Equivariant Euler characteristic

Jesper M. Møller[1]

[1]Københavns Universitet

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Euler characteristics of centralizer subcategories

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Local Euler characteristics at p of an A-group G

Let G be a finite A-group and p a prime (such that $p \mid |G|$).

- $\mathcal{S}_G^p$ ($\mathcal{S}_G^{p+*}$) is the poset of (nontrivial) p-subgroup of G
- $C_{\mathcal{S}_G^p}(A)$ is the poset of A-p-subgroups $H \leq G$
- $N_G(H)$, $H \in C_{\mathcal{S}_G^p}(A)$, is a p-local A-subgroup
- $C_{\mathcal{S}^{p+*}_{N_G(H)/H}}(A)$, poset of nontrivial A-p-subgroups of $N_G(H)/H$
- $\widetilde{\chi}(C_{\mathcal{S}^{p+*}_{N_G(H)/H}}(A))$ is a p-local Euler characteristic of G
- $C_G(A)$ is the subgroup of G fixed by A
- $|G_p|$ the number of p-singular elements in G
- $|G|_p$ the p-part of the group order

$$G \circlearrowleft A \qquad H \leq G, H^A = H \qquad N_G(H)/H \circlearrowleft A$$

The p-local Euler characteristics are globally constrained

Global constraints on local Euler characteristics

Theorem:

1. There is an inclusion-exclusion principle for the number $|C_G(A)_p|$ of p-singular A-centralized elements of G:

$$\sum_{H\in C_{\mathcal{S}_G^p}(A)} -\widetilde{\chi}(C_{\mathcal{S}^{p+*}_{N_G(H)/H}}(A))|C_H(A)| = |C_G(A)_p|$$

2. For any A-normalized p-subgroup K of G

$$\sum_{K\leq H\in C_{\mathcal{S}_G^p}(A)} -\widetilde{\chi}(C_{\mathcal{S}^{p+*}_{N_G(H)/H}}(A)) = 1$$

3. $|C_G(A)|_p \mid \widetilde{\chi}(C_{\mathcal{S}_G^{p+*}}(A))$

Global constraints for trivial action

When A acts trivially on G the global constraints are

1. $\sum_{H \in \mathcal{S}_G^p} -\widetilde{\chi}(\mathcal{S}_{N_G(H)/H}^{p+*})|H| = |G_p|$ (inclusion-exclusion)
2. $\sum_{K \leq H \in \mathcal{S}_G^p} -\widetilde{\chi}(\mathcal{S}_{N_G(H)/H}^{p+*}) = 1$
3. $|G|_p \mid \widetilde{\chi}(\mathcal{S}_G^{p+*})$ (Brown's theorem)

Corollary: Brown's theorem of 1975 ($|G|_p \mid \widetilde{\chi}(\mathcal{S}_G^{p+*})$) and Frobenius' theorem of 1907 ($|G|_p \mid |G_p|$) are equivalent.

Corollary: $|\Sigma(q)_p| = |\Sigma(q)|_p^2$ for an untwisted finite group of Lie type $\Sigma(q)$ in defining characteristic p, $\Sigma = A, B, \dots, G$.

Reality check: $G = \mathrm{GL}_3(\mathbf{F}_2) = A_2(\mathbf{F}_2)$, $p = 2$

- $|G| = 168 = 8 \cdot 3 \cdot 7$, $|G|_2 = 8$, $|G_2| = 64 = |G|_2^2$
- 6 conjugacy classes of 2-subgroups
- $4 = 2^{|\Pi(A_2)|}$ conjugacy classes of 2-radical ($H = O_2 N_G(H)$) 2-subgroups (Borel–Tits)

1. $$\sum_{[H]\in[\mathcal{S}_G^p]} -\widetilde{\chi}(\mathcal{S}^{2+*}_{N_G(H)/H})|G : N_G(H)||H| = 64$$
2. $$\sum_{[H]\in[\mathcal{S}_G^p]} -\widetilde{\chi}(\mathcal{S}^{2+*}_{N_G(H)/H})||G : N_G(H)| = 1$$
3. $8 = |G|_2 \mid \widetilde{\chi}(\mathcal{S}_G^{2+*}) = -8$

$\lvert H\rvert$	1	2	4	4	4	8	
$\lvert G : N_G(H)\rvert$	1	21	7	7	21	21	
$-\widetilde{\chi}(\mathcal{S}^{2+*}_{N_G(H)/H})$	8	0	-2	-2	0	1	
$-\widetilde{\chi}(\mathcal{S}^{2+*}_{N_G(H)/H})\lvert G : N_G(H)\rvert\lvert H\rvert$	8	0	-56	-56	0	168	64
$-\widetilde{\chi}(\mathcal{S}^{2+*}_{N_G(H)/H})\lvert G : N_G(H)\rvert$	8	0	-14	-14	0	21	1

Motivation for studying Euler characteristics of centralizer subcategories

Why are Euler characteristics of centralizer subcategories relevant?

