Introduction

Joint with Frank Lutz

Picture of 7 vertex torus $\frac{1}{3}\binom{7}{2} = \frac{2}{3}\cdot\frac{7\cdot 6}{2} = 14$

Complete 1-skeleton

(3,2)-coloring

Kingfisher

2-chromatic numbers and Steiner Triple Systems.

$(164)(37)$
sta → st0
stb → st1

Tiling in green and yellow triangles.

$T = S^- \cup S^+,\ S^- \cap S^+ = \emptyset$ (tilings)

$chr^1(T) = 7$

$S = (15)(36)(47) =$
What is Aut(K)

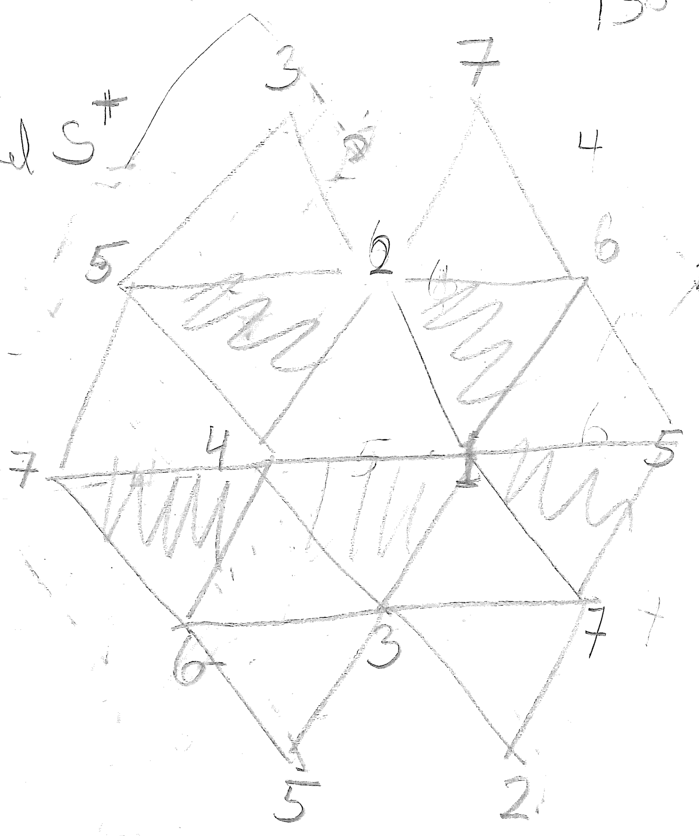
$chr^2(T) = 3$

Any edge is contained in exactly one green and exactly one yellow triangle

267 354

134 → 527
136 → 563

The exists an involution automorphism's S of K interchanging S^- and S^+



2-chromatic numbers of manifolds and Steiner triple systems

Chromatic numbers of simplicial complexes

V = vertex set of K $\quad K = \{\sigma\}, \sigma \subseteq V$

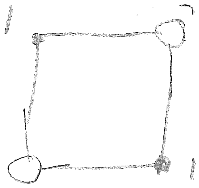
K ASC finite

$\mathrm{chr}^s(K) = r :\Leftrightarrow$ There exists a function $\mathrm{col}: V \to [r]$ such that there are no monochrome s-dimensional simplices in K and r is as small as possible

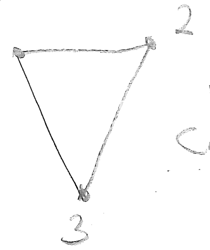
$$\mathrm{chr}^s(K) = \mathrm{chr}^s(K^s) \qquad \mathrm{chr}^s(K) = 1 \text{ if } s > \dim(K)$$

$$\mathrm{chr}^s(K) \leq |V|$$

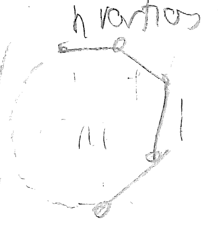
Example



$\mathrm{chr}^1(K) = 2$



$\mathrm{chr}^1(K) = 3$

$\mathrm{chr}^1\Big($  $\Big) = \begin{cases} 2 & \text{if } n \text{ even} \\ 3 & \text{if } n \text{ odd} \end{cases}$ $\qquad \mathrm{chr}^1(\text{complete graph on } n \text{ vertices}) = n$

Show pictures of $(r,2)$-colourings

The function $(s, K) \to \mathrm{chr}^s(K)$ is

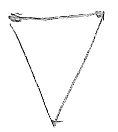
1) increasing in K: $K_1 \subseteq K_2 \quad \mathrm{chr}^s(K_1) \leq \mathrm{chr}^s(K_2)$

