Mikkel Strunge

Expansion in groups

Coarse geometry and analytic aspects

Advisor: Magdalena Musat

February 2017

Master thesis in Mathematics
University of Copenhagen

Abstract

In this thesis we study finitely-generated groups as geometric objects, where the distance between points is defined by some word-length measured on the Cayley graph of the group. We establish connections between analytic properties of groups (amenability, Kazhdan's property (T) and Haagerup's property (H)) and large-scale (coarse) geometry properties of certain families of graphs (the so-called *box spaces*) constructed inside the group. We also introduce and study *expander graphs*, which are highly connected finite graphs that play an important role in computer science and pure mathematics. While existence of expanders follows from probabilistic arguments, explicit constructions require deep mathematical arguments. We prove Margulis' result that box spaces of residually finite property (T) groups, such as $SL_3(\mathbb{Z})$, are expanders.

Resumé

I dette speciale studerer vi geometrien af endeligt frembragte grupper. Vi definerer afstanden mellem punkter ved længden af ord i den tilhørende Cayleygraf. Vi etablerer forbindelser mellem analytiske egenskaber for grupper (amenabilitet, Kazhdan's (T) og Haagerup's (H)) og geometrien af familier af grafer, kaldet box-spaces. Vi vil også introducere expander-grafer, som spiller en stor rolle i datalogi såvel som i ren matematik. Eksistensen af sådanne grafer følger fra probabilistiske argumenter, men en eksplicit konstruktion kræver dybe matematiske argumenter. Vi beviser Margulis' resultat, der siger, at box-spaces af residually finite grupper med Kazhdan's (T), såsom $SL_3(\mathbb{Z})$, er expander-grafer.

Contents

Introduction

The study of infinite finitely-generated groups as geometric objects has become an important and very active research topic due to the fundamental work of Gromov in the 80's. By now there is lot of fruitful interaction between group theory, operator algebras and other fields of mathematics, including number theory, combinatorics and graph theory.

A finitely generated group can be viewed as a metric space, where the distance between points is defined by some "word-length" measured on a certain graph, namely the Cayley graph, associated to the group.

The overall theme of this thesis is to give a comprehensive introduction to the fundamental ideas of large scale (or coarse) geometry, and use them to establish connections between analytic properties of groups (including amenability, Kazhdan's property (T) and Haagerup's property (H), or a-T-menability, in Gromov's terminology) and coarse geometry properties of certain families of graphs constructed inside the group. More precisely, these are the so-called *box spaces* of the given finitely-generated group. (If Γ is such a group and S is a finite (symmetric) generating set, the *box space* of Γ with respect to a sequence of finite index normal subgroups having trivial intersection is the coarse disjoint union of the Cayley graphs associated to the corresponding quotients in Γ.)

In particular, we will prove the following results for an infinite finitely-generated residually finite group Γ: Amenability of Γ is equivalent to box spaces having property (A), a coarse geometry invariant introduced by G. Yu. Respectively, if the box space of Γ coarsely embeds into a Hilbert space, then Γ has property (H). (The converse is not true.)

We also introduce and discuss a special class of graphs, namely *expander graphs*, which are highly connected finite graphs that have already played an important role in computer science (as basic building blocks for various networks and in computation theory), and in recent years have provided a lot of applications in pure mathematics.

Following Bollobás, we will show that *random* d-regular graphs are expanders. (This is a result originally due to Pinsker, 1972.) However, providing *explicit* constructions requires deep mathematical theories. Building on seminal work of Kazhdan, Margulis gave in 1973 the first explicit examples of expanders, namely, box spaces of residually finite property (T) groups, such as $SL_3(\mathbb{Z})$. We will prove this result, following ideas of Alon and Milman.

We also discuss briefly a weaker analytical property of Γ ensuring that the associated box spaces form an expander family. This is property (τ) of Lubotzky and Zimmer. This approach gave rise to further concrete examples of expanders, due to Bourgain-Gamburd and Sarnak-Lubotzky-Phillips, using intricate number theory. We should also mention the remarkable recent developments concerning *approximate groups*, introduced by Tao, and their applications to expanders, including results of Breuillard, Green and Tao, using methods from additive combinatorics.

The thesis is organized as follows:

In Chapter 1, we consider metric spaces, for which we develop the notion of large scale (coarse) geometry. We then introduce and discuss a coarsely invariant property, called property (A). Last, we investigate how property (A) of a metric space relates to coarse embeddings into Hilbert spaces.

In Chapter 2, we introduce graphs, and in particular we give the definition of expander graphs. We then show results concerning expansion properties of graphs. Along the way, we show that expander families do not coarsely embed into any Hilbert space, and conclude that they do not posses property (A), either. We end the chapter by showing that almost every d-regular graph, $d \geq 3$, on n vertices is an expander, as n tends to infinity.

In Chapter 3, we introduce Cayley graphs, which will give us a way to view groups as graphs. Also, it provides us with a natural way of considering growth of groups, which we relate to amenability in Chapter 4. To study analytic properties of groups in Chapter 4, we make use of unitary representation and cocycles, which we introduce here.

In Chapter 4, we introduce analytic properties of groups. We start with the notion of amenability. By weakening the conditions of amenability, we introduce Haagerup's property (H). We then introduce Kazhdan's property (T), which, as we shall see, in some sense is the opposite to amenability and property (H). We give various examples and results relating these properties to one another.

In Chapter 5, we introduce box spaces of residually finite groups. This construction provides us with interesting examples of metric spaces with and without property (A). We then investigate the connection between analytic properties of the group and large scale geometric properties of the box spaces.

Each chapter ends with a section called *Literature*, containing the primary source of literature used for the chapter.

My interest for the topics in this thesis was first developed in a course on approximation properties for groups and C^*-algebras. Here I was exposed, for the first time, to concepts such as property (T) and property (H). When my supervisor, Magdalena Musat, suggested that I did a thesis on expanders, I was intrigued. I did not know much about graph theory, but the connection between deep group theoretic results and computer science seemed surprising and interesting. At first, it was my intention to discuss this relationship in the thesis. However, as the thesis evolved, the main focus shifted from expander graphs to geometric and analytic properties of groups.

Before I started writing the thesis, I participated in a Masterclass titled "Expanders and rigidity of groups actions" held in Copenhagen (May 2016). Among the speakers were both Lubotzky and Osajda. This was my first meeting with expanders and coarse geometry, and I liked it! In the Summer 2016, I participated in the conference "YMC^*A" in Münster. Here Willett was a main speaker, and since I now was familiar with the concepts, the lectures

gave fruitful insights into the world of coarse geometry and expanders.

Regarding the available literature on the topics herein, I greatly appreciated Khukhro's well-written PhD-thesis [15]. Also Willett's notes on property (A) [25] have been of great help. The book 'Large Scale Geometry' by Yu and Nowak [19] was a good introduction to the topic of large scale geometry. Also the notes by Hoory, Linial and Wigderson [12] were very interesting, as they established a connection between computer science and expander graphs.

Acknowledgement

I would like to thank my family and friends for their support and help during the process. In particular, I would like to thank Pernille Julie Viuff Sand for standing by my side this whole time. Kristian Knudsen Olesen for valuable tex-nichal support. Also, I would like to thank Pelle Kaas Vad for making me many dinners, while I was busy writing. Last, but not least, I would like to thank my advisor Magdalena Musat for her valuable advices and effort in the making of this thesis.

Chapter 1

Metric spaces and coarse geometry

In this chapter we introduce the most basic facts of metric spaces and coarse geometry. We then investigate a metric property, called property (A), introduced by Guoliang Yu in [26]. Last we show a result connecting property (A) and coarse embeddings.

1.1 Metric spaces

Definition 1.1.1. A (non-empty) set X equipped with a map $d_X : X \times X \to \mathbb{R}$ satisfying the following properties

(i) $d_X(x,y) \geq 0$, $d_X(x,y) = 0 \Leftrightarrow x = y$,

(ii) $d_X(x,y) = d_X(y,x)$,

(iii) $d_X(x,z) \leq d_X(x,y) + d_X(y,z)$,

is called a *metric space* and is denoted by the pair (X, d_X).

For ease of notation, we will sometimes denote the metric space (X, d_X) just by X and the map d_X by d. We define the open ball of radius $r > 0$ centered at $x \in X$ as $B(x,r) := \{y \in X \mid d(x,y) < r\}$.

Definition 1.1.2. For a metric space X, we say that X is

- uniformly discrete, if there exists $\varepsilon > 0$ such that $d(x,y) \geq \varepsilon$ for all $x \neq y \in X$.
- locally finite, if X is uniformly discrete, and for all $x \in X$ and $r \geq 0$, the ball $B(x,r)$ is finite, i.e., $|B(x,r)| < \infty$.
- bounded, if X is locally finite, and for all $r \geq 0$, there exists $n \in \mathbb{N}$ such that $|B(x,r)| \leq n$, for all $x \in X$.

We define the diameter of a metric space (or of a subspace) to be the greatest distance between any two points, that is, $\mathrm{diam}(X) = \sup\{d(x,y) \mid x, y \in X\}$.

Definition 1.1.3. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ satisfying

$$d_X(x, y) = d_Y(f(x), f(y)),$$

for all $x, y \in X$, is called an *isometry*.

1.2 Coarse geometry

We now introduce the notion of coarse geometry. The idea behind coarse geometry is, roughly speaking, to study geometric objects from far away. When geometric objects are viewed at a great distance, all the small differences will disappear. Thus coarse geometry is in some sense the opposite of a topological study of a space. For instance, if we consider the integers, they look like the real numbers from a great distance. See Figure 1.1 for an illustration of this.

Figure 1.1: The integers $\mathbb{Z}$ viewed from increasing distance.

We start with a little bit of notation. Let X be a set, and let $E \subseteq X \times X$ be a subset. The inverse of E is defined as

$$E^{-1} := \{(y, x) \in X \times X \mid (x, y) \in E\}.$$

If $F \subseteq X \times X$ is another subset, we define the product $E \circ F$ of E and F by

$$E \circ F := \{(x, y) \in X \times X \mid \exists z \in X : (x, z) \in E \text{ and } (z, y) \in F\}$$

Definition 1.2.1. A *coarse structure* on a set X is a collection $\mathcal{E}$ of subsets of $X \times X$, which contains the diagonal and is closed under the formation of subsets, inverses, products and finite unions. A set equipped with a coarse structure is called a *coarse space*.

Notice that $\mathcal{E}$ is also closed under intersections, since it is closed under taking subsets. We will illustrate this definition with a few examples.

Example 1.2.2. Let X be a set. The *maximal* coarse structure $\mathcal{E}$ on X is the collection of all subsets of $X \times X$.

Example 1.2.3. Let X be a metric space. Let $\mathcal{E}$ be the collection of all subsets of $X \times X$ such that $\sup\{d(x, y) | x, y \in E\} < \infty$. It is quite easy to see that $\mathcal{E}$ satisfies the conditions of the definition, but for good measure we will check this. First note that $d(x, x) = 0 < \infty$, which means that $\mathcal{E}$ contains the diagonal. Let $E \in \mathcal{E}$ be some element in $\mathcal{E}$. For any $E' \subseteq E$, we have that

$$\sup_{(x,y)\in E'} \{d(x, y)\} \le \sup_{(x,y)\in E} \{d(x, y)\} < \infty,$$

which shows that $E' \in \mathcal{E}$ so that $\mathcal{E}$ is closed under taking subsets. Since $d(x, y) = d(y, x)$, for all $x, y \in \mathcal{E}$, we see that $\mathcal{E}$ is closed under inverses. That $\mathcal{E}$ is closed under products follows from the triangle inequality. To see that $\mathcal{E}$ is closed under finite unions, let $E_1, E_2 \in \mathcal{E}$. We have

$$\sup_{(x,y)\in E_1\cup E_2} \{d(x, y)\} \leq \sup_{(x,y)\in E_1} \{d(x, y)\} + \sup_{(x,y)\in E_2} \{d(x, y)\},$$

which is finite, and thus $E_1 \cup E_2 \in \mathcal{E}$. Hence any finite union $\bigcup_{i=1}^{n} E_i = E \in \mathcal{E}$.

Example 1.2.4. Let X be a metric space and let $\mathcal{E}$ be the collection of all those subsets $E \subseteq X \times X$ such that, for all $\varepsilon > 0$, there exists a compact set $K \subseteq X$ such that $d(x, y) < \varepsilon$ whenever $(x, y) \in E \setminus K \times K$. Clearly $\mathcal{E}$ satisfies the condition in the definition of being a coarse structure. This coarse structure is called the C_0 coarse structure on X.

Example 1.2.5. Let X be a set, and let $\mathcal{E}$ be the collection of all those subsets of $X \times X$ containing finitely many points off the diagonal. This coarse structure is called the *discrete* coarse structure.

Let $\mathcal{E}$ and $\mathcal{E}'$ be two coarse structures on a set X. If $\mathcal{E} \subseteq \mathcal{E}'$, we say that $\mathcal{E}$ is finer than $\mathcal{E}'$, or, equivalently, that $\mathcal{E}'$ is coarser than $\mathcal{E}$. Notice that this notion is very similar to the notion of topologies on a set being finer or coarser, except that the inclusion is reversed, i.e., a coarse structure is finer if it is contained in a 'coarser' coarse structure. As with topologies, we want to have a notion of a generating set for a coarse structure. The following proposition will justify this notion.

Proposition 1.2.6. *Let S be a family of subsets of $X \times X$. Then there exists a unique coarse structure $\mathcal{E}$ on X such that $\mathcal{E}$ contains S and is finer than all other coarse structures on X containing S. Such a coarse structure is said to be generated by S.*

Proof. Let $\{\mathcal{E}_i\}_{i\in I}$ be a collection of coarse structures on a set X containing S. Notice that this collection is non-empty since the maximal coarse structure always contains S. Thus $\mathcal{E} = \bigcap_{i\in I} \mathcal{E}_i$ is a coarse structure satisfying the properties. □

We will now look at coarse maps between metric spaces.