Because of their connection to equivariant Euler characteristics!

Many familiar posets are equivariant posets and their equivariant Euler characteristics carry interesting information. Here are some examples:

Equivariant Euler characteristics of finite A-categories

finite poset (finite category) $\longrightarrow \mathcal{C} \circlearrowleft A \longleftarrow$ group acting on $\mathcal{C}$

$$\chi(\mathcal{C}), \qquad \widetilde{\chi}(\mathcal{C}) = \chi(\mathcal{C}) - 1$$

$$\chi_r(\mathcal{C}, A) = \frac{1}{|A|} \sum_{X \in C_r(A)} \chi(C_{\mathcal{C}}(X)) \in \mathbf{Q}, \qquad r = 1, 2, \dots$$

$$\widetilde{\chi}_r(\mathcal{C}, A) = \frac{1}{|A|} \sum_{X \in C_r(A)} \widetilde{\chi}(C_{\mathcal{C}}(X)) \in \mathbf{Q}, \qquad r = 1, 2, \dots$$

$C_r(A)$ the set of commuting r-tuples $X = (x_1, \dots, x_r)$ of A-elements

$C_{\mathcal{C}}(X)$ the subcategory of $\mathcal{C}$ fixed (centralized) by all autofunctors of the r-tuple $X = (x_1, \dots, x_r)$

$$\chi_r(\mathcal{C}, A) = \sum_{[x] \in [A]} \chi_{r-1}(C_{\mathcal{C}}(x), C_A(x)) \qquad \text{(recursion)}$$

Equivariant posets

Many familiar posets are A-posets

Poset	A
Brown poset $\mathcal{S}_G^{p+*}$	G
Partition poset Π_n	Σ_n
Boolean poset B_n	Σ_n
Subspace poset $L_n(\mathbf{F}_q)$	$\mathrm{GL}_n(\mathbf{F}_q)$

and have equivariant (reduced) Euler characteristics $\chi_r(\mathcal{C}, A)$ for $r = 1, 2, \ldots$.

$$\chi_1(\mathcal{C}, A) = \widetilde{\chi}_1(\mathcal{C}, A) + 1$$
$$\chi_2(\mathcal{C}, A) = \widetilde{\chi}_2(\mathcal{C}, A) + k(A)$$
$$\chi_3(\mathcal{C}, A) = \widetilde{\chi}_3(\mathcal{C}, A) + \sum_{[x]\in[A]} k(C_A(x))$$

Equivariant Euler characteristics of Brown posets

Nontrivial p-subgroups of G ordered by inclusion $\longrightarrow \mathcal{S}_G^{p+*} \circlearrowleft G \longleftarrow$ conjugation action

G finite group

p prime number dividing the order of G

$z_p(G)$ number of irreducible **C**-reps of p-defect 0

0. $\widetilde{\chi}(\mathcal{S}_G^{p+*}) = 0 \iff \exists P \in \mathcal{S}_G^{p+*} \colon P \trianglelefteq G$ (Quillen Conjecture)
1. $\widetilde{\chi}_1(\mathcal{S}_G^{p+*}, G) = 0$ (Webb Theorem)
2. $\widetilde{\chi}_2(\mathcal{S}_G^{p+*}, G) = -z_p(G)$ (Alperin Weight Conjecture)
3. $\widetilde{\chi}_3(\mathcal{S}_G^{p+*}, G) =?$