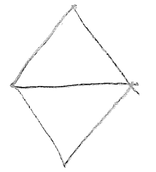
2) decreasing in s: $s_1 \leq s_2 \quad \mathrm{chr}^{s_1}(K) \geq \mathrm{chr}^{s_2}(K)$

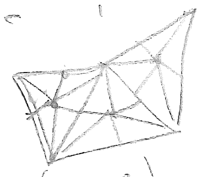
3) $|V| \geq \mathrm{chr}^1(K) \geq \mathrm{chr}^2(K) \geq \ldots \geq \mathrm{chr}^d(K) \geq \mathrm{chr}^{d+1}(K) = 1$, $d = \dim K$

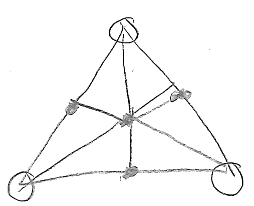
How does chr^s behave under subdivision?

1) $K \quad \mathrm{sd}(K) \qquad \mathrm{chr}^s(\mathrm{sd} K) = 2$







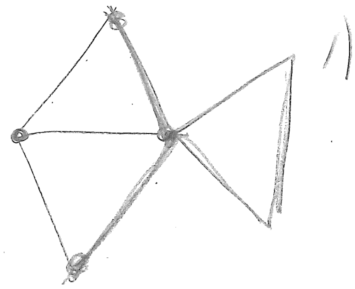


$\mathrm{chr}^s(\mathrm{sd} K) = 2$

Chromatic numbers of manifolds

$$\mathrm{lk}_K(v) = \{\sigma \in K \mid v \notin \sigma\} \cap \mathrm{AS}(\{\sigma \in K \mid v \in \sigma\})$$

$\mathrm{st}_K(v) = v * \mathrm{lk}_K(v)$

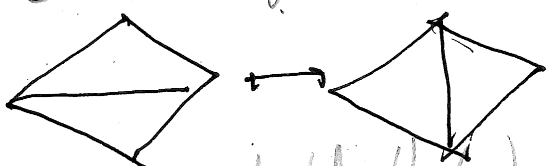


~~topologists~~
like manifolds

B^d = all subsets of d_+ $\qquad d_+ = d \cup \{d\}$

S^d = all proper subsets of d_+

Two simplicial complexes are PL-isomorphic if they have isomorphic subdivisions.



Def A combinatorial d-manifold is a simplicial complex K such that $\mathrm{lk}_K(v)$ is PL-isomorphic to S^{d-1}.

Pachner moves

A PL-manifold is an equivalence class of combinatorial d-manifolds

$\mathrm{chr}^s(\mathbb{M}) = r$: in any subdivision of $\mathbb{M}$ there is a coloring $c: V(M) \to r$ that

Def $\mathbb{M}$ PL-manifold

$$\mathrm{chr}^s \mathbb{M} = \sup\{\mathrm{chr}^s(M) \mid M \in \mathbb{M}\}$$

We have seen that subdivisions can change the chromatic number drastically, so that they are no ~~[illegible]~~ simplices in M. and r is as small as possible.

1-dimensional manifolds

$\mathbb{S}^1$ PL-manifold $\quad \mathbb{S}^1 = \{\, n\text{-gons} \,\}$

$$\mathrm{chr}^1(\mathbb{S}^1) = 3$$

$\mathrm{chr}^s(\mathbb{M}) = r$:

For any manifold M that is in the class of $\mathbb{M}$ there exists a coloring $c: M \to r$

Do we know any chromatic number of any other manifold?

~~3-manifolds~~? Spheres We do know that ~~chr~~²

Lemma $chr^s(S^d) = \infty$ when $d \geq 3$ and $s \leq \lceil d/2 \rceil$

$chr^1(S^3) = \infty$: well known family cyclic polytopes

$chr^2(S^3) = \infty$ - there are triangulations of S^3 with arbitrarily

high 2-chromatic number.

The literature has examples with $chr^3(S^3) = 4$ (10 vertices)

Example There exists a triangulated 3-sphere on 167 vertices

with ~~$chr^2 > 4$~~ $6 \geq chr^2 > 4$.

What is $chr^3(S^3)$?

Find the function $d \to chr^d(S^d)$.

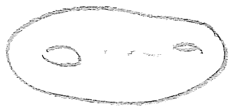
$chr^2(S^2) = 2$

$chr^3(S^3) = ?$

Chromatic number 2-dimensional manifolds.

