Objective

Study the p-completed classifying space $BG_p^\wedge$ where G is a finite group and p a prime.

- $BG_p^\wedge = B\{1\}$ if $p \nmid |G|$
- $BP_p^\wedge = BP$ when P is a p-group
- $BG_p^\wedge$ is generally not aspherical
- $G = \pi_1(BG) \to \pi_1(BG_p^\wedge) = G_p = G/O^pG$ is the universal homomorphism from G into a p-group
- $[BP, BG_p^\wedge] = \mathrm{Rep}(P, G)$
- $BC_G(P)_p^\wedge \simeq \mathrm{map}(BP, BG_p^\wedge)$
- $BZ(P) \simeq \mathrm{map}(BP, BG_p^\wedge)$ if $P \leq G$ is p-centric.

Definition 1 *A p-subgroup $P \leq G$ is p-centric if $Z(P) \in \mathrm{Syl}_p C_G(P)$.*

Algebraization of $BG_p^\wedge$

Definition 2 *The p-fusion category of G is the finite category $\mathcal{F}_p(G)$ with all p-subgroups of G as objects. The morphism sets*

$$\mathcal{F}_p(G) = N_G(P,Q)/C_G(P)$$

consist of all group homomorphisms induced by conjugation in G.

There is a functor

$$\mathrm{Mono}_p^c(-,G)\colon \mathcal{F}_p(G)^{\mathrm{op}} \to \mathbf{GRPOID} \qquad (1)$$

such that $\mathrm{Mono}_p^c(P,G) \simeq Z(P)$ when $P \leq G$ is p-centric.

$\mathrm{Mono}_p^c(P,G)$ is the connected groupoid whose objects are monomorphisms $P \rightarrowtail G$ conjugate to the inclusion $P \hookrightarrow G$. The morphisms between two inclusions $P \overset{j_1}{\underset{j_2}{\rightrightarrows}} G$ is

$$\{g \in G \mid {}^g j_1 = j_2\}/O^p C_G(j_1)$$

In particular, $\mathrm{Mono}_p^c(P,G)(j,j) = C_G(P)_p$.

Theorem 3 (Dwyer–Zabrodsky) (1) *is an algebraization of $BG_p^\wedge = |\int_{\mathcal{F}_p^c(G)} \mathrm{Mono}_p^c(-,G)|$.*

The Martino–Priddy conjecture. Alperin's fusion theorem.

p-local finite groups

A p-local finite group is a generalized $BG^\wedge_p$.

Definition 4 *Let S be a finite p-group. A saturated fusion system over S is a finite category whose objects are the subgroups of S and whose morphism sets are sets of group homomorphisms satisfying certain axioms formulated by Puig.*

$\mathcal{F}_p(G)$ is a saturated fusion system over $S \in \mathrm{Syl}_p(G)$.

Definition 5 *Let S be a finite p-group and $\mathcal{F}$ a saturated fusion system over S. A p-local finite group over $\mathcal{F}$ is a functor*

$$L\colon \mathcal{F}^{\mathrm{op}} \to \mathbf{GRPOID}$$

into the category of groupoids such that $L(P)$ is a connected groupoid and $L(P) \simeq Z(P)$ for all p-centric objects P of $\mathcal{F}$. The classifying space BL of the p-local finite group L is the p-completion of $|\int_{\mathcal{F}^c} L|$.

Any finite group determines a p-local finite group.

Theorem 6 (Broto–Levi–Oliver) *The functor* $L\colon \mathcal{F}^{\mathrm{op}} \to \mathbf{GRPOID}$ *is an algebraization of* BL.

p-local Chevalley groups

Let (now) G be a connected Lie group, p a prime, q a power of another prime, and τ a graph automorphism of G. Let

$$1BG(\tau\psi^q) = \text{holim}(\, BG \xrightarrow{(1,\tau\psi^q)} BG \times BG \xleftarrow{(1,1)} BG\,)$$

be the homotopy fixed points of the automorphism $\tau\psi^q$ of BG. (All spaces are completed at p.)