Class equation interpretation of Webb's Theorem (χ_1)

$$\sum_{[x]\in[G]} |G : C_G(x)| = \sum_{[x]\in[G]} |[x]| = |G|$$

$$\sum_{[x]\in[G]} \chi(C_{\mathcal{S}_G^{p+*}}(x))|G : C_G(x)| = |G|$$

$$\sum_{[x]\in[G]} \widetilde{\chi}(C_{\mathcal{S}_G^{p+*}}(x))|G : C_G(x)| = 0$$

$$p \mid |x| \implies \widetilde{\chi}(C_{\mathcal{S}_G^{p+*}}(x)) = 0$$

For the simple group $G = \mathrm{GL}_3(\mathbf{F}_2)$ of order 168:

$\lvert x\rvert$	1	2	3	4	7	7	$\cdot$
$\lvert G : C_G(x)\rvert$	1	21	56	42	24	24	
$\widetilde{\chi}(C_{\mathcal{S}_G^{2+*}}(x))$	-8	0	1	0	-1	-1	0
$\widetilde{\chi}(C_{\mathcal{S}_G^{3+*}}(x))$	27	3	0	-1	-1	-1	0
$\widetilde{\chi}(C_{\mathcal{S}_G^{7+*}}(x))$	7	-1	1	-1	0	0	0

Alperin's Weight Conjecture (χ_2)

The (Knörr–Robinson formulation of the) Alperin Weight Conjecture

$$-\widetilde{\chi}_2(\mathcal{S}_G^{p+*}, G) = z_p(G)$$
$$\sum_{A \in \mathcal{S}_G^{p'+\text{abelian}}} -\widetilde{\chi}(C_{\mathcal{S}_G^{p+*}}(A))\varphi_2(A) = z_p(G)|G|$$

is true for

- G with cyclic p-Sylow subgroup
- G solvable
- G with a nontrivial normal p-subgroup ($\implies z_p(G) = 0$)
- $\mathrm{GL}_n(\mathbf{F}_q)$ where p is the characteristic of $\mathbf{F}_q$
- The Mathieu groups $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$ and the Janko groups J_1, J_2, J_3 at all primes p dividing the group order (computer verifications)

Alperin's Weight Conjecture for $GL_3(\mathbf{F}_2)$, $p = 3$

$G = GL_3(\mathbf{F}_2)$, $p = 3$, $z_p(G) = 3$, $|G| = 168$
G contains six classes of abelian subgroups of order prime to 3

A	1	2	4	7	2×2	2×2
$\lvert G : N_G(A)\rvert$	1	21	21	8	7	7
$\varphi_2(A)$	1	3	12	48	6	6
$-\widetilde{\chi}(C_{\mathcal{S}_G^{p+*}}(A))$	-27	-3	1	1	1	1
$-\widetilde{\chi}\varphi_2\lvert G : N_G(A)\rvert$	-27	-189	252	384	42	42

The sum of the numbers of the bottom row is

$$\sum_{A \in \mathcal{S}_G^{p'+\text{abelian}}} -\widetilde{\chi}(C_{\mathcal{S}_G^{p+*}}(A))\varphi_2(A) = 504 = 3 \cdot 168 = z_p(G) \cdot |G|$$

Why?

$\widetilde{\chi}_3(\mathcal{S}_G^{2+*}, G)$ for alternating and symmetric groups (χ_3)

n	4	5	6	7	8	9	10
$\widetilde{\chi}_3(\mathcal{S}_{A_n}^{2+*}, A_n)$	0	-8	-24	2	-32	-20	-42
$\widetilde{\chi}_3(\mathcal{S}_{\Sigma_n}^{2+*}, \Sigma_n)$	0	-2	-12	2	-10	-11	-16

- Let p be the characteristic of $\mathbf{F}_q$. Then

$$\widetilde{\chi}_2(\mathcal{S}_{\mathrm{GL}_n(\mathbf{F}_q)}^{p+*}, \mathrm{GL}_n(\mathbf{F}_q)) = \widetilde{\chi}_2(L_n^*(\mathbf{F}_q), \mathrm{GL}_n(\mathbf{F}_q))$$

  as the Brown poset $\mathcal{S}_{\mathrm{GL}_n(\mathbf{F}_q)}^{p+*}$ and the building $L_n^*(\mathbf{F}_q)$ are $\mathrm{GL}_n(\mathbf{F}_q)$-homotopy equivalent
- $z_p(\mathrm{GL}_n(\mathbf{F}_q)) = q - 1$ where p is the characteristic of $\mathbf{F}_q$
- $\chi_r(L_n^*(\mathbf{F}_q), G)$ are the equivariant Euler characteristics of an n-dimensional $\mathbf{F}_q$-representation of G