Definition 1.2.7. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is called

- *bornologous* if for each pair of points $x, x' \in X$ and all $R > 0$, there exists $S > 0$ such that $d_X(x, x') < R$ implies $d_Y(f(x), f(x')) < S$.
- *proper* if for each pair of points $x, x' \in X$ and all $R > 0$, there exists $S > 0$ such that $d_Y(f(x), f(x')) < R$ implies that $d_X(x, x') < S$.

If f satisfies both conditions, it is called a coarse map.

Two coarse maps $f, g : X \to Y$ are said to be *close* if there exists $n \in \mathbb{N}$ such that $d_Y(f(x), g(x)) < n$, for all $x \in X$. This leads us to the definition of coarse equivalence.

Definition 1.2.8. Two metric spaces (X, d_X) and (Y, d_Y) are said to be *coarsely equivalent*, if there exist coarse maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are close to the identities on Y and X, respectively.

We will now introduce the notion of coarse embeddings, which is very similar to the notion of coarse maps.

Definition 1.2.9. A map $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces satisfying

- for all $R > 0$, there exists $S > 0$ such that whenever $x, x' \in X$ satisfy $d_X(x, x') < R$ then $d_Y(f(x), f(x')) < S$,
- for all $R > 0$, there exists $S > 0$ such that for all $x, x' \in X$ with $d_Y(f(x), f(x')) < S$ we have $d_X(x, x') < R$,

is called a *coarse embedding*.

At first glance the two definitions seems identical. However they do differ at one point, namely the second condition of a coarse embedding is a uniform condition. In particular any coarse embedding is a coarse map. Notice that a coarse embedding is in some sense a large scale inclusion. We now give an equivalent description of coarse embeddings.

Proposition 1.2.10. *A map $f : (X, d_X) \to (Y, d_Y)$ is a coarse embedding if and only if there exist non-decreasing functions $\rho_\pm : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{n\to\infty} \rho_-(n) = \infty$, satisfying*

$$\rho_-(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq \rho_+(d_X(x, x')), \tag{1.1}$$

for all $x, x' \in X$.

Proof. Suppose that $f : X \to Y$ is a coarse embedding. For each $R \geq 0$ we define

$$\begin{aligned}\rho_-(R) &= \inf\{d_Y(f(x), f(x')) \mid d_X(x, x') \geq R\},\\ \rho_+(R) &= \sup\{d_Y(f(x), f(x')) \mid d_X(x, x') \leq R\}.\end{aligned}$$

Then both maps are non-decreasing, since the sets we take infimum over become smaller and the sets we take supremum over become larger, as R grows. To see that (1.1) hold, we note that for any $R > 0$ satisfying $d_X(x, x') = R$, we have

$$\rho_-(d_X(x, x')) = \inf\{d_Y(f(x), f(x')) \mid d_X(x, x') \geq R\} \leq d_Y(f(x), f(x')),$$

and

$$d_Y(f(x), f(x')) \leq \sup\{d_Y(f(x), f(x')) \mid d_X(x, x') \leq R\} = \rho_+(d_X(x, x')).$$

To see that $\rho_-(R) \to \infty$ as $R \to \infty$, we note that if $x, x' \in X$ satisfy $d_X(x, x') \geq R$ for some R, then there exists $S > 0$ such that $d_Y(f(x), f(x')) > S$. Using that ρ_- is non-decreasing, we conclude that $\rho_-(R) \to \infty$, as wanted.

Assume now that there exist two maps $\rho_\pm$ satisfying the above conditions. To see that f satisfies the first condition, let $R > 0$ be given and assume that $d_X(x, x') < R$. By assumption, $d_Y(f(x), f(x')) \le \rho_+(d_X(x, x'))$, so letting $S = \rho_+(d_X(x, x')) + 1$ shows that $d_Y(f(x), f(x')) < S$. The other condition follows similarly. $\square$

We end this section by introducing the notion of coarse disjoint unions of metric spaces.

Definition 1.2.11. Let $\{(X_i, d_i)\}_{i\ge1}$ be a sequence of finite metric spaces. The *coarse disjoint union* $X = \bigsqcup X_i$ is turned into a metric space by defining a metric d on X satisfying

- d restricted to X_i coincides with d_i, for all $i \ge 1$,
- $d(x_i, x_j) \ge \max\{\mathrm{diam}(X_i), \mathrm{diam}(X_j)\}$, for $x_i \in X_i$, $x_j \in X_j$ and $i \ne j$.

We can picture a coarse disjoint union of finite metric spaces as a sequence of disjoint metric spaces of increasing distance, displayed on a string passing through one element $x_i \in X_i$ for each $i \ge 1$.

1.3 Property (A)

We are now ready to introduce property (A). It was first introduced by G. Yu in [26] and it turns out to be a large scale invariant property. We will make this more precise throughout this section, but let us start by defining property (A).

Definition 1.3.1. A discrete metric space X is said to have *property (A)* if for all $R, \varepsilon > 0$, there exists a family $\{A_x\}_{x\in X}$ of non-empty, finite subsets of $X \times \mathbb{N}$ such that for every $x, y \in X$, the following conditions hold

1. $\frac{|A_x \oplus A_y|}{|A_x \cap A_y|} < \varepsilon$, whenever $d(x, y) \le R$,
2. there exists $S > 0$ such that $A_x \subseteq B(x, S) \times \mathbb{N}$,

where $A \oplus B$ denotes the symmetric difference of A and B, i.e., $A \oplus B = (A\setminus B) \cup (B \setminus A)^\dagger$.

As we shall see, the first condition mimics the notion of amenability of a group. This is by no means a coincidence, since property (A) is a somewhat weaker condition than amenability. At first glance this comparison does not make sense since amenability is a property for groups and property (A) is a property for metric spaces. However, as we shall see later, it is possible to look at groups from a geometric point of view, and then it becomes a natural question to ask, if the geometric structure of amenable groups have property (A). However, before we can show this, we need to investigate property (A). We warm up by showing a (almost) trivial fact regarding property (A).

---

†The reason for choosing this notation instead of the more commonly used $A \triangle B$ is twofold: We reserve the symbol $\triangle$ for the Laplace operator. The other reason is that two sets $A, B \subseteq X$ of some universal set X can be viewed in terms of their power sets, i.e., $\mathcal{P}(A), \mathcal{P}(B) \subseteq \mathcal{P}(X)$. Now power sets can be viewed algebraically as $(\mathbb{Z}/2\mathbb{Z})^{|X|}$, and then the symmetric difference is exactly the direct sum.

Example 1.3.2. Any finite metric space has property (A).

Proof. Let (X, d) be a finite metric space. Set $A_x = X \times \{1\}$, for all $x \in X$. Then, for all $\varepsilon, R > 0$ and $d(x, y) \leq R$, we have

$$\frac{|A_x \oplus A_y|}{|A_x \cap A_y|} = \frac{0}{|X|} = 0 < \varepsilon,$$

and by choosing $S \geq |X|$ we are done. $\square$

We now show that property (A) is a large scale invariant, i.e., that property (A) is invariant under coarse equivalence.

Proposition 1.3.3. *Property (A) is invariant under coarse equivalence.*

Proof. Let (X, d_X) and (Y, d_Y) be coarsely equivalent metric spaces, i.e., there exist coarse maps $f : X \to Y$ and $g : Y \to X$ satisfying the conditions. Assume that X has property (A), and let $R, \varepsilon > 0$ be given. We wish to exhibit a family of finite sets $\{B_y\}_{y \in Y} \subseteq Y \times \mathbb{N}$ such that the conditions of property (A) are satisfied. By assumption g is bornologous, so there exists $R' > 0$ such that if $d_Y(y, y') < R$, then $d_X(g(y), g(y')) < R'$. Let now $\{A_x\}_{x \in X} \subseteq X \times \mathbb{N}$ be a family of subsets satisfying the definition of property (A) for $R', \varepsilon > 0$ and some $S > 0$. For fixed $y_0 \in Y$, we define for each $y \in Y$

$$M_{y_0} := \{(f^{-1}(y) \times \mathbb{N}) \cap A_{g(y_0)}\} \subseteq X \times \mathbb{N}$$

and

$$N_{y_0} := \{y \in Y \mid (f^{-1}(y) \times \mathbb{N}) \in A_{g(y_0)}\} \subseteq Y.$$

Notice that

$$\bigcup_{y \in Y} M_{y_0} = \bigcup_{y \in Y} \{(f^{-1}(y) \times \mathbb{N}) \cap A_{g(y_0)}\} = \{(\bigcup_{y \in Y} f^{-1}(y) \times \mathbb{N}) \cap A_{g(y_0)}\} = A_{g(y_0)}.$$

Let $n_{y_0} = |M_{y_0}|$ and define

$$B_{y_0} := \bigcup_{y \in N_{y_0}} \{(y, 1), (y, 2), \dots, (y, n_{y_0})\} = \bigcup_{y \in Y} \{(y, 1), (y, 2), \dots, (y, n_{y_0})\},$$

and notice that that $|B_{y_0}| = |A_{g(y_0)}|$. This allows us to construct a family of finite sets $\{B_y\}_{y \in Y}$, which will be our candidate. To show this, we see that for $y, y' \in Y$ we have

$$|B_y \cap B_{y'}| \geq |A_{g(y)} \cap A_{g(y')}|,$$

which implies that

$$\begin{aligned} |A_{g(y)} \oplus A_{g(y')}| &= |A_{g(y)}| + |A_{g(y')}| - 2|A_{g(y)} \cap A_{g(y')}| \\ &= |B_y| + |B_{y'}| - 2|A_{g(y)} \cap A_{g(y')}| \\ &\leq |B_y| + B_{y'}| - 2|B_y \cap B_{y'}| \\ &= |B_y \oplus B_{y'}|. \end{aligned}$$

Thus, if $d_Y(y, y') < R$, then $d_X(g(y), g(y')) < R'$ and we get from our assumption that

$$\frac{|B_y \oplus B_{y'}|}{|B_y \cap B_{y'}|} \leq \frac{|A_{g(y)} \oplus A_{g(y')}|}{|A_{g(y)} \cap A_{g(y')}|} < \varepsilon,$$

which proves the first condition of property (A). To prove that $\{B_y\}_{y \in Y}$ satisfy the other condition, let $(y', n) \in B_y$. Then $(f^{-1}(y') \times \mathbb{N}) \cap A_{g(y)} \neq \emptyset$, which means we can pick (x, m) in this set. Since the sets $\{A_x\}_{x \in X}$ satisfy property (A) by assumption, we get that $d(g(y), x) \leq S$. Using that f is bornologous, we get $d(fg(y), y') \leq S'$, for some $S' > 0$. Now we use that $f \circ g$ is close to the identity, which implies that there exists $S'' > 0$ such that $d(y, y') \leq S''$. This shows that the sets $\{B_y\}_{y \in Y}$ satisfy the second condition of property (A). $\square$

We will now show a result called the *Higson-Roe condition*. To do this, we will need a few lemmas.

Lemma 1.3.4. *For any finite sets A and B, we have*

$$||A| - |B|| \leq |A \oplus B|.$$

Proof. Since A and B are finite, we may assume that $|A| = n$ and $|B| = m$ with $n \geq m$. This implies that $|A \cap B| \leq |B| = m$, and we see that

$$|A \oplus B| = |A| + |B| - 2|A \cap B| \geq n + m - 2m = n - m = ||A| - |B||. \qquad \square$$

Lemma 1.3.5. *Let X be a metric space with property (A). Let $R, \varepsilon > 0$ be given, and let $\{A_x\}_{x \in X}$ be a family of sets satisfying the conditions of property (A), for some $S > 0$. Then for all $x, y \in X$ with $d(x, y) \leq R$, the following hold:*

- $1 \leq \frac{|A_x|}{|A_x \cap A_y|} < 1 + \varepsilon$,
- $\frac{1}{1+\varepsilon} < \frac{|A_x|}{|A_y|} < 1 + \varepsilon$.

Proof. It is clear that for any two sets A and B we have $A \setminus B \subseteq A \oplus B$. Let $d(x, y) \leq R$, then by property (A), we get

$$|A_x \setminus A_y| \leq |A_x \oplus A_y| < \varepsilon |A_x \cap A_y|,$$

and thus

$$\frac{|A_x \cap A_y|}{|A_x \cap A_y|} \leq \frac{|A_x|}{|A_x \cap A_y|} = \frac{|A_x \setminus A_y| + |A_x \cap A_y|}{|A_x \cap A_y|} < \frac{\varepsilon |A_x \cap A_y| + |A_x \cap A_y|}{|A_x \cap A_y|} = \varepsilon + 1,$$

which shows the first condition. Clearly this also hold for A_y, and thus combining the two we get the second condition. $\square$

In words, this lemma tells us that property (A) ensures that the cardinality of any two sets A_x and A_y from the given family are almost equal when x and y are close. We are now able to give an equivalent characterization of property (A).