M_g orientable

N_g nonorientable

g	0	1	2	g
M_g	S^2	T		
$\chi(M_g)$	2	0	-2	$2-2g$
$chr^1 = H$	4	7	8	

g	1	2	3	$\cdots\ g$
N_g	$\mathbb{RP}^2$	KB	$\mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$	
$\chi(N_g)$	1	0	-1	$2-g$
$chr^1 = H$	6	6	7	

The Heawood number of a surface $S \overset{def}{=}$ max $chr^1(G)$ of any graph G embedded in the surface S = max $chr^1(K)$ of any triangulation of the surface = $chr^1(S)$

$\boxed{\text{4-color thm + map color thm}}$

Prop $$chr^1(S) = H(S) \overset{Thm}{=} \left\lfloor \frac{7+\sqrt{49-24\chi(S)}}{2} \right\rfloor$$

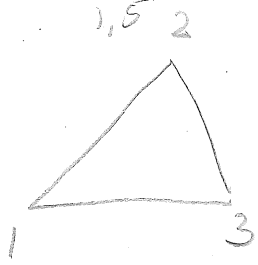
(exception that $H(KB) = 6$ and not 7)

FILL OUT THIS FORM

g	0	1	2	g
$\mathbb{M}_g$	S^2	T	$T\#T$	$T\#\cdots\#T$
$\chi(\mathbb{M}_g)$	2	0	-2	$2-2g$
chr^1	4	7	8	
chr^2	2	3,4?		

g	1	2	3
$\mathbb{N}_g$	$\mathbb{RP}^2$	KB	$\mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2$
$\chi(\mathbb{N}_g)$	1	0	-1
chr^1	6	6	7
chr^2	3	3	3,4

The 2-chromatic number of a surface



Rough upper bound: $chr^2(\mathbb{S}) \le \left\lceil \frac{1}{2} chr^1(\mathbb{S}) \right\rceil$ 1,5

Rough lower bound: $\mathbb{M}_{g+1} = \mathbb{M}_1 \# \mathbb{M}_g$

$$chr^2(\mathbb{M}_{g+1}) \ge chr^2(\mathbb{M}_g^\circ)$$

If $M_g \in \mathbb{M}_g$ then $chr^2(\mathbb{M}_g) \ge chr^2(M_g)$.

This gives

$chr^2(\mathbb{S}^2) = 2$

$chr^2(\mathbb{N}_1) = 3$

$chr^2(\mathbb{M}_1) \in \{3, 4\}$ We do not know the 2-chromatic number of the surface!

$chr^2(\mathbb{M}_1^\circ), chr^2(\mathbb{N}_1^\circ) \ge 3 \Rightarrow chr^2(\mathbb{S}) \ge 3$ if $\mathbb{S} \ne \mathbb{S}^2$

This is about all that is known!

Going through the library of known triangulations one can find examples of surface triangulations with $chr^2 = 4$:

There is a combinatorial $\mathbb{M}_{20}$ containing K_{19} with $chr^2(M_{20}) = 4$.

For some time we conjectured that $chr^2(\mathbb{S}) \le 4$. But this is ~~fa~~ false.

<u>Thm</u> $\sup\{chr^2(\mathbb{S})\} = \infty$

Find a concrete example of a surface with $chr^2 = 5$. !

There seem to be none in the library!

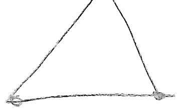
FUNDAMENTAL PROBLEM: How do we determine the chromatic number of a given triangulation?

Steiner Triple System

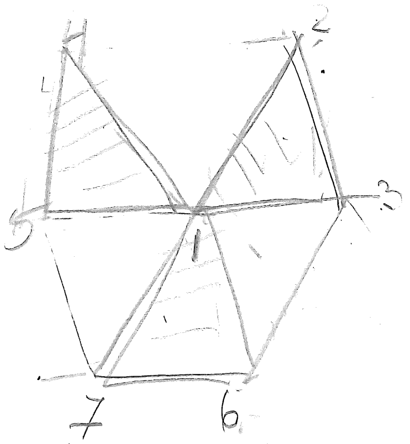
A STS on L is a 2-dim simplicial complex such that
1. the 1-skeleton is the complete graph on L
2. every 1-simplex is contained in a unique 2-simplex.

$STS(L)$ = the set of all STS on L

Thm If $STS(L) \neq \emptyset$ then $|L| \equiv 1,3 \mod 6$

$|L| = 3$  B^2 is a STS

$|L| = 7$



123 145 167
246 257 347
356

A Steiner triple system is half of a surface.

Definition A (orientable) transversal to $S \in STS(L)$ is a permutation $T \in \Sigma(L)$ such that S and S^T have no common 2-simplices and $S \cup S^T$ is a (orientable) surface. $|L| = n$

If $S \cup S^T$ is an orientable surface then

$$\chi(S \cup S^T) = n - \binom{n}{2} + \frac{2}{3}\binom{n}{2} = n + \frac{1}{3}\binom{n}{2}$$

is odd if $n \equiv 1,9 \mod 12$ and even if $n \equiv 3,7 \mod 12$

How can we tell if $S \cup S^T$ is a surface?