Equivariant Euler characteristics of partition posets

Partitions of $\{1, \ldots, n\}$ ordered by refinement $\longrightarrow \Pi_n \circlearrowleft \Sigma_n \longleftarrow$ obvious action

Remove smallest and largest element:

$$\Pi_n^* = \Pi_n - \{\{\{1\}, \ldots, \{n\}\}, \{\{1, \ldots, n\}\}\}$$

0. $\widetilde{\chi}(\Pi_n^*) = (-1)^{n-1}(n-1)!$ (Stanley)
1. $\widetilde{\chi}_1(\Pi_n^*, \Sigma_n) = 0$ (me!)
2. $\widetilde{\chi}_2(\Pi_n^*, \Sigma_n) = \mu(n) - \mu(n/2)$
3. $\widetilde{\chi}_3(\Pi_n^*, \Sigma_n) = -4, -4, 5, -6, 16, -8, -2, \ldots$ (not in OEIS)
   $\chi_3(\Pi_n^*, \Sigma_n) = 0, 4, 26, 33, 108, 162, 358, \ldots$ (not in OEIS)

$$\mu(n) - \mu(n/2) = -2, -1, 1, -1, 2, -1, 0, 0, \ldots (\text{A092673})$$

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Equivariant Euler characteristics of Boolean lattices

Subsets of $\{1,\dots,n\}$ ordered by inclusion $\longrightarrow B_n \circlearrowleft \Sigma_n \longleftarrow$ obvious action

Remove smallest and largest element:

$$B_n^* = B_n - \{\emptyset, \{1,\dots,n\}\}$$

0. $\widetilde{\chi}(B_n^*) = (-1)^n$ (Stanley)
1. $\widetilde{\chi}_1(B_n^*, \Sigma_n) = 0$
2. $\widetilde{\chi}_2(B_n^*, \Sigma_n) = p^*_{\mathrm{even}}(n) - p^*_{\mathrm{odd}}(n)$
3. $\widetilde{\chi}_3(B_n^*, \Sigma_n) = -3, -1, 0, 10, 8, 12, 1, -28, \dots$ (not in OEIS)
   $\chi_3(B_n^*, \Sigma_n) = 1, 7, 21, 49, 100, 182, 361, \dots$ (not in OEIS)

$p^*_{\mathrm{even}}(n)$: The number of partitions of n with en even number of distinct blocks.

$$p^*_{\mathrm{even}}(n) - p^*_{\mathrm{odd}}(n) = -1, 0, 0, 1, 0, 1, 0, 0, 0, \dots \text{(A010815)}$$

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Subspaces of $\mathbf{F}_q^n$ ordered by inclusion (q is a prime power) $\longrightarrow$ $L_n(\mathbf{F}_q)$ $\circlearrowleft$ $\mathrm{GL}_n(\mathbf{F}_q)$ $\longleftarrow$ obvious action

Remove smallest and largest element:

$$L_n^*(\mathbf{F}_q) = L_n(\mathbf{F}_q) - \{0, \mathbf{F}_q^n\}$$

0. $\widetilde{\chi}(L_n^*(\mathbf{F}_q)) = (-1)^n q^{\binom{n}{2}}$ (Stanley)
1. $\widetilde{\chi}_1(L_n^*(\mathbf{F}_q), \mathrm{GL}_n(\mathbf{F}_q)) = 0$
2. $\widetilde{\chi}_2(L_n^*(\mathbf{F}_q), \mathrm{GL}_n(\mathbf{F}_q)) = -(q-1)$ (Thévenaz)
3. $\widetilde{\chi}_3(L_n^*(\mathbf{F}_2), \mathrm{GL}_n(\mathbf{F}_2)) = -4, -12, -32, -80, -192, \dots$ (multiple entries in OEIS)

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Summary

- There are global constraints on the p-local Euler characteristics $\widetilde{\chi}(C_{\mathcal{S}^{p+*}_{N_G(H)/H}}(A))$ defined for A-p-subgroups $H \leq G$
- $\chi_1(\mathcal{C}, A) = \chi(|\mathcal{C}|/A)$ is suprisingly often 1
- $\chi_2(\mathcal{C}, A)$ may carry crucial information
- $\chi_3(\mathcal{C}, A)$ is bewildering
- $\chi_r(\mathcal{C}, A)$ for $r > 3$ is terra incognita