Proposition 1.3.6. *The first condition in Definition 1.3.1 can be replaced by*

$$\frac{|A_x \oplus A_y|}{|A_x|} < \varepsilon.$$

Proof. To see this, assume first that X is a metric space with property (A). Let $\varepsilon, R > 0$ be given, and let $\{A_x\}_{x\in X}$ be a family of sets satisfying the conditions of property (A). Then we see that

$$\frac{|A_x \oplus A_y|}{|A_x|} = \frac{|A_x \oplus A_y|}{|A_x \cap A_y|} \cdot \frac{|A_x \cap A_y|}{|A_x|} < \varepsilon,$$

where we have used that

$$\frac{|A_x \cap A_y|}{|A_x|} \leq 1.$$

Now assume that $\{A_x\}_{x\in X}$ is a family of finite sets satisfying property (A) with condition (1) replaced by $\frac{|A_x \oplus A_y|}{|A_x|} < \varepsilon$ for some given $\varepsilon > 0$. Then by the previous lemma we get

$$\frac{|A_x \oplus A_y|}{|A_x \cap A_y|} = \frac{|A_x \oplus A_y|}{|A_x|} \cdot \frac{|A_x|}{|A_x \cap A_y|} \leq \frac{|A_x \oplus A_y|}{|A_x|} \cdot \frac{1}{1+\varepsilon} < \frac{\varepsilon}{1+\varepsilon} < \varepsilon. \qquad \square$$

We are now almost ready to show the Higson-Roe condition. Given a discrete set X, we define for $1 \leq p < \infty$ the following spaces

$$\ell^p(X) := \left\{ f : X \to \mathbb{C} \,\middle|\, \|f\|_p = \left(\sum_{x\in X} |f(x)|^p \right)^{1/p} < \infty \right\}.$$

Further, we denote by $\ell^p(X)_{1,+}$ the set of positive functions in $\ell^1(X)$ with norm 1. That is,

$$\ell^p(X)_{1,+} := \{f \in \ell^1(X) \mid \|f\|_p = 1, \text{ and } f \geq 0\}.$$

Lemma 1.3.7. *Let $f \in \ell^1(X)_{1,+}$ be finitely supported and let $\varepsilon > 0$ be given. Then there exists $n \in \mathbb{N}$ and $\tilde{f} \in \ell^1(X)_{1,+}$ such that $\tilde{f} : X \to \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$ and $\|f - \tilde{f}\|_1 < \varepsilon$.*

Proof. Let $A \subseteq X$ be the finite subset where f is non-zero, and let $\varepsilon > 0$ be given. Since A is finite, we can enumerate its elements as $x_1, x_2, \ldots, x_k$. For each $x_i \in A$ we can find $n_i \in \mathbb{N}$ such that

$$|f(x_i) - \frac{m_i}{n_i}| < \frac{\varepsilon}{k}, \qquad |f(x_i) - \frac{m_i+1}{n_i}| < \frac{\varepsilon}{k},$$

for some $m_i \in \{0, 1, \ldots, n_i\}$. Note that this also holds for all $n \geq n_i$ by choosing m_i accordingly bigger. Now we choose $n = \max\{n_1, n_2, \ldots, n_k\}$ and define $\tilde{f}(x_i) = \frac{m_i}{n}$. This leads to

$$\|f - \tilde{f}\| = \sum_{i=1}^{k} |f(x_i) - \tilde{f}(x_i)| < \sum_{i=1}^{k} \frac{\varepsilon}{k} = \varepsilon. \qquad \square$$

Theorem 1.3.8. *Let X be a uniformly discrete metric space with bounded geometry. Then X has property (A) if and only if for all $R, \varepsilon > 0$, there exist a map $\xi : X \to \ell^1(X)_{1,+}$ and a number $S > 0$ such that*

- $\|\xi_x - \xi_y\| < \varepsilon$ *when* $d(x, y) \leq R$,
- $\operatorname{supp}(\xi_x) \subseteq B(x, S)$.

Proof. Assume first that X has property (A). Let $R, \varepsilon > 0$ be given, and let $\{A_x\}_{x \in X}$ be a family of sets satisfying the definition of property (A) for some $S > 0$. For fixed $z \in X$, let $A_x^z := A_x \cap (\{z\} \times \mathbb{N})$, so that A_x^z is the vertical section of A_x at z. For each $x \in X$ we define $\xi_x \in \ell^1(X)_{1,+}$ by

$$\xi_x(z) = \frac{|A_x^z|}{|A_x|}.$$

To see that ξ_x in fact lies in $\ell^1(X)_{1,+}$, we start by noting that $\xi_x \geq 0$ and that the support is contained in $B(x, S)$, since, by assumption, each A_x is. Furthermore we see that

$$\|\xi_x\| = \sum_{z \in X} \xi_x(z) = \sum_{z \in X} \frac{|A_x^z|}{|A_x|} = 1.$$

Thus it only remains to show that $\|\xi_x - \xi_y\| < \varepsilon$. To this end we observe that

$$\begin{aligned}
\|(|A_x|\xi_x) - (|A_y|\xi_y)\| &= \sum_{z \in X} |(|A_x|\xi_x(z)) - |A_y|\xi_y(z)| \\
&= \sum_{z \in X} ||A_x^z| - |A_y^z|| \\
&\leq \sum_{z \in X} |A_x^z \oplus A_y^z| \\
&= |A_x \oplus A_y|,
\end{aligned}$$

where we have used Lemma 1.3.4 for the inequality. Thus we see that

$$\left\| \xi_x - \frac{|A_y|}{|A_x|}\xi_y \right\| \leq \frac{|A_x \oplus A_y|}{|A_x|} \leq \frac{|A_x \oplus A_y|}{|A_x \cap A_y|},$$

which leads to

$$\begin{aligned}
\|\xi_x - \xi_y\| &= \left\| \xi_x - \frac{|A_y|}{|A_x|}\xi_y + \frac{|A_y|}{|A_x|}\xi_y - \xi_y \right\| \\
&\leq \left\| \xi_x - \frac{|A_y|}{|A_x|}\xi_y \right\| + \left\| \frac{|A_y|}{|A_x|}\xi_y - \xi_y \right\| \\
&\leq \frac{|A_x \oplus A_y|}{|A_x \cap A_y|} + \left| \frac{|A_y|}{|A_x|} - 1 \right| \|\xi_y\| \leq 2\varepsilon,
\end{aligned}$$

where we in the last inequality have used Lemma 1.3.5.
To show the other direction, let $R, \varepsilon > 0$ be given, and assume that there exists a function $\xi : X \to \ell^1(X)_{1,+}$ satisfying the conditions. By Lemma 1.3.7 we can replace each ξ_x by some function $\tilde{\xi}_x \in \ell^1(X)_{1,+}$ such that $\tilde{\xi}_x : X \to \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$ and $\|\xi_x - \tilde{\xi}_x\| < \varepsilon$. We have to show that n is independent of x in order to get a uniform bound. Recall that $\mathrm{supp}(\xi_x) \subseteq B(x, S)$, for some fixed S. Thus we have that $|\mathrm{supp}(\xi_x)| \leq |B(x, S)|$, and since X has bounded geometry, there exists $m \in \mathbb{N}$ such that $|B(x, S)| \leq m$, for all $x \in X$. Thus we can choose some uniform $N \in \mathbb{N}$ only depending on ε. We can now define the sets A_x by

$$A_x = \{(z, i) \in X \times \mathbb{N} \mid \xi_x(z) \geq \frac{i}{N}\}.$$

For each $x \in X$ we see that $A_x \subseteq B(x, S) \times \mathbb{N}$ and $|A_x| = N < \infty$. Notice now that

$$\|\tilde{\xi}_x - \tilde{\xi}_y\| = \sum_{z \in X} \left| \frac{k_z}{N} - \frac{l_z}{N} \right| = \frac{1}{N} \sum_{z \in X} |k_z - l_z| = \frac{1}{N} |A_x \oplus A_y|,$$

where $k_z, l_z \in \{0, 1, \dots, N-1, N\}$ is the denominator of the evaluation of $\tilde{\xi}_x$ at z. By the triangle inequality we then achieve

$$\frac{|A_x \oplus A_y|}{|A_x|} = \frac{|A_x \oplus A_y|}{N} = \|\tilde{\xi}_x - \tilde{\xi}_y\| \leq \|\tilde{\xi}_x - \xi_x\| + \|\xi_x - \xi_y\| + \|\xi_y - \tilde{\xi}_y\| \leq 3\varepsilon.$$

Using Lemma 1.3.6 we see that this indeed implies property (A). $\square$

We end this section with a lemma, which allows us to express property (A) for a coarse disjoint union of metric spaces in terms of a local condition.

Lemma 1.3.9. *Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of finite metric spaces. The coarse disjoint union of X_i has property (A) if and only if for all $R, \varepsilon > 0$ and for all but finitely many $i \in \mathbb{N}$, the space X_i has a Higson-Roe function ξ^i for ε and R satisfying $\mathrm{supp}(\xi^i_x) \subseteq B(x, S)$, where $S > 0$ is independent of i.*

Proof. Suppose that for all $\varepsilon, R > 0$ there exists $i_0 \in \mathbb{N}$ such that for all $i > i_0$ there exist Higson-Roe functions $\xi^i : X_i \to \ell^1(X_i)_{1,+}$ with $\mathrm{supp}(\xi^i_x) \subseteq B(x, S)$, for some $S > 0$ independent of i. We wish to show that the coarse disjoint union $\bigsqcup_{i \in \mathbb{N}} X_i$ has property (A). Let $Y = \bigsqcup_{i=1}^{i_0} X_i$ and $Z = \bigsqcup_{i > i_0} X_i$. Define, for each $1 \leq i \leq i_0$, $A_{x_i} := X_i \times \{1\}$ for all $x_i \in X_i$. Then $\{A_{x_i}\}_{x_i \in Y}$ is a finite collection of finite sets, and this collection satisfies the definition of property (A) for any $S_1 \geq \max\{\mathrm{diam}(X_1), \mathrm{diam}(X_2), \dots, \mathrm{diam}(X_{i_0})\}$. By assumption, there exist Higson-Roe functions ξ^i for all $i > i_0$. Defining $\xi : Z \to \ell^1(Z)_{1,+}$ by $x \mapsto \xi^i_x$ for $x \in X_i$, we get a Higson-Roe function on Z. Thus Theorem 1.3.8 tells us that there exists a family $\{B_x\}_{x \in Z}$, satisfying the definition of property (A), for some $S_2 > 0$. This means that the family $\{A_{x_i}\}_{x_i \in Y} \cup \{B_x\}_{x \in Z}$ is a family of finite sets satisfying the definition of property (A), for $S \geq \max\{S_1, S_2\}$.
The converse follows by restricting the (global) Higson-Roe function to each X_i. $\square$

We will now turn our attention to coarse embeddings into Hilbert spaces. In particular, we will show some results connecting property (A) and such coarse embeddings.

1.4 Coarse embeddings

In this section we investigate which spaces it is possible to coarsely embed into a Hilbert space. We first look at a space which can be coarsely embedded into a Hilbert space.

Example 1.4.1. The Banach space $\ell^1(\mathbb{N})$ coarsely embeds into a Hilbert space. To see this, let H be the infinite dimensional Hilbert space given by

$$H = \left(\bigoplus_{i=1}^{\infty} L^2(\mathbb{R}, m))\right),$$

where m is the Lebesgue measure. Define $f : \ell^1(\mathbb{N}) \to H$ by

$$f(x) = f((x_1, x_2, \ldots)) = \bigoplus_{i=1}^{\infty} \varphi(x_i),$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is given by $\varphi(x) = 1_{[0,x]}$ when $x \geq 0$ and $\varphi(x) = 1_{[x,0]}$ when $x \leq 0$. Thus we see that for $x, y \in \ell^1(\mathbb{N})$ we have

$$\|f(x) - f(y)\| = \sqrt{\|x - y\|},$$

which shows that f is a coarse embedding with $\rho_-(t) = \rho_+(t) = t^{1/2}$.

Lemma 1.4.2. *Let X be a metric space with property (A). For every $R, \varepsilon > 0$, there exist a map $\xi : X \to \ell^2(X \times \mathbb{N})$ and $S > 0$ such that for all $x, y \in X$,*

- $\|\xi_x\|_2 = 1$,
- $\|\xi_x - \xi_y\| < \varepsilon$ *when* $d(x, y) \leq R$,
- $\mathrm{supp}(\xi_x) \subseteq B(x, S) \times \mathbb{N}$.

Proof. Let $\varepsilon, R > 0$ be given, and let $\{A_x\}_{x \in X}$ be a family of sets satisfying property (A) for $\delta = \varepsilon^2$ and R. Define ξ_x by

$$\xi_x = \frac{1_{A_x}}{\sqrt{|A_x|}},$$

for each $x \in X$. Then clearly $\mathrm{supp}(\xi_x) = A_x \subseteq B(x, S) \times \mathbb{N}$ by assumption. Furthermore we see that

$$\|\xi_x\|^2 = \sum_{x \in X} \left(\frac{1_{A_x}}{\sqrt{|A_x|}}\right)^2 = \frac{1}{|A_x|} \sum_{x \in A_x} 1_{A_x} = 1,$$

so that $\|\xi_x\| = 1$. Thus it only remains to show that the second condition is satisfied. To this end note that

$$|A_x| + |A_y| = 2|A_x \cap A_y| + |A_x \oplus A_y| = 2|A_x \cap A_y| + \frac{|A_x \oplus A_y||A_x \cap A_y|}{|A_x \cap A_y|} < (2 + \delta)|A_x \cap A_y|,$$

whenever $d(x,y) \leq R$. Using this, we can estimate

$$\langle \xi_x, \xi_y \rangle = \sum_{x \in X} \frac{1_{A_x} \cdot 1_{A_y}}{\sqrt{|A_x||A_y|}} = \frac{|A_x \cap A_y|}{\sqrt{|A_x||A_y|}} \geq \frac{2|A_x \cap A_y|}{|A_x| + |A_y|} \geq \frac{2}{2+\delta}.$$

Note that $\langle \xi_x, \xi_y \rangle$ is real, which implies that $\langle \xi_x, \xi_y \rangle = \langle \xi_y, \xi_x \rangle$. Thus we see that

$$\begin{aligned}
\|\xi_x - \xi_y\|^2 &= \langle \xi_x - \xi_y, \xi_x - \xi_y \rangle \\
&= \langle \xi_x, \xi_x \rangle + \langle \xi_y, \xi_y \rangle - \langle \xi_x, \xi_y \rangle - \langle \xi_y, \xi_x \rangle \\
&= 1 + 1 - \langle \xi_x, \xi_y \rangle - \langle \xi_y, \xi_x \rangle, \\
&\leq 2 - 2\left(\frac{2}{2+\delta}\right) \leq \delta.
\end{aligned}$$

Since $\varepsilon = \sqrt{\delta}$, we are done. □

Having showed this lemma, we are now ready to introduce our first result relating property (A) and embeddability into Hilbert spaces.