Theorem 7 (Friedlander) *$BG(\tau\psi^q)$ is the p-completed classfying space of a finite group.*

Let X be a connected p-compact group.

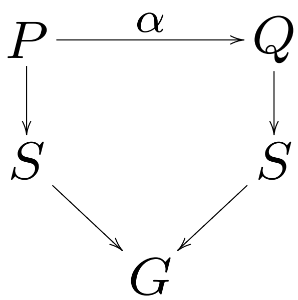
Theorem 8 (Broto–M) *$BX(\tau\psi^q)$ is the p-completed classfying space of a p-local finite group (that sometimes is not finite group).*

The p-fusion category of a finite group G

Let G be a finite group and $S < G$ a Sylow p-subgroup. $\mathcal{F}_p(G)$ is the category with objects all subgroups P, Q of S. The morphism

$$\mathcal{F}_p(G)(P,Q)$$

consists of all group homomorphisms $\alpha\colon P \to Q$ such that



commutes up to inner automorphisms of G. Thus

$$\mathcal{F}_p(G)(P,Q) = N_G(P,Q)/C_G(P)$$

The Sylow p-subgroup S of G is an object of $\mathcal{F}_p(G)$ and $\mathcal{F}_p(G)(S,S) = N_G(S)/C_G(S)$.

The p-fusion category of a space X

Let X be a space, S a p-group and $BS \longrightarrow X$ a map.

$\mathcal{F}_S(X)$ is the category with objects all subgroups P, Q of S. The morphisms

