$$

Equivalently : $$\sum (-1)^J |P_I : P_J| = |U_\emptyset : U_I|$$

Möbius inversion (?)

(the Möb function is ... +1, ... part)

We now work on Relation 1:

$$|(P_I)_p| = \sum_{I \supseteq J} (-1)^{|J|} |U_\emptyset : U_J| |P_I : P_J| |U_J|$$

$$= |U_\emptyset| \sum_{I \supseteq J} (-1)^{|J|} |P_I : P_J|$$

$$= |U_\emptyset| |U_\emptyset : U_I| = |U_\emptyset|^2 / |U_I| = |P_I|_p^2 / |O_p(P_I)|$$

$$= |K|_p^2 / |O_p(P_I)| = |K|_p^2 / |U_I|$$

This is a formula for any parabolic P_I in K.

Thm $|(P_I)_p| = |P_I|_p |L_I|_p$ Jean Michel: See book

Cross characteristic case

FIRST: Polynomial identities

$$\left| SL_n(\mathbb{F}_q)_p / SL_n(\mathbb{F}_q) \right| = \sum_{\lambda \vdash n} T(\lambda) \prod_{b \in \lambda} (q^b - 1)_p$$

$T(\lambda) = \#$ elts in Σ_n of cycle type λ

Interpretation of ②

Polynomial identities

What are these formulas for Chevalley homotopy groups (Jason: for spetses)

K · Chevalley group (untwisted) (But twisted also possible)

$I \subseteq \Pi$ The Poincaré polynomial of I is

$$W_I(q) = \prod \frac{q^d - 1}{q - 1}$$

where the product is over the degrees d of W. Then

$$|P_I : P_J| = \frac{W_I(q)}{W_J(q)} \qquad |U_\emptyset : U_J| = q^{|\Sigma_J^+|}$$

Fundamental relation ② is

$$\sum_{J \subseteq I} (-1)^{|J|} \frac{W_I(q)}{W_J(q)} q^{|\Sigma_J^+|} = 1 \qquad \sum_{J \subseteq I} (-1)^{|J|} \frac{W_I(q)}{W_J(q)} = q^{|\Sigma_I^+|}$$

Relation 2 is an extension to all groups of these identities

Observed (for $I = \Pi$) by Björner–Brenti 2005. (and they are q-analogs of an identity by Witt 1941). Thus fundamental relation ② is the generalization to arbitrary groups of Witt–Björner–Brenti identity)

BACK TO
CROSS-CHARACTERISTIC
CASE

The Leinster-Euler char of a ~~matrix~~ square matrix

ζ square matrix with $\mathbb{Q}$-coeff $\zeta = (\zeta(a,b))$

$$(\zeta(a,b))\begin{pmatrix}\vdots\\ k^b\\ \vdots\end{pmatrix} = \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} \qquad (\cdots k_a \cdots)(\zeta(a,b)) = (1 \cdots 1)$$

weighting $\qquad$ coweighting

<u>Defn</u> ζ has Euc if ζ has a we + cowe. The Euc of ζ is

$$\chi(\zeta) = \sum k^b = \sum k_a, \quad \tilde{\chi}(\zeta) = \chi(\zeta) - 1$$

The we/cowe of a cat. $\mathcal{C}$ is the we/cowe of

$$\zeta(\mathcal{C})(a,b) = |\mathcal{C}(a,b)|$$

$$\tilde{\chi}(\mathcal{C}) = \tilde{\chi}(\zeta(\mathcal{C}))$$

The we for $\mathcal{S}_G^{p+*}$ is $k^K_{\mathcal{S}} = -\tilde{\chi}(\mathcal{S}^{p+*}_{O_G(K)})$.

Quillen's thm: $k^K_{\mathcal{S}} \neq 0 \Rightarrow O_p O_G(K) = 1 \Leftrightarrow K$ is p-radical

This indicates: $\mathcal{S}_G^{p+*+\mathrm{rad}} \hookrightarrow \mathcal{S}_G^{p+*}$ is a h.e. THIS IS TRUE

The we for $\mathcal{O}_G^{p}$ is $k^K_{\mathcal{O}} = \frac{|K|}{|G|} k^K_{\mathcal{S}}$ also live only on p-radical subgroups

and $\mathcal{O}_G^{p+\mathrm{rad}} \hookrightarrow \mathcal{O}_G^{p}$ is a homotopy equiv.

Thm The we for $\mathcal{O}_G^p$ is
(JNM + Marchin Vdal)

$$k_{\mathcal{O}}^{K} = \frac{-\tilde{\chi}\left(\mathcal{S}_{\mathcal{O}_G(K)}^{p+*}\right)}{|G:K|}$$

and the Ehe ~~of~~ is

$$\chi(\mathcal{O}_G^p) = \frac{|G_p|}{|G|}$$

The S-matrix of $\mathcal{O}_G$ in the TOM_G become

$$|\mathcal{O}_G(H,K)| = |N_G(H,K)/K| = |(K\backslash G)^H| = TOM_G(H,K)$$