Theorem 1.4.3. *A metric space X with property (A) can be coarsely embedded into some Hilbert space.*

Proof. For each $n \in \mathbb{N}$, let $\varepsilon = 2^{-n}$ and $R = n$ be given and choose ξ^n satisfying the conditions in the lemma above. Then there exists an increasing sequence S_n of positive numbers such that $\operatorname{supp}(\xi^n_x) \subseteq B(x, S_n) \times \mathbb{N}$. This implies that if $d(x,y) \geq 2S_n$, then

$$\|\xi^n_x - \xi^n_y\| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

Fix $z \in X$ and define $f : X \to \bigoplus_{n=1}^{\infty} \ell^2(X \times \mathbb{N})$ by

$$f(x) = \bigoplus_{n=1}^{\infty} (\xi^n_x - \xi^n_z).$$

We have to show that f is well defined and that it is a coarse embedding. Consider $k \in \mathbb{N}$ satisfying $k - 1 < d(x,y) \leq k$. When $n \geq k$, we have $\|\xi^n_x - \xi^n_y\| < \frac{1}{2^n}$. Thus we see that

$$\begin{aligned}
\|f(x) - f(y)\|^2 &= \|\bigoplus_{n=1}^{\infty} (\xi^n_x - \xi^n_z) - \bigoplus_{n=1}^{\infty} (\xi^n_y - \xi^n_z)\|^2 \\
&= \sum_{n=1}^{\infty} \|\xi^n_x - \xi^n_y\|^2 \\
&\leq 2k + 1,
\end{aligned}$$

which leads to

$$2k + 1 = 2(k - 1 + 1) + 1 < 2(d(x,y) + 1) + 1 = 2d(x,y) + 3.$$

Note that if $z = y$, the calculations above shows that f is well defined. Now we see that

$$\|f(x) - f(y)\| \leq (2d(x,y) + 3)^{1/2},$$

which means we have obtained an upper bound

$$\rho_+(t) = (2t + 3)^{1/2}.$$

Thus it only remains to find a lower bound. To this end, let for all $k \in \mathbb{N}$,

$$\varphi(k) = \sup_{n \in \mathbb{N}} \{2S_n \leq k - 1\}.$$

Then we see that

$$\|f(x) - f(y)\|^2 = \sum_{n=1}^{\varphi(k)} \|\xi_x^n - \xi_y^n\|^2 + \sum_{n=\varphi(k)+1}^{\infty} \|\xi_x^n - \xi_y^n\|^2 = \sum_{n=1}^{\varphi(k)} 2 + \sum_{n=\varphi(k)+1}^{\infty} \|\xi_x^n - \xi_y^n\|^2 \geq 2\varphi(k).$$

We know that S_n is an increasing sequence, and thus $\varphi(k)$ is a non-decreasing function with $\varphi(k) \to \infty$ as $k \to \infty$. Now we define ρ_- to be the piecewise linear extension of $\sqrt{\varphi}$ to the real numbers, and we have the desired result:

$$\rho_-(d(x,y)) \leq d(f(x), f(y)) \leq \rho_+(d(x,y)). \qquad \square$$

The construction used in the proof above can be traced back to an article by Bekka, Cherix and Valette [2].

1.5 Kernels and coarse embeddings

We now introduce a certain type of maps called *kernels*. We introduce kernels as maps into the real numbers, but the theory can be generalized to complex numbers. As a consequence, all Hilbert spaces in this section will be assumed to be real. We start with the definition.

Definition 1.5.1. Let X be a set. A map $k : X \times X \to \mathbb{R}$ is a kernel. We say that k is

(i) *symmetric* if $k(x, y) = k(y, x)$ for all $x, y \in X$,

(ii) of *positive type* if for all finite sequences $x_1, \ldots, x_n$ of elements of X and $\lambda_1, \ldots, \lambda_n$ of real numbers

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j k(x_i, x_j) \geq 0.$$

If, furthermore, $k(x, x) = 1$, for all $x \in X$, we say that k is normalized.

(iii) of *negative type* if for all finite sequences $x_1, \ldots, x_n$ of elements of X and $\lambda_1, \ldots, \lambda_n$ of real numbers such that $\sum_{i=1}^n \lambda_i = 0$, one has

$$\sum_{i,j=1}^n \lambda_i \lambda_j k(x_i, x_j) \leq 0.$$

If, furthermore, $k(x,x) = 0$, for all $x \in X$, we say that k is normalized.

The following lemma shows some basic facts about kernels.

Lemma 1.5.2. *Let X be a set and $k : X \times X \to \mathbb{R}$ a kernel.*

1. *If k is constant, then k is of negative type,*
2. *If k is of the form $k(x,y) = f(x)f(y)$, where $f : X \to \mathbb{R}$ is any map, then k is of positive type.*

Proof. Suppose that for all $(x,y) \in X \times X$ we have $k(x,y) = c$, for some $c \in \mathbb{R}$. Then for any finite sequence $x_1, \ldots, x_n$ of elements of X and any sequence $\lambda_1, \ldots, \lambda_n$ of real numbers satisfying $\sum_{i=1}^n \lambda_i = 0$, we have

$$\sum_{i,j=1}^n \lambda_i \lambda_j k(x_i, x_j) = \sum_{i,j=1}^n \lambda_i \lambda_j c = 0.$$

Thus k is a symmetric, normalized kernel of negative type.
Let $f : X \to \mathbb{R}$ be some map, and suppose that $k(x,y) = f(x)f(y)$. Then

$$\sum_{i,j=1}^n \lambda_i \lambda_j k(x_i, x_j) = \sum_{i,j=1}^n \lambda_i \lambda_j f(x_i) f(x_j) = \left(\sum_{i=1}^n \lambda_i f(x_i)\right)\left(\sum_{i=1}^n \lambda_i f(x_i)\right) \geq 0.$$

Thus k is a symmetric kernel of positive type. $\square$

The next theorem shows that there exists a connection between Hilbert spaces and kernels of positive or negative type.

Theorem 1.5.3. *Let k be a symmetric kernel on a set X. Then there exists a (real) Hilbert space H such that*

(i) if k is of positive type, then there exists a map $\varphi : X \to H$ such that for all $x, y \in X$,

$$k(x,y) = \langle \varphi(x), \varphi(y) \rangle.$$

(ii) if k is of negative type and normalized, i.e., $k(x,x) = 0$, for all $x \in X$, there exists a map $\psi : X \to H$ such that for all $x, y \in X$,

$$k(x,y) = \|\psi(x) - \psi(y)\|^2.$$

A kernel as in *(ii)* is sometimes called a *conditionally negative definite kernel*.

Proof. Suppose that $k : X \times X \to \mathbb{R}$ is of positive type. Let V be the vector space of finitely supported functions $f : X \to \mathbb{R}$. Define a bilinear form (or a pseudo-inner product) on V by

$$\langle f, g\rangle = \sum_{x,y\in X} k(x,y)f(x)g(y).$$

This bilinear form is positive semi-definite, due to k being of positive type. Consider the set $E = \{f \in V \mid \langle f, f\rangle = 0\} \subseteq V$, and notice that for any $t \in \mathbb{R}$, and $f, g \in V$ we have

$$0 \leq \langle tf + g, tf + g\rangle = t^2\langle f, f\rangle + 2t\langle f, g\rangle + \langle g, g\rangle.$$

This is a quadratic polynomial in t, and since it is non-negative, it has at most one real root. This implies that the discriminant is less than or equal to zero, i.e., $(2\langle f, g\rangle)^2 - 4\langle f, f\rangle\langle g, g\rangle \leq 0$, and rearranging we get the Cauchy-Schwarz inequality. Thus we can consider the quotient $^V/_E$, and our bilinear form becomes a genuine inner product on this space. Let H be the completion of $^V/_E$ in the induced norm to get a Hilbert space. Consider the map $\varphi : X \to H$ given by $\varphi(x) = \delta_x$, where δ_x is the Dirac function at x. Then φ satisfies the condition in the theorem, since

$$\langle\varphi(x), \varphi(y)\rangle = \sum_{x,y\in X} k(x,y)\varphi(x)\varphi(y) = \sum_{x,y\in X} k(x,y)\delta_x\delta_y = k(x,y).$$

Suppose now that $k : X \times X \to \mathbb{R}$ is a normalized kernel of negative type. Let W be the vector space of finitely supported functions $f : X \to \mathbb{R}$ satisfying $\sum_{x\in X} f(x) = 0$. This time we define a bilinear form by

$$\langle f, g\rangle = -\frac{1}{2}\sum_{x,y\in X} k(x,y)f(x)g(y).$$

Again we quotient out by $F = \{f \in W \mid \langle f, f\rangle = 0\}$ and complete to get a Hilbert space H. For any fixed point $x_0 \in X$, we define $\psi : X \to H$ by $\psi(x) = \delta_x - \delta_{x_0}$. Then

$$\|\psi(x) - \psi(y)\|^2 = \langle\delta_x - \delta_y, \delta_x - \delta_y\rangle = -\frac{1}{2}k(x,x) - \frac{1}{2}k(y,y) + k(x,y) = k(x,y),$$

as wanted. $\square$

The next result tells us when it is possible to coarsely embedded a coarse space into a Hilbert space.

Corollary 1.5.4. *Let X be a coarse space. Then X can be coarsely embedded into a Hilbert space if and only if there exist a symmetric, normalized kernel k of negative type on X and non-decreasing maps $\rho_\pm : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\rho_-(t) \to \infty$ as $t \to \infty$, satisfying*

$$\rho_-(d(x,y)) \leq k(x,y) \leq \rho_+(d(x,y)),$$

for all $x, y \in \mathbb{R}_+$.

Proof. Suppose first that $f : X \to H$ is a coarse embedding of X into some (real) Hilbert space H. Then Proposition 1.2.10 tells us that there exist non-decreasing maps $\rho_\pm : \mathbb{R} \to \mathbb{R}$ such that

$$\rho_-(d_X(x,y)) \leq d_H(f(x), f(y)) \leq \rho_+(d_X(x,y).$$

Thus we have to show that $d_H(f(x), f(y)) = \|f(x) - f(y)\|^2$ is a symmetric, normalized kernel of negative type. To see this, we let $x_1, \dots, x_n \in X$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ satisfy $\sum_{i=1}^n \lambda_i = 0$. Then we see that

$$\begin{aligned}
&\sum_{i,j=1}^n \lambda_i \lambda_j \|f(x_i) - f(x_j)\|^2 \\
&\quad = \sum_{i,j=1}^n \lambda_i \lambda_j \left(\|f(x_i)\|^2 + \|f(x_j)\|^2 - 2\langle f(x_i), f(x_j)\rangle \right) \\
&\quad = \sum_{i=1}^n \lambda_i f(x_i) \left(\sum_{j=1}^n \lambda_j \right) + \sum_{j=1}^n \lambda_j f(x_j) \left(\sum_{i=1}^n \lambda_i \right) - 2 \sum_{i,j=1}^n \langle f(x_i), f(x_j)\rangle \\
&\quad = -2 \left\langle \sum_{i=1}^n \lambda_i f(x_i), \sum_{i=1}^n \lambda_i f(x_i) \right\rangle \leq 0.
\end{aligned}$$

Clearly, the map defined by $k(x,y) = \|f(x) - f(y)\|^2$ is a symmetric, normalized kernel of negative type.
On the other hand, assume that there exist a symmetric, normalized kernel $k : X \times X \to \mathbb{R}$ of negative type and non-decreasing maps $\rho_\pm : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the conditions above. From Theorem 1.5.3 we get a Hilbert space H and a function $\psi : X \to H$ satisfying $k(x,y) = \|\psi(x) - \psi(y)\|^2$. By assumption,

$$\sqrt{\rho_-(d_X(x,y))} \leq \|\psi(x) - \psi(y)\|^2 \leq \sqrt{\rho_+(d_X(x,y))},$$

showing that ψ is a coarse embedding. $\square$

There is a similar result for positive type kernels (see [25, Theorem 3.2.8]), but we will only need the result for negative type kernels.

1.6 Literature

Sections 1 through 4 of this chapter are mostly inspired by [19]. However, some of the ideas, and in particular the last section regarding kernels, are taken from [25] and [20].

Chapter 2

Graphs

In this chapter we introduce the notion of graphs. A graph is a rich structure which is studied in many areas of mathematics and in many other sciences as well. We are in particular interested in a certain type of graphs called *expanders*. Expanders first appeared in a paper of Pinsker in 1973, although some research by Larry Guth showed that Barzdin and Kolmogorov studied a concept equivalent to expanders before Pinsker. Expanders are sparse, highly connect finite graphs. This definition seems almost contradictory since sparse means that there are “few” edges, and highly connected means that there are “many” connections. We will of course make this more precise in this chapter. Even though Pinsker was the first to show the existence of expanders, the first explicit construction is due to Margulis, who constructed expanders using the notion of box spaces of residually finite groups with Kazhdan’s property (T). The reasons for the study of expander graphs are plentyful. Among others, they play an important role in various aspects of computer science and hence the strong interest to make explicit constructions.

2.1 Introduction to Graphs and expanders

We start with the definition of a graph:

Definition 2.1.1. A *graph* is an ordered pair $G = (V, E)$ consisting of a set $V(G)$ of vertices and a (multi) set $E(G)$ of edges equipped with two maps $s, r : E \to V$, called the source and range map, respectively.

For any edge $e \in E$, the image of the source map $s(e) \in V$ is the vertex at which the edge begins, and the image of the range map $r(e) \in V$ is the vertex at which the edge ends. As for any mathematical object there is a lot of standard terminology. We will go through the most important notions and state them together in a definition below. The reader acquainted with graph theory may skip directly to Definition 2.1.2 below.
In the following, let $G = (V, E)$ be a graph. The number of vertices of a graph is denoted by $|V|$ and the number of edges by $|E|$. If $|V| + |E| < \infty$, we say that G is a finite graph.