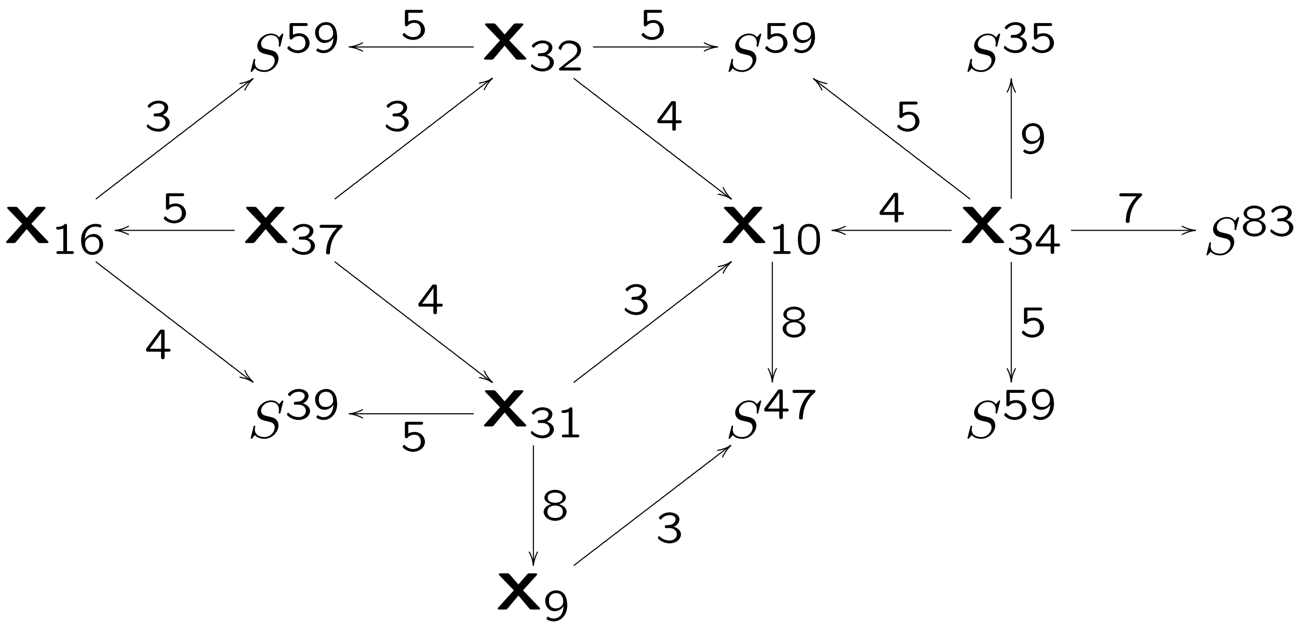
$$\mathcal{F}_S(X)(P, Q)$$

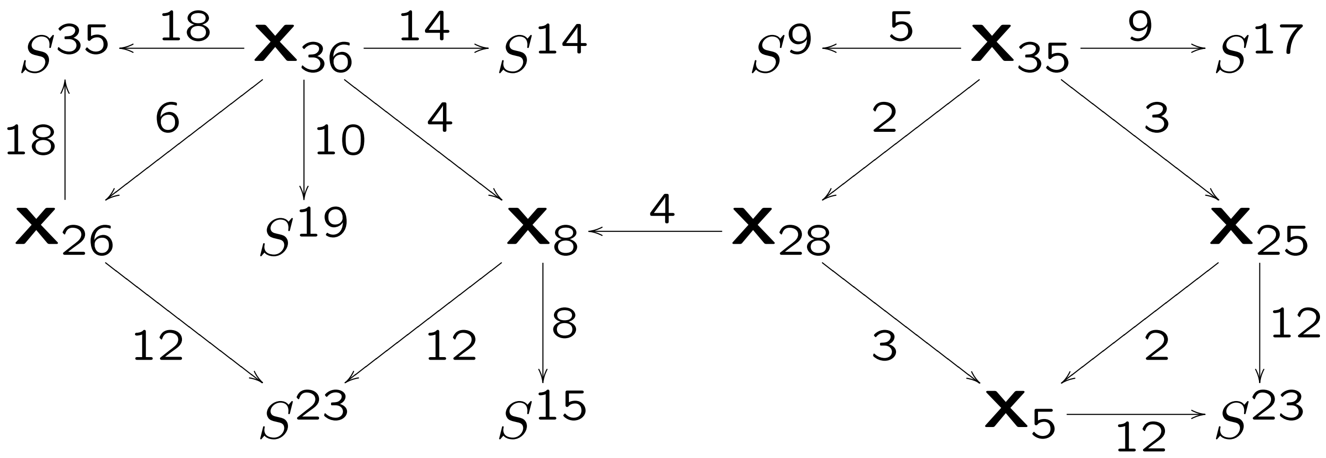
are all group homomorphisms $\alpha\colon P \longrightarrow Q$ such that

$$\begin{array}{ccc} BP & \xrightarrow{B\alpha} & BQ \\ \downarrow & & \downarrow \\ BS & & BS \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

commutes up to homotopy.

Fixed points p-compact groups





Examples of fusion categories

Let $p > 2$ be a prime and q a prime power not divisible by p. Assume that $\langle -1, q \rangle$ has order $r > 2$ in $(\mathbf{Z}/p)^\times$. Then

$$\mathcal{F}_p E_7(q) = \begin{cases} \mathcal{F}_p \mathbf{X}_8(q^4) & r = 4 \\ \mathcal{F}_p \mathbf{X}_{26}(q^6) & r = 6 \\ \mathcal{F}_p S^{2r-1}(q^r) & r \in \{8, 10, 12, 14, 18\} \\ 1 & \text{otherwise} \end{cases}$$

$$\mathcal{F}_p E_8(q) = \begin{cases} \mathcal{F}_p \mathbf{X}_{31}(q^4) & r = 4 \\ \mathcal{F}_p \mathbf{X}_{32}(q^6) & r = 6 \\ \mathcal{F}_p \mathbf{X}_9(q^8) & r = 8 \\ \mathcal{F}_p \mathbf{X}_{16}(q^{10}) & r = 10 \\ \mathcal{F}_p \mathbf{X}_{10}(q^{12}) & r = 12 \\ \mathcal{F}_p S^{2r-1}(q^r) & r \in \{14, 18, 20, 24, 30\} \\ 1 & \text{otherwise} \end{cases}$$