Steiner quasigroup $\mu : L \times L \to L$

$\mu(x,x) = x$

$\mu(x,x) = x$

$\mu(x,y) = \mu(y,x)$ $\quad x \neq y \quad \{x, y, \mu(x,y)\} \in S$

$\mu(x, \mu(x,y)) = y$ $\quad \{y, x, \mu(y,x)\} = \{x, \mu(x,y), \mu(x, \mu(x,y))\}$

$|L| = 2m+1$

Prop T is a transversal to S if and only if $\mu^T(x,y) = \mu(x^{-T}, y^{-T})^T$

$\sigma(x,T) : y \to \mu(x, \mu(x^{-T}, y^{-T})^T)$

has cycle structure $1^1 m^2$ and T is an orientable transversal if and

A local orientation at x one of the two m-cycles at x.

T is an orientable transversal if local orientations exist s.t. ....

If T is an orientable transversal cond depends on the class of T in

$\Sigma(\mu) T \Sigma(\mu) \in \Sigma(\mu) \backslash \Sigma(L) / \Sigma(\mu)$ $\quad |\Sigma(\mu) T \Sigma(\mu)| = \frac{|\Sigma(\mu)|^2}{|\Sigma(\mu) \cap \Sigma(\mu)^T|}$

A family of Steiner triple systems

Why not any field: $x, y,\ z = -(x+y)$
$-(x+y)$ is 0 if $x = -y$.

$d \geq 3 \qquad \mathbb{F} = \mathbb{F}_{2^d} \qquad x \neq y$

$PG(2^d) = \{ \{x, y, z\} \in \mathbb{F}_{2^d}^{\times} \mid x + y + z = 0 \}$

$x \qquad x+y \qquad \mathbb{F}_2\{x,y\} \qquad 0$

$\forall x \neq y : P(x+y) = P(x) + P(y)$

What is $\Sigma(PG(2^d)) = ?$

$P(0) = 0$

$L = \mathbb{F}^{\times} \qquad |L| = 2^d - 1 = n \qquad x^{2^{d}-1} = 1$

$\Sigma(PG(2^d)) = GL(d, \mathbb{F}_2) \qquad G_2(\mathbb{F}_{2^d})$

Any permutation T of $\mathbb{F}$ can be executed by a polynomial P of degree $\deg P \leq 2^d - 2$: $x^T = x^P$

permutation polynomial

Surface polynomials:

Example: If $(r, |\mathbb{F}^{\times}|) = 1$ then x^r is a permutation monomial.

A permutation polynomial $P \in GL(d, \mathbb{F}_2) \setminus \Sigma_0(\mathbb{F}_{2^d}) / GL(d, \mathbb{F}_2)$ is a transversal to $PG(2^d)$ if and only if

$$\forall x \in \mathbb{F}_{2^d}^{\times} : y \longrightarrow x + (x^{-P} + y^{-P})^{P}$$

has cycle structure $1^2 (2^{d-1} - 1)^2$.

Problem: Find the transversals to $PG(2^d)$!

$d = 3, 4 \qquad 2^3 = 8,\ 2^4 = 16 \qquad$ OK

$d = 5 \qquad 2^d = 32 \qquad P = x^5$

$d = 6 \qquad 2^d = 64 \qquad ? \qquad \mathrm{chr}^2(PG(64)) = 5$

$d = 7 \qquad 2^d = 128 \qquad P = x^7 \qquad \mathrm{chr}^2(PG(128)) \geq 5$

$PG(2^7) \cup PG(2^7)^7$ is a homeomorphic surface of genus 2542 on 127 vertices with $6 \geq \mathrm{chr}^2 \geq 5$.

Help from outside:

d	1	2	3	4	5	6	
2^d	2	4	8	16	32	64	128

- $chr^2(PG(2^d)) \to \infty$ for $d \to \infty$
- $3 \le chr^2(PG(2^d)) \le chr^2(PG(2^{d+1})) \le chr^2(PG(2^d))+1$
- $chr^2(PG(2^4)) = 3$ $\quad chr^2(PG(\underset{64}{2^6})) = 5$

So $chr^2(PG(\underset{2^5}{32})) \ge 4$ and $chr^2(PG(128)) \ge 5$
($2^4 = 16$)

- $PG(8)$ ~~one~~ 1 transversal X^3 orientable
- $PG(16)$ 4 transversals, one is orientable $T = aX^{11} + X^6 + X$
  $a^4 + a + 1 = 0$
- $PG(32)$ X^5 is a nonorientable transversal $\to$ nonorientable surface of genus 126 with $chr^2 = 4$
- $PG(64)$ no transversals are known
- $PG(128)$ X^7 is ~~an~~ a nonorientable transversal $\to$ nonorientable surface of genus 2542 with $chr^2 = 5$