We call G *undirected* if whenever $(x, y) \in E$ it follows that $(y, x) \in E$. Equivalently, one could say that whenever $s(e) = v$, for some $e \in E$ and $v \in V$, there exists $e' \in E$ such that $r(e') = v$. A graph which is not undirected is called *directed*. A directed graph is called *oriented* if there are no symmetric edges, i.e., whenever $(x, y) \in E$ then $(y, x) \notin E$. Given an undirected graph, it is possible to make it into a directed graph, by assigning an orientation to each edge. That is, for each pair of edges $(x, y), (y, x) \in E$ of the undirected graph, we choose one pair and delete the other. For a vertex $x \in V$ we define the *neighbourhood* of x as $N_x = \{y \in G \mid (x, y) \in E\}$, and the *degree* of x as $d_x = |N_x|$. If all vertices of the graph have the same degree, we say that G is *regular*. In particular, if the degree of each vertex is d, we say that G is d-regular. A *loop* is an edge which begins and ends at the same vertex, i.e., $s(e) = r(e)$. If the map $e \mapsto (s(e), r(e)) \in V^2$ is injective, we say that G has no multiple edges. A graph is called *simple* if it contains no loops and no multiple edges. If the graph contains loops or multiple edges (or both), it is called a *multigraph*. Two vertices $x, y \in V$ are called *adjacent* if there exists an edge $e \in E$ such that $s(e) = x$ and $r(e) = y$. A *path* in a graph is a list of distinct adjacent vertices, or, equivalently a string of distinct edges $e_1 e_2 \cdots e_n$ with $r(e_i) = s(e_{i+1})$, $i = 1, \dots, n-1$. A path consisting of l edges is said to be a path of length l. A *cycle* of length l, also called an *l-cycle*, is a path of length l which starts and ends at the same vertex. We say that a graph is *connected* if there is a path connecting any two vertices. By convention, a graph consisting of a single vertex is connected. A *component* is a maximal connected subset of G. Note that if G is connected, then G has exactly one component. To sum up the most important notions, we have the following definition:

Definition 2.1.2. Let $G = (V, E)$ be a graph. The graph G is called

- *finite*, if $|V| + |E| < \infty$.
- *undirected*, if whenever $(x, y) \in E$, then $(y, x) \in E$.
- *oriented*, if it is directed and whenever $(x, y) \in E$, then $(y, x) \notin E$.
- *regular*, if all vertices have the same degree.
- *simple*, if it contains no loops, nor multiple edges.
- *connected*, if any two vertices are connected.

We continue by introducing a little more notation. For subsets $S, T \subset V$, we denote by $E(S, T) = \{(x, y) \in E \mid x \in S, y \in T\} = \{E \cap (S \times T)\}$ the set of edges from S to T. Note that $E(S, T)$ is a set of directed edges. By $E(S)$ we denote the set of edges contained in S, i.e., $E(S) = \{(x, y) \in E \mid x, y \in S\}$. Let $\overline{S} = V \setminus S$ denote the complement of S[†] and $\partial S = E(S, \overline{S})$ the boundary set of edges of S. Note that $\partial S = \partial \overline{S}$, whenever G is

[†] Note that we here use the bar notation to denote complement and not the closure. We will later use the bar to also denote the closure, but it should always be clear from the context if it means closure or complement.

simple. It is in fact very easy to turn a graph into a metric space. To see this, let $G = (V, E)$ be a connected graph and let $d : G \times G \to \mathbb{R}$ be defined by

$$d(x, y) = \min\{l(\pi) \in \mathbb{N}_0{}^{\dagger} \mid \pi \text{ is a path connecting } x \text{ and } y\}.$$

Clearly d is a metric and it is known as the *path metric*. An important consequence of this is that it makes sense to consider the *coarse disjoint union* of a sequence of (finite) graphs. We proceed by defining the expansion rate of a graph:

Definition 2.1.3. Let $G = (V, E)$ be a graph, and let X be a component of G. We define the (discrete isoperimetric) *Cheeger constant* as

$$h(G) = \inf_{S \subseteq X} \left(\frac{|\partial S|}{\min\{|S|, |\overline{S}|\}} \right),$$

where the infimum runs over all non-empty subsets of the components of G.

It is called the *discrete isoperimetric Cheeger constant* because it is a discrete version of the isoperimetric Cheeger constant for compact Riemannian manifolds. We say that the graph G is an *expander* graph if there exists $\varepsilon > 0$ such that $h(G) \geq \varepsilon$. If G is a finite, d-regular expander for some $\varepsilon > 0$, we say that G is an (n, d, ε)-expander, where n is the number of vertices in G.

Definition 2.1.4. A sequence of finite connected d-regular graphs $\{G_i = (V_i, E_i)\}$ is called a *family of expanders* if $|V_i| \to \infty$, as $i \to \infty$, and there exists $\varepsilon > 0$ such that $h(G_i) \geq \varepsilon$, for all $i \in \mathbb{N}$.

This means that the Cheeger constants of the finite graphs are uniformly bounded away from zero. Note that if $G = \bigsqcup G_i$ is the coarse disjoint union of finite connected d-regular graphs G_i, with $|V_i| \to \infty$ as $i \to \infty$, and there exists some $\varepsilon > 0$ such that $h(G_i) \geq \varepsilon$ for all i, then G is a family of expanders. We will use expanders to construct metric spaces which do not coarsely embed into any Hilbert space. However, we need more properties of graphs before we can carry out this construction.

2.2 Regular graphs

Two useful tools in the study finite graphs (and expanders) are the associated *adjacency matrix* A_G and the *incidence matrix* B_G.

Definition 2.2.1. Let $G = (V, E)$ be a finite graph on n vertices. The adjacency matrix A_G, or just A if no ambiguity will arise, is the $n \times n$ matrix whose entries are defined by

$$a_{i,j} = \text{the number of edges from vertex } i \text{ to } j, \text{ where } 1 \leq i, j \leq n,$$

and the normalized adjacency matrix is $\tilde{A}_G = \frac{1}{d_x} A_G$.

†We use $\mathbb{N}_0$ to denote the set $\{0\} \cup \mathbb{N}$.

Note that if G is an undirected graph, then A_G is symmetric. It is a well known fact that any real and symmetric $n \times n$ matrix has n real (not necessarily distinct) eigenvalues $\lambda_0(A) \geq \lambda_1(A) \geq \ldots \geq \lambda_{n-1}(A)$ with corresponding eigenvectors $v_0, \ldots, v_{n-1}$. If we normalize the eigenvectors and recall that v_i is orthogonal to v_j, whenever $\lambda_i \neq \lambda_j$, we see that the eigenvectors form an orthonormal basis for the vector space associated with the (different) eigenvalues of A_G. The set of eigenvalues of A_G is referred to as the *spectrum* of G. In the case when G is d-regular, it follows from the Perron-Frobenius theorem that all the eigenvalues are contained in $[-d, d]$, and the greatest eigenvalue is $\lambda_0 = d$ with corresponding eigenvector $\frac{1}{\sqrt{n}}(1, \ldots, 1)$. We will be interested in the second greatest eigenvalue λ_1, since this eigenvalue tells us a lot about the structure of the graph. The number

$$\lambda_0(A) - \lambda_1(A) = d - \lambda_1,$$

is referred to as the *spectral gap*. It is worth noting that if G is connected, then $\lambda_0 > \lambda_1$ and hence $d - \lambda_1 > 0$, i.e., the spectral gap is non-zero.

Definition 2.2.2. Let $G = (V, E)$ be a finite directed graph on n vertices and m edges. We define the incidence matrix B_G, or just B if the underlying graph is clear, as the $n \times m$ matrix defined by

$$b_{i,j} = \begin{cases} 1 & \text{if } s(j) = i \neq r(j), \\ -1 & \text{if } r(j) = i \neq s(j), \\ 0 & \text{else.} \end{cases}$$

If G is undirected, the incidence matrix is obtained by replacing each -1 with 1 and loops get the value 2.

Following ideas from [12], we use the incidence matrix to construct the discrete version of the *Laplace operator* $\triangle$:

The discrete Laplace operator

The Laplace operator $\triangle$ is usually defined as the divergence of the gradient of a differentiable function f. Thus we start by defining the discrete version of the gradient $\nabla(f)$, and then we define the divergence. In the following, let $G = (V, E)$ be a finite undirected graph, and assign an orientation to each of the edges turning G into an oriented graph. The specific orientation does not matter as long it is stays fixed.

The usual interpretation of the gradient is a generalization of the derivative, and hence it measures the changes of a function on a surface or manifold. We want a similar measurement in the discrete case, so we want to measure the change at each vertex. To do this, let $f : V \to \mathbb{R}$ be a function of the vertices of G. We define the gradient $\nabla(f)$ to be the operator that maps f, viewed as a row vector, to fB_G. Given an edge $e \in E$ which goes from a vertex x to a vertex y with respect to the given orientation, i.e., with $s(e) = x$ and $r(e) = y$, we see that

$$(fB_G)(e) = f(x) - f(y).$$

Now we are ready to define the discrete version of divergence. The divergence measures the accumulated outflow at a given point, and this should, of course, be reflected in the discrete version. Let $g : E \to \mathbb{R}$ be a function of the edges of G, and define the divergence as the operator mapping g, considered as a column vector, to $B_G g$. Thus we see that given a vertex $x \in V$, we get

$$(Bg)(x) = \sum_{s(e)=x} g(e) - \sum_{r(e)=x} g(e).$$

We now define the discrete Laplace operator for finite, connected d-regular graphs by the formula $\triangle = BB^T = dI - A$. That is, given a function $f : V \to \mathbb{R}$, the Laplacian maps f to $\triangle f = BB^T f$. For $x, y \in V$, the entries in $\triangle$ are given by

$$l_{x,y} = \begin{cases} -1 & \text{if } (x,y) \in E, \\ d_x & \text{if} x = y, \\ 0 & \text{else}, \end{cases}$$

where $l_{x,y}$ are the entries of $\triangle$. This means that the diagonal consists of the degree of each vertex, and the other entries are either 0 or -1. In particular, if G is d-regular, the degree of each vertex is d and $\triangle = dI - A$. Furthermore, since $\triangle$ is real-valued and symmetric, $\triangle$ has real eigenvalues. We may also write $\triangle$ in quadratic form, yielding

$$f^T \triangle f = \sum_{x \in V} \sum_{y \in V} l_{x,y} f(x) f(y) = \sum_{(x,y) \in E} (f(x) - f(y))^2,$$

from which we see that $\triangle$ is positive semi-definite. This implies that all the eigenvalues are non-negative, and since $\triangle = dI - A$, the eigenvalues are contained in $[0, 2d]$. The eigenvalue zero is obtained if and only if $f : V \to \mathbb{R}$ is a constant function. Furthermore, the spectral gap $d - \lambda_1(A)$ is the smallest non-zero eigenvalue of $\triangle$. An interesting property for the expansion of a graph is the close relation between the spectral gap and the Cheeger constant. This result is known as the Cheeger inequalities, and it is due to Tanner [24], Dodziuk [7], and independently to Alon and Milman [1]. The proof we give here follows [12].

Theorem 2.2.3. *Let G be finite, connected d-regular graph. Then*

$$\frac{1}{2}(d - \lambda_1(G)) \leq h(G) \leq \sqrt{2d(d - \lambda_1(G))}. \tag{2.1}$$

We will prove the theorem in two steps; one for each inequality. But before we prove the inequalities, we need one last concept, namely the (real) *Rayleigh quotient.*

Definition 2.2.4. Given a real symmetric matrix A and a non-zero vector x, the *Rayleigh quotient* $R(A, x)$ is defined as

$$R(A, x) = \frac{x^T A x}{x^T x}.$$

We note that the Rayleigh quotient satisfies $R(A, x) \in [\lambda_{n-1}(A), \lambda_0(A)]$ with

$$R(A, v_0) = \lambda_0 \geq R(A, x) \geq R(A, v_{n-1}) = \lambda_{n-1},$$

where $\lambda_0 \geq \ldots \geq \lambda_{n-1}$ are the eigenvalues of A and v_i are the corresponding eigenvectors. Using the Rayleigh quotients, we can now show the inequalities.

Proof (of the lefthand side of (2.1)). Rewriting the inequality we get $\lambda_1 \geq d - 2h$. To show this inequality, let $S \subseteq V$ be a proper subset of the vertices, i.e., S is non-empty and $S \neq V$. We know that the first eigenvector of a d-regular graph is the vector $(1, \ldots, 1)$, so we need to find a non-zero vector f orthogonal to $(1, \ldots, 1)$, such that the Rayleigh quotient $R(A_G, f)$ satisfies $\frac{f^T A_G f}{\|f\|^2} \geq d - 2h$. To do this, we consider f given by $f(x) = 1_S(x)|\overline{S}| - 1_{\overline{S}}(x)|S|$, where 1_S is the characteristic function of S. Then we see that the norm of f is given by $\|f\|^2 = |S||\overline{S}|^2 + |\overline{S}||S|^2 = n|S||\overline{S}|$, and

$$f^T A_G f = \sum_{i=1}^{n} f(x_i) \sum_{j=1}^{n} a_{i,j} f(x_j) = 2(|E(S)||\overline{S}|^2 + |E(\overline{S})||S|^2 - |S||\overline{S}||\partial S|).$$

Since G is d-regular, we know that each vertex has exactly d edges, and thus we can write

$$2|E(S)| = d|S| - |\partial S| \qquad\qquad 2|E(\overline{S})| = d|\overline{S}| - |\partial S|.$$

Substituting this into $f^T A_G f$ shows us that

$$f^T A_G f = (d|S| - |\partial S|)|\overline{S}|^2 + (d|\overline{S}| - |\partial S|)|S|^2 - 2|S||\overline{S}||\partial S| = d|S||\overline{S}|n - |\partial S|n^2.$$

Thus we see that

$$\frac{f^T A_G f}{\|f\|^2} = \frac{nd|S||\overline{S}| - n^2|\partial S|}{n|S||\overline{S}|} = d - \frac{n|\partial S|}{|S||\overline{S}|} \geq d - 2\frac{|\partial S|}{\min\{|S|, |\overline{S}|\}} \geq d - 2h,$$

since $\max\{|S|, |\overline{S}|\} \geq \frac{n}{2}$ and h is the infimum over all such sets S. □

The proof of the other inequality is given in more generality in the next section. Most of what we have done so far has been for finite connected d-regular graphs, but we can generalize this to any (possible infinite) graph with multiple edges, loops and any number of edges between the vertices, with the only assumptions that G is locally finite and has finite components. To do this, we change our point of view to graphs viewed as ℓ^2-spaces.

2.3 Graphs as ℓ^2-spaces

Throughout this section let $G = (V, E)$ be any locally finite graph, i.e., $d_x < \infty$, for all $x \in V$. We consider the *real* $\ell^2(V)$ space equipped with the measure $\mu(\{x\}) = d_x$, and standard inner product, i.e., given two functions f and g on V, we set

$$\langle f, g \rangle_V = \sum_{x \in V} d_x f(x) \overline{g(x)}.$$

In connection with the normalized adjacency matrix, we define the local average operator $A : \ell^2(V) \to \ell^2(V)$ by

$$(Af)(x) = \frac{1}{d_x} \sum_{y \in N_x} f(y).$$

We see that A is self-adjoint by the following computations

$$\begin{aligned}
\langle Af, g\rangle_V &= \sum_{x\in V} d_x \left(\frac{1}{d_x} \sum_{y\in N_x} f(y) \right) \overline{g(x)} \\
&= \sum_{x\in V} \sum_{y\in N_x} f(y)\overline{g(x)} \\
&= \sum_{y\in V} d_y f(y) \left(\frac{1}{d_y} \sum_{x\in N_y} \overline{g(y)} \right) \\
&= \langle f, Ag\rangle_V,
\end{aligned}$$

where we have used the assumption of locally finiteness to manipulate the sums. Using the Cauchy-Schwarz inequality twice, we further see that

$$\begin{aligned}
|\langle Af, g\rangle_V| &= \left| \sum_{x\in V} d_x \left(\frac{1}{d_x} \sum_{y\in N_x} f(y) \right) \overline{g(x)} \right| \\
&= \left| \sum_{y\in V} d_y f(y) \left(\frac{1}{d_y} \sum_{x\in N_y} \overline{g(x)} \right) \right| \\
&\leq \sum_{y\in V} |f(y)| \sqrt{d_y} \left(\sum_{x\in N_y} |g(x)|^2 \right)^{1/2} \\
&\leq \left(\sum_{y\in V} |f(y)|^2 d_y \right)^{1/2} \left(\sum_{y\in V} \sum_{x\in N_y} |g(y)|^2 \right)^{1/2} \\
&= \|f\|\|g\|,
\end{aligned}$$

showing that $\|A\| \leq 1$. Note that $Af = f$ if and only if f is constant on each component of G. We now generalize the discrete Laplacian to locally finite graphs by setting

$$\triangle := I - A.$$

Note that we here use the local average operator in place of the adjacency matrix. As before, the Laplacian is a self-adjoint, positive operator, and we have the norm estimate $\|\triangle\| = \|I - A\| \leq \|I\| + \|A\| \leq 1 + 1 = 2$. Furthermore, we note that the kernel of $\triangle$ is spanned by the characteristic functions of the components of G, which implies that if G

is connected (consists of exactly one component), then zero is an eigenvalue of multiplicity one. Note that this is the same result as in the previous section. The orthogonal complement of the kernel, denoted by $\ell_0^2(V)$, is the space of so-called balanced function. That is, for each component $X \subseteq G$ we have

$$\ell_0^2(V) = \{f \in \ell^2(V) \mid \langle f, 1_X \rangle_V = 0\} = \{f \in \ell^2(V) \mid \sum_{x \in X} d_x f(x) = 0\}.$$

Theorem 2.3.1. *Let $h(G)$ be the Cheeger constant of G (cf. Definition 2.1.3). Then*

$$\frac{1}{2}\lambda \leq h(G) \leq \sqrt{2k\lambda},$$

where λ is the infimum of the positive eigenvalues of $\triangle$, and $k = \max\{d_x \mid x \in V\}$. Note that if G is d-regular, then $k = d$.

Proof. We start by proving that $\lambda \leq 2h$. Let $X \subseteq G$ be a component and $S \subseteq X$ a proper subset. Consider $f \in \ell_0^2(V)$ given by $f(x) = \alpha 1_S(x) + \beta 1_{\overline{S}}(x)$, for $\alpha, \beta \in \mathbb{R}$. Since we want $f \in \ell_0^2(V)$, it must satisfy the following equation

$$0 = \langle f, 1_X \rangle_V = \sum_{x \in X} d_x(\alpha 1_S(x) + \beta 1_{\overline{S}}(x)) = (2|E(S)| + |\partial S|)\alpha + (2|E(\overline{S})| + |\partial \overline{S}|)\beta,$$

from which we see that

$$\frac{\beta}{\alpha} = \frac{-(2|E(S)| + |\partial S|)}{2|E(\overline{S})| + |\partial \overline{S}|}.$$

For ease of notation, we will follow the notation of [23]. Thus, let $e(S) = |E(S, G)| = 2|E(S)| + |\partial S|$ and we get $\frac{\beta}{\alpha} = \frac{-e(S)}{e(\overline{S})}$. Since f is not constant, it follows that

$$\lambda \leq \frac{\langle \triangle f, f \rangle_V}{\langle f, f \rangle_V}, \tag{2.2}$$

since λ is the infimum over all the non-zero eigenvalues. For $x \in S$, we have

$$\begin{aligned}
\triangle f(x) &= \alpha - (Af)(x) \\
&= \alpha - \frac{1}{d_x}\left(\sum_{y \in N_x \cap S} \alpha + \sum_{y \in N_x \cap \overline{S}} \beta\right) \\
&= \alpha - \frac{|N_x \cap S|}{d_x}\alpha - \frac{|N_x \cap \overline{S}|}{d_x}\beta \\
&= \frac{|N_x \cap \overline{S}|}{d_x}(\alpha - \beta).
\end{aligned}$$

Similar calculations show that for $x \in \overline{S}$,

$$\triangle f(x) = \frac{|N_x \cap S|}{d_x}(\beta - \alpha).$$

From this we can now calculate $\langle \triangle f, f\rangle_V$. Note that we only need to consider $x \in X$ since f is zero for x outside of X. We have, accordingly,

$$\begin{aligned}
\langle \triangle f, f\rangle_V &= \sum_{x\in X} d_x \triangle f(x)\overline{f(x)} \\
&= \sum_{x\in S} d_x \triangle f(x)\overline{f(x)} + \sum_{x\in \overline{S}} d_x \triangle f(x)\overline{f(x)} \\
&= \sum_{x\in S} |N_x \cap \overline{S}|(\alpha-\beta)\alpha + \sum_{x\in \overline{S}} |N_x \cap \overline{S}|(\beta-\alpha)\beta \\
&= |\partial S|(\alpha^2 - \alpha\beta) + |\partial \overline{S}|(\beta^2 - \alpha\beta) \\
&= |\partial S|(\alpha-\beta)^2.
\end{aligned}$$

Now we are ready to calculate the denominator $\langle f, f\rangle_V$ of (2.2). Indeed,

$$\begin{aligned}
\langle f, f\rangle_V &= \sum_{x\in X} d_x f(x)\overline{f(x)} \\
&= \sum_{x\in S} d_x \alpha^2 + \sum_{x\in \overline{S}} d_x \beta^2 \\
&= (2|E(S)| + |\partial S|)\alpha^2 + (2|E(\overline{S})| + |\partial \overline{S}|)\beta^2 \\
&= e(S)\alpha^2 + e(\overline{S})\beta^2.
\end{aligned}$$

Combining the above we get

$$\lambda \le \frac{\langle \triangle f, f\rangle_V}{\langle f, f\rangle_V} = \frac{(\alpha-\beta)^2}{e(S)\alpha^2 + e(\overline{S})\beta^2}|\partial S| = \frac{\left(1-\frac{\beta}{\alpha}\right)^2}{e(S) + e(\overline{S})\frac{\beta^2}{\alpha^2}}|\partial S|.$$

Substituting our value of $\frac{\alpha}{\beta}$ into this, we get

$$\lambda \le \frac{|\partial S|\left(1+\frac{e(S)}{e(\overline{S})}\right)^2}{e(S) + e(\overline{S})\left(\frac{-e(S)}{e(\overline{S})}\right)^2} = \frac{|\partial S|(e(S)+e(\overline{S}))^2}{e(S)e(\overline{S})^2 + e(S)^2 e(\overline{S})} = |\partial S|\frac{e(S)+e(\overline{S})}{e(S)e(\overline{S})},$$

from which we see that

$$\lambda \le \frac{|\partial S|}{e(S)} + \frac{|\partial S|}{e(\overline{S})} \le 2\frac{|\partial S|}{\min\{e(S), e(\overline{S})\}} \le 2\frac{|\partial S|}{\min\{|S|, |\overline{S}|\}} = 2h,$$

which proves the first inequality.
For the second inequality, let f be an eigenfunction of $\triangle$ with corresponding eigenvalue $\lambda_f \le \lambda + \varepsilon$, for some $\varepsilon > 0$, i.e., $\triangle f(x) = \lambda_f f(x)$, for all $x \in V$. We may assume that $f : G \to \mathbb{R}$ is supported on a component $X \subseteq G$ with $|X| = n$. We now define

$$S^+ = \{x \in V \mid f(x) > 0\} = \{x \in X \mid f(x) > 0\}.$$

Note that we may assume that $|S^+| \leq \frac{n}{2}$ by choosing $-f$, if necessary. We now define $g(x) = 1_{S^+}(x)f(x)$, for $x \in V$. Then we see that for $x \in S^+$,

$$\begin{aligned}
\triangle f(x) &= f(x) - \frac{1}{d_x}\sum_{y\in N_x} f(y) \\
&= g(x) - \frac{1}{d_x}\sum_{y\in N_x\cap S^+} f(y) - \frac{1}{d_x}\sum_{y\in N_x\cap\overline{S^+}} f(y) \\
&= \triangle g(x) + \frac{1}{d_x}\sum_{y\in N_x\cap\overline{S^+}} (-f(y)) \\
&\geq \triangle g(x).
\end{aligned}$$

By construction, $g(x) = 0$ for $x \notin S^+$, and since f is an eigenfunction of $\triangle$, we get

$$\lambda_f \langle g, g\rangle_V = \lambda_f \sum_{x\in S^+} d_x g(x)^2 = \sum_{x\in S^+} d_x \triangle f(x) g(x) \geq \sum_{x\in S^+} d_x \triangle g(x) g(x) = \langle \triangle g, g\rangle_V,$$

which shows that $\lambda + \varepsilon \geq \lambda_f \geq \frac{\langle \triangle g, g\rangle_V}{\langle g,g\rangle_V}$. Now we want to estimate $\langle \triangle g, g\rangle_V$ in a different way. We start by defining

$$M = \sum_{x\in V} d_x \frac{1}{d_x}\sum_{y\in N_x} |g(x)^2 - g(y)^2| = \sum_{x\in V}\sum_{y\in N_x} |g(x)^2 - g(y)^2| = \sum_{(x,y)\in E} |g(x)^2 - g(y)^2|.$$

If we recall the definition of the discrete gradient, we note that $M = \nabla g^2$. Using the Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned}
M &= \sum_{(x,y)\in E} |g(x)^2 - g(y)^2| \\
&= \sum_{(x,y)\in E} |g(x) + g(y)||g(x) - g(y)| \\
&\leq \left(\sum_{(x,y)\in E} (g(x) + g(y))^2\right)^{1/2} \left(\sum_{(x,y)\in E} (g(x) - g(y))^2\right)^{1/2}.
\end{aligned}$$

We will calculate these sums one by one. First we have

$$\sum_{(x,y)\in E} (g(x) + g(y))^2 \leq 2\sum_{(x,y)\in E} (g(x)^2 + g(y)^2) = 2\sum_{x\in V} d_x g(x)^2 + 2\sum_{y\in V} d_y g(y)^2 = 4\langle g, g\rangle_V,$$

while for the second one we have

$$\begin{aligned}
&\sum_{(x,y)\in E}(g(x)-g(y))^2\\
&\quad=\sum_{(x,y)\in E}(g(x)^2+g(y)^2-2g(x)g(y))\\
&\quad=\sum_{(x,y)\in E}(g(x)[g(x)-g(y)])-\sum_{(x,y)\in E}(g(y)[g(x)-g(y)])\\
&\quad\le\sum_{x\in V}kg(x)\sum_{y\in N_x}(g(x)-g(y))-\sum_{y\in V}kg(y)\sum_{x\in N_y}(g(x)-g(y))\\
&\quad=k\left(\sum_{x\in V}d_xg(x)\frac{1}{d_x}\sum_{y\in N_x}(g(x)-g(y))-\sum_{y\in V}d_yg(y)\frac{1}{d_y}\sum_{x\in N_y}(g(x)-g(y))\right)\\
&\quad=2k\langle\triangle g,g\rangle_V.
\end{aligned}$$

Putting all this together, we have shown that

$$M^2\le 4\langle g,g\rangle_V 2k\langle\triangle g,g\rangle_V\le 8k\lambda_f\langle g,g\rangle_V^2\le 8k(\lambda+\varepsilon)\langle g,g\rangle_V^2,$$

where we have used the estimate that $\lambda_f\langle g,g\rangle_V\ge\langle\triangle g,g\rangle_V$. Recalling from vector analysis that $\nabla(fg)=f\nabla g+g\nabla f$, we get that $\nabla g^2=2g\nabla g$. Motivated by this, we will estimate M in a different way. First recall that $|S^+|\le\frac{n}{2}$, which means that g takes at most $\frac{n}{2}$ different values. Denote these values by $0=\alpha_0<\alpha_1<\ldots<\alpha_r$, and let $L_i=\{x\in V\mid g(x)\ge\alpha_i\}$ be the level set of the vertices. Note that $L_0=V$ and $|L_i|\le\frac{n}{2}$ for $i>0$. Last we let $\delta_{x,y}$ denote the number of edges between vertices x and y. Then we can rewrite, for all $x,y\in V$,

$$M=2\sum_{g(x)>g(y)}\delta_{xy}(g(x)^2-g(y)^2)=2\sum_{i=1}^{r}\sum_{g(x)=\alpha_i}\sum_{g(y)<\alpha_i}\delta_{xy}(g(x)^2-g(y)^2).$$

Next note that $L_0\supseteq L_1\supseteq\ldots\supseteq L_r$, hence $\partial L_i=\{(x,y)\in E\mid x\in L_i, y\in L_j, j<i\}$. Furthermore, we note that for any edge $(x,y)\in E$ with $g(x)=\alpha_i$ and $g(y)=\alpha_j$, $i>j$, we have

$$g(x)^2-g(y)^2=\alpha_i^2-\alpha_j^2=(\alpha_i^2-\alpha_{i-1}^2)+(\alpha_{i-1}^2-\alpha_{i-2}^2)+\cdots+(\alpha_{j+1}^2-\alpha_j^2),$$

which means that $(x,y)\in\partial L_i$. This allows us to rewrite

$$M=2\sum_{i=1}^{r}|\partial L_i|(\alpha_i^2-\alpha_{i-1}^2).$$

Now note that the definition of the Cheeger constant gives us $|\partial L_i|\ge h|L_i|$ for $i>0$ (since $\frac{|\partial L_i|}{|L_i|}\ge h$), which leads to

$$M\ge 2h\sum_{i=1}^{r}|L_i|(\alpha_i^2-\alpha_{i-1}^2)=2h(|L_r|\alpha_r^2+\sum_{i=1}^{r-1}\alpha_i^2(|L_i|-|L_{i+1}|)).$$

It is clear that $x \in L_i \setminus L_{i+1}$ if and only if $g(x) = \alpha_i$, and thus we get

$$M \leq 2h \sum_{i=1}^{r-1} \sum_{g(x)=\alpha_i} d_x \alpha_i^2 + 2h \sum_{g(x)=\alpha_r} d_x \alpha_r^2 = 2h \sum_{x \in V} d_x g(x)^2 = 2h \langle g, g \rangle_V,$$

and finally putting everything together, we see that

$$2h\langle g, g\rangle_V \leq A \leq 2\sqrt{2k(\lambda + \varepsilon)}\langle g, g\rangle_V$$

for all $\varepsilon > 0$, which concludes the proof. □

Having established these inequalities, we are ready to relate coarse geometric concepts to graphs. In particular, we are interested in the relationship between property (A) and families of expander graphs.

2.4 Coarse embeddings and property (A)

In this section we investigate geometric properties of graphs in terms of coarse embeddings and property (A). We start with a few examples of graphs with property (A).

Example 2.4.1. Any finite graph has property (A).

Example 2.4.2. Any tree† $T = (V, E)$ has property (A) .

Proof. Let $T = (V, E)$ be an infinite tree, and let $\varepsilon, R > 0$ be given. Fix a vertex $x_0 \in V$, and let γ_0 be an infinite path $x_0, x_1, \ldots$ starting at x_0 with $x_i \neq x_j$, for all $i \neq j$. Such a path is sometimes called a *geodesic ray*. For each $x \in V$, let γ_x denote the (unique) path starting at x and following γ_0 for an infinite distance. For each $x \in V$, we then define

$$A_x = \{(y, 1) \in V \times \mathbb{N} \mid y \in \gamma_x, d(x, y) \leq 3R/\varepsilon + 1\}.$$

Note that $3R/\varepsilon \leq |A_x| \leq 3R/\varepsilon + 1$, for all $x \in V$. Furthermore, we see that $|A_x \oplus A_y| \leq 2R$, whenever $d(x, y) \leq R$, which implies that $|A_x \cap A_y| \geq 3R/\varepsilon - R$. Thus, we have the following estimate

$$\frac{|A_x \oplus A_y|}{|A_x \cap A_y|} \leq \frac{2R}{3R/\varepsilon - R} < \varepsilon,$$

showing that $T = (V, E)$ has property (A). □

In particular, $\mathbb{Z}$ viewed as a graph with the usual metric $d(n, m) = |m - n|$, is an infinite tree, and hence it has property (A). Note also that Theorem 1.4.3 tells us that each of these graphs are embeddable into the Hilbert space $H = \ell^2(V)$, where V is the vertex set of the graph in question. The example showing that infinite trees have property (A) becomes even more interesting, when we begin the study of groups. This is because the *Cayley graph*

† A tree is an undirected, connected graph, in which any two vertices can be connected by *exactly* one path.

associated to a free group on n generators is an infinite tree.
The remainder of this section is devoted to show that there exist graphs without property (A). We begin with the following lemma.

Lemma 2.4.3. *Let $G = (V, E)$ be a finite connected d-regular graph with $\lambda_1(G) \geq \lambda > 0$. Let $f : V \to \mathbb{R}$ be a map from the vertices of G, then*

$$\sum_{x \in V} \|f(x) - m(f)\|^2 \leq \frac{1}{\lambda} \sum_{(x,y) \in E} \|f(x) - f(y)\|^2,$$

where $m(f) = \frac{1}{|V|} \sum_{x \in V} f(x)$ is the mean of f.

Comparing m to the local averaging operator defined in Section 2.3 we see that m is a global average.

Proof. Let $\lambda_1 \leq \lambda_2, \ldots, \lambda_n$ denote the eigenvalues of the Laplace operator $\triangle$. Since $\ell^2(V)$ is finite dimensional, we can decompose $\ell^2_0(V)$ (the orthogonal complement of $\ker(\triangle)$), into the direct sum $\bigoplus_{i=1}^m E_i$ of eigenspaces E_i for some $m \leq n$. This means that we can write any element $f \in \ell^2(V)$ as $f_1 + f_2 + \cdots + f_m$, where each f_i belongs to E_i. Furthermore, we have

$$\langle \triangle f, f \rangle = \sum_{i=1}^{m} \langle \lambda_i f, f \rangle = \sum_{i=1}^{m} \lambda_i \|f\|^2 \geq \lambda \|f\|^2,$$

by assumptions on λ. If we consider $m(f)$ as a constant map $V \to \mathbb{R}$, we have that $m(f) \in \ker(\triangle)$, which implies that $f - m(f) \in \ell^2_0(V)$. Thus,

$$\lambda \sum_{x \in V} |f(x) - m(f)|^2 = \lambda \|f - m(f)\|^2 \leq \langle \triangle(f - m(f)), f - m(f) \rangle = \langle \triangle f, f \rangle.$$

However, $\langle \triangle f, f \rangle = \sum_{(x,y) \in E} (f(x) - f(y))^2 = \sum_{(x,y) \in E} \|f(x) - f(y)\|^2$, yielding the desired result. □

This lemma tells us that the total variance of f from its mean value is bounded by λ^{-1} multiplied by the sum of variances over each edge. With this in mind, we are ready to show the main theorem of this section.

Theorem 2.4.4. *The coarse disjoint union of expanders $\{X_n = (V_n, E_n)\}_{n \in \mathbb{N}}$ do not coarsely embed into any Hilbert space.*

Proof. Let $X = \bigsqcup X_n$ be the coarse disjoint union of a family of d-regular expanders $\{X_n = (V_n, E_n)\}_{n \in \mathbb{N}}$. Let $\lambda_n = \lambda_n(X)$ be the spectral gap for each of the graphs satisfying $\lambda_n \geq \lambda$, for some $\lambda > 0$. Assume, for contraction, that $f : X \to H$ is a coarse embedding into some Hilbert space H. Recall that a coarse embedding is both proper and bornologous. As f is proper, we know that for any vertex $x \in V$ and $r > 0$, there exists

$s > 0$ such that for all $y \in V$ with $d_H(f(x), f(y)) < r$, we have $y \in B_X(x, s)$. This means that there exists $k \in \mathbb{N}$ such that at most $k = |B_X(x, s)|$ vertices gets mapped into $B_H(f(x), r)$. Furthermore, due to f being bornologous, there exists $c > 0$ such that for any two vertices $x, y \in V$ with $(x, y) \in E$, we have $d_H(f(x), f(y)) < c$, or, equivalently $\|f(x) - f(y)\|_H < c$. Let f_n be the restriction of f to V_n for each $n \in \mathbb{N}$, and translate the origin such that

$$\sum_{x \in V_n} f_n(x) = 0,$$

which implies that the average $m(f)$ becomes trivial. Using Lemma 2.4.3, we see

$$\begin{aligned}\sum_{x \in V_n} \|f_n(x)\|^2 &= \sum_{x \in V_n} \|f_n(x) - m(f_n)\|^2 \\ &\leq \frac{1}{\lambda} \sum_{(x,y) \in E_n} \|f_n(x) - f_n(y)\|^2 \\ &\leq \frac{c^2 |V_n|}{\lambda}.\end{aligned}$$

This shows that at most $|V_n|/2$ terms on the left hand side can be larger than $2c^2/\lambda$ in absolute value, which means that f_n maps at least $|V_n|/2$ vertices into the ball $B_H(0, 2c^2/\lambda)$. This implies that $k_{2c^2/\lambda}$ is unbounded, which is a contradiction. Thus such an f cannot exist. $\square$

We deduce the following interesting corollary.

Corollary 2.4.5. *The coarse disjoint union of expanders does not have property (A).*

We now turn our attention to random graphs, since, quite surprisingly, almost every d-regular graph on n vertices has Cheeger constant uniformly bounded away from zero, as $n \to \infty$. The next section is devoted to proving this result.

2.5 Random graphs

We end this chapter with the study of random graphs. Our goal is to show that *almost every* d-regular graph with $d \geq 3$ on $n > d$ vertices has Cheeger constant strictly greater than zero. This means that almost every d-regular graph is an expander graph. In order to show this, we need the notion of *random* graphs. Let $\mathcal{G}_{n,m}$ denote the family of all (labelled) graphs on n vertices and m edges. A random graph $G \in \mathcal{G}_{n,m}$ on n vertices and m edges is obtained by taking the empty graph on n vertices, and then connecting the vertices by edges at random, where any two vertices have equal probability of being connected. Now, let $G(n, d)$ denote a random d-regular graph on n vertices chosen uniformly at random from the set of all d-regular graphs on n vertices. Note that such a graph has nd edges. Our goal is to show that

$$\lim_{n \to \infty} P[h(G(n, d)) > 0] = 1.$$

In the following we assume that $dn = m$ is even. In order to show results concerning random d-regular graphs, we need a way to take samples uniformly at random from the set of all d-regular graphs. To do this, we introduce Bollobás' *configuration model*.

The Bollobás configuration model

For fixed $d, n \in \mathbb{N}$ with $3 \leq d < n$ and $nd = m$ even, we let $W = \bigcup_{i=1}^{n} W_i$ be a fixed set of m vertices, where $|W_i| = d$, for each $i = 1, \ldots, n$. A configuration M is a partition of W into m pairs of vertices, called a *matching*. Note that we can view such a matching M as a subset of $W \times W$, so it makes sense to talk about the elements of$M$ as edges. See Figure 2.1 for an illustration of a configuration with $n = 4$ and $d = 3$. The number of such matchings $N(m)$ of a set with m elements is

$$N(m) = (m-1)(m-3)\cdots(3)(1) := (m-1)!!,$$

from which it follows that

$$N(m) = \frac{m!}{2^{m/2}(m/2)!}.$$

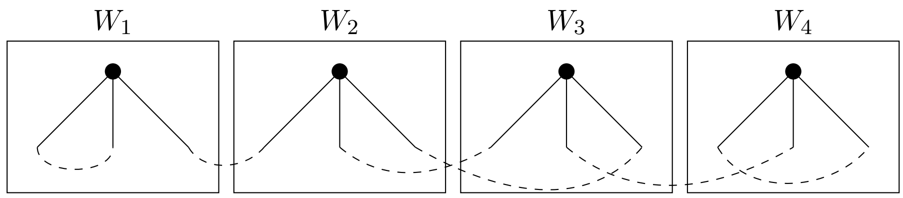

Figure 2.1: A configuration for a 3-regular graph on 4 vertices.

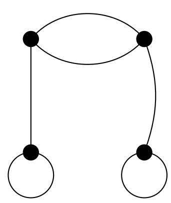

Figure 2.2: The 3-regular graph arising from a matching of the configuration in Figure 2.1.

Recall Sterling's Formula,

$$m! \sim \sqrt{2\pi m}\left(\frac{m}{e}\right)^m,$$

where $\sim$ denotes the asymptotic behaviour, i.e., the ratio between the terms tends to 1, as m tends to infinity. We then get

$$N(m) \sim \frac{\sqrt{2\pi m}\left(\frac{m}{e}\right)^m}{2^{m/2}\sqrt{2\pi(m/2)}\left(\frac{(m/2)}{e}\right)^{(m/2)}} = \sqrt{2}\left(\frac{m}{e}\right)^{(m/2)}.$$

Such a matching M gives rise to a d-regular (multigraph) on n edges, i.e., we put an edge between vertex i and j if and only if M has an edge connecting W_i and W_j. Note that we allow loops and multiple edges. Let $G^*(n,d)$ denote a (random) d-regular graph on n vertices constructed from a matching. See Figure 2.2 for a continuation of the example from before. One could hope that this process constructs a random multigraph on n vertices in a uniform way. However, it turns out that the probability that a certain graph arises depends on the number of loops and multiple edges. Luckily, the probability that $G^*(n,d)$ is simple (has no loops and no multiple edges) tends to $e^{-(d^2-1)/4}$, as n tends to infinity. Thus, for any $d \geq 3$, the configuration model gives rise to a simple, d-regular graph, with probability bounded away from zero. Furthermore, each of the simple, d-regular graphs obtained in this way occur equiprobable. To see this, we notice that a simple d-regular graph arises from exactly $(d!)^n$ of the matchings. This means that if we condition on $G^*(n,d)$ being simple, we get a d-regular graph on n vertices chosen uniformly at random, which is $G(n,d)$.
Our next goal is to show that

$$\lim_{n\to\infty} P[G^*(n,d) \text{ is simple}] = e^{-(d^2-1)/4}.$$

To show this, we will consider the probability that a random graph $G^*(n,d)$ has cycles of length l. Recall that an l-cycle is a path of length l, which starts and ends at the same vertex. In particular, we are interested in $l = 1, 2$, since $l = 1$ is the case of loops and $l = 2$ is the case of multiple edges. Let the random variable $Z_{l,n}$ be the number of l-cycles in $G^*(n,d)$.

Lemma 2.5.1. *For any integer $1 \leq l$, the expected number of l-cycles in $G^*(n,d)$ satisfies*

$$\lim_{n\to\infty} E[Z_{l,n}] = \frac{(d-1)^l}{2l}.$$

Proof. First note that for a given (random) graph $G^*(n,d)$, the number of l-cycles can be identified with sets of l edges $\{e_1,\ldots,e_l\}$ from the matching M, such that there exists a sequence of l distinct vertices $(v_1,\ldots,v_l)$ of the graph, where each e_i connects vertex v_i to v_{i+1}, and e_l connects v_l to v_1. Since we are interested in the expected number of l-cycles in any random d-regular graph on n vertices, we have to look at the number of possible ways to construct such sequences from all the possible matchings. Thus, let a_l denote the number of sets of l pairwise disjoint edges $\{e_1,\ldots,e_l\} \subseteq W \times W$ such that there exists a sequence $(W_1,\ldots,W_l)$ with each W_i being unique and e_i connecting W_i with W_{i+1}, and e_l connecting W_i with W_1. In order to find this number, we let b_l denote the number of *sequences* of l disjoint edges $(e_1,\ldots,e_l)$ such that there exist l distinct $(W_1,\ldots,W_l)$ with e_i connecting the sets as described above. This set corresponds to $2a_l$. To see this, we start at some W_i, of which there are l, and then we follow the l edges in a circle back to W_i. Since we have to choose the *direction* in which we traverse the circle, we end up with $b_l = 2a_l$. To calculate the number b_l, we note that we have n sets to choose from. After one is chosen, we have $n-1$ sets to choose from, and continuing in this way, we get $n(n-1)\cdots(n-l+1) := (n)_l$ possible ways of choosing the l sets W_i. Each W_i has exactly d elements, and we need to choose one element which connects W_i to W_{i-1}, and one element which connects W_i to W_{i+1}. This can be done in $d(d-1)$ ways, and since we have l sets, we conclude that

$$b_l = (n)_l(d(d-1))^l.$$

From this we see that

$$a_l = \frac{(n)_l(d(d-1))^l}{2l}.$$

We have already calculated the number of possible matchings, $N(m)$, so the probability that a particular set of l pairwise disjoint edges appear in a random matching is

$$p_l = ((m-1)(m-3)\cdots(m-2l+1))^{-1} = ((nd-1)(nd-3)\cdots(nd-2l+1))^{-1}.$$

Thus, the expected number of l-cycles in $G^*(n,d)$ is

$$E[Z_{l,n}] = a_l p_l = \frac{(n_l)(d(d-1))^l}{2l(nd-1)\cdots(nd-2l+1)} \sim \frac{(d-1)^l}{2l},$$

as $n \to \infty$, proving the lemma. □

The next lemma will help us to determine the expected number of ordered pairs of distinct l-cycles in $G^*(n,d)$.

Lemma 2.5.2. *Let H be a fixed connected multigraph with more edges than vertices. Then the expected number of copies of H in $G^*(n,d)$ is $O(\frac{1}{n})$*[†].

Proof. Let H be a connected multigraph on m vertices and k edges with $m < k$. Let a_H be the number of sets of k edges in $W \times W$, such that each set produces a copy of H in $G^*(n,d)$. There are $\binom{n}{m}$ ways to choose the vertices for the copy of H, and given this choice, the number of copies of H depends only upon d. This implies that $a_H = O(n^m)$. From this we see that the expected number of copies of H in $G^*(n,d)$ is

$$a_H p_k = O(n^{m-k}) = O\left(\frac{1}{n^s}\right) = O\left(\frac{1}{n}\right),$$

for some $s \geq 1$, completing the proof. □

[†]Recall that for functions $f, g : \mathbb{R} \to \mathbb{R}$ we say that f is $O(g)$, written $f = O(g)$, if there exist constants $k \geq 0$ and $c > 0$ such that $|f(x)| \leq c|g(x)|$, for all $x \geq k$.

Lemma 2.5.3. *For any integer $1 \leq l$, the expected number of ordered pairs of l distinct l-cycles in $G^*(n,d)$ satisfies*

$$\lim_{n\to\infty} E[Z_{l,n}\cdot(Z_{l,n}-1)] = \lim_{n\to\infty} E[(Z_{l,n})_2] = \left(\frac{(d-1)^l}{2l}\right)^2.$$

Proof. Let $Y = (Z_{l,n})_2$ be the number of ordered pairs of distinct l-cycles in $G^*(n,d)$. We split Y into $Y' + Y''$, where Y' is the number of ordered pairs of *vertex-disjoint* l-cycles in $G^*(n,d)$, and Y'' is the number of ordered pairs of non-vertex-disjoint l-cycles. Note that each unordered pair of non-vertex-disjoint l-cycles corresponds to a connected multigraph H with more edges than vertices. Lemma 2.5.2 then tells us that the expected numbers of such copies in $G^*(n,d)$ is $O(\frac{1}{n})$, i.e., it tends to zero as n tends to infinity. Thus, it suffices to show that

$$\lim_{n\to\infty} E[Y'] = \left(\frac{(d-1)^l}{2l}\right)^2.$$

To see this, we use the same technique as we used in the proof of Lemma 2.5.1. Let $a_{l,l}$ be the total number of sets of ordered pairs of $2l$ pairwise disjoint edges, i.e., the sets of the form $(\{e_1,\dots,e_l\},\{f_1,\dots,f_l\})$, such that there exists a sequence $((W_{e_i})_{i=1}^l,(W_{f_i})_{i=1}^l)$ with each W_i being unique and e_i connecting W_{e_i} with $W_{e_{i+1}}$, and f_i connecting W_{f_i} to $W_{f_{i+1}}$. Then, by a similar argument as before, we get

$$a_{l,l} = \frac{(n)_{2l}(d(d-1))^{2l}}{(2l)^2}.$$

Since the probability that all $2l$ edges in such a pair appear in a random matching is p_{2l}, we see that

$$E[Y'] = a_{l,l}p_{2l} = \frac{(n)_{2l}(d(d-1))^{2l}}{(2l)^2(nd-1)\cdots(nd-4l+1)} \sim \left(\frac{(d-1)^l}{2l}\right)^2,$$

as $n\to\infty$, proving the lemma. □

Generalizing the above results, we get that for any fixed $k_1,k_2,\dots,k_m \in \mathbb{N}_0$, we have

$$\lim_{n\to\infty} E[(Z_{1,n})_{k_1}(Z_{2,n})_{k_2}\cdots(Z_{m,n})_{k_m}] = \prod_{i=1}^m \lambda_i^{k_i},$$

where $\lambda_i = \frac{(d-1)^i}{2i}$. In order to show the desired result, we now introduce an approximation theorem for joint Poisson distributions [8].

Theorem 2.5.4. *Let $X_{i,1}, X_{i,2},\dots$ be a sequence of bounded random variables taking values in $\mathbb{N}_0$, for each $1 \leq i \leq m$. If there exists $\lambda_1,\lambda_2,\dots,\lambda_m \geq 0$ such that for any fixed $r_1,r_2,\dots,r_m \in \mathbb{N}_0$, it holds that*

$$E[(X_{1,n})_{r_1}(X_{2,n})_{r_2}\cdots(X_{m,n})_{r_n}] \to \prod_{i=1}^m \lambda_i^{r_i},$$

as $n \to \infty$. Then, as $n \to \infty$, $P[X_{1,n}, X_{2,n}, \ldots, X_{m,n}] \to P[Y_1, Y_2, \ldots, Y_m]$, where Y_i is an independent Poisson random variable with $E[Y_i] = \lambda_i$, for each $1 \leq i \leq m$.

From this theorem it follows that for fixed $m \in \mathbb{N}$, we have

$$\lim_{n\to\infty} P[Z_{1,n}, Z_{2,n}, \ldots, Z_{m,n}] = P[Y_1, Y_2, \ldots, Y_m],$$

where the Y_i's are independent Poisson random variables with $E[Y_i] = \lambda_i$, for each i. So in particular we have that $(Z_{1,n}, Z_{2,n}) \to (Y_1, Y_2)$, as $n \to \infty$. From this we can calculate

$$\begin{aligned}\lim_{n\to\infty} P[G^*(n,d) \text{ is simple}] &= \lim_{n\to\infty} P[(Z_{1,n}, Z_{2,n}) = (0,0)] \\ &= P[(Y_1, Y_2) = (0,0)] \\ &= P[Y_1 = 0]P[Y_2 = 0] \\ &= e^{-\lambda_1} e^{-\lambda_2} \\ &= e^{-(d^2-1)/4}.\end{aligned}$$

As mentioned earlier, our main goal of this section is to show that almost every random d-regular graph on n vertices is an expander, as n tends to infinity. In order to prove this, we need a way to connect random d-regular graphs $G(n,d)$ and graphs arising from the configuration model $G^*(n,d)$. This connection is established by noting that if $G^*(n,d)$ has a certain property with high probability, then so does $G(n,d)$. Indeed, if $P[G^*(n,d) \notin \mathcal{A}] \to 0$, as $n \to \infty$, for some property $\mathcal{A}$, then, by the Bayes theorem, we get

$$\lim_{n\to\infty} P[G(n,d) \notin \mathcal{A}] = \lim_{n\to\infty} \frac{P[G^*(n,d) \notin \mathcal{A},\ G^*(n,d) \text{ is simple}]}{P[G^*(n,d) \text{ is simple}]} = 0. \tag{2.3}$$

Here we have used that $P[G^*(n,d) \text{ is simple}] > 0$, and that $G^*(n,d)$ is uniformly distributed when $G^*(n,d)$ is simple. Having this established, we are now ready to prove our main theorem of this section, due to Bollobás [3].

Theorem 2.5.5. *Let $d \geq 3$, and let $\eta \in (0,1)$ be such that*

$$2^{4/d} < (1-\eta)^{1-\eta}(1+\eta)^{1+\eta}.$$

Then almost every d-regular graph has Cheeger constant at least $(1-\eta)d/2$, as $n \to \infty$, that is,

$$\lim_{n\to\infty} P\left[h(G(n,d)) \geq \frac{(1-\eta)d}{2}\right] = 1.$$

Proof. We have shown that

$$\lim_{n\to\infty} P[G^*(n,d) \text{ is simple}] = e^{-(d^2-1)/4},$$

so by (2.3), it suffices to prove that $h(G^*(n,d)) \geq (1-\eta)d/2$ almost surely. Let $P[s,k]$ denote the probability that $G^*(n,d)$ contains a set S on s vertices, such that there are exactly

k edges, each of which has precisely one vertex in $\cup_{i\in S}W_i$, i.e., there are precisely k edges between S and $\overline{S}$. Let k_s be the largest integer less than $(1-\eta)ds/2$ such that $ds-k_s$ is even. Notice that for $0 \le k \le k_s$, it follows that $h(G^*(n,d)) < (1-\eta)ds/2$, since the subset S satisfies

$$\frac{|\partial S|}{\min\{|S|, |\overline{S}|\}} = \frac{k}{s}.$$

Thus, by Boole's inequality, we have

$$P\left[h(G^*(n,d) < \frac{(1-\eta)d}{2}\right] \le \sum_{s=1}^{\lfloor n/2 \rfloor} \sum_{0\le k\le k_s} P[s,k],$$

where k must satisfy that $ds-k$ is even. Here $\lfloor\cdot\rfloor$ denotes the floor function, i.e., $\lfloor j \rfloor$ is the greatest integer less than (or equal to) j. Since a configuration is a partition of a set of dn elements into pairs, we deduce that

$$P[s,k] \le \binom{n}{s}\binom{ds}{k}\binom{d(n-s)}{k}k!\frac{N(ds-k)N(d(n-s)-k)}{N(dn)} := P_0[s,k].$$

We briefly explain the terms in the above product. First, we have to choose s vertices from the set of n vertices. Each of these vertices has d edges, due to regularity, and of these ds edges we have to choose k. Each of the k edges must end in the complement of S, so we get $d(n-s)$ edges to choose from. Given all these choices, we get a number, say, r, of possible ways to choose our set of k edges, and we now have $k!$ ways to pick the edges. Last, we have the number of ways to do the matchings divided by the total number of matchings, i.e., the number of matchings of $ds-k$ elements multiplied by the number of matchings in the complement of S, $d(n-s)-k$, and then divided by the total number of matchings $N(dn)$. Notice that for $0 \le k < k' \le k_s$ and $1 \le s \le \lfloor n/2 \rfloor$, we have $P_0[s,k] \le P_0[s,k']$. Thus,

$$\sum_{s=1}^{\lfloor n/2 \rfloor} \sum_{0\le k\le k_s} P_0[s,k] \le \sum_{s=1}^{\lfloor n/2 \rfloor} sP_0[s,k_s],$$

where on the left hand side, k must satisfy that $ds-k$ is even. So it suffices to show that

$$\sum_{s=1}^{\lfloor n/2 \rfloor} sP_0[s,k_s] = o(1)^\dagger.$$

To do this, we show that

1. $P_0[s,k_s] = o(1)$, for $1 \le s \le 100$,
2. $P_0[s,k_s] = o(n^{-2})$ for $100 \le s \le n/2$,

†Recall that for functions $f,g : \mathbb{R} \to \mathbb{R}$ we say that f is $o(g)$, written $f = o(g)$, if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f(x)| \le \varepsilon|g(x)|$, for all $x \ge N$. Note that if $f = o(g)$, then $f = O(g)$.

The first statement is easily checked. For the second statement we note that there exists a constant $c > 0$ such that

$$P_0[s, k_s] \leq cP_0[\lfloor n/2 \rfloor, k_{\lfloor n/2 \rfloor}],$$

whenever, say, $100 \leq s \leq \lfloor n/2 \rfloor$. Hence it suffices to prove the statement for $s = \lfloor n/2 \rfloor$. To avoid unnecessary difficulties, we will now assume that n is even. We then have

$$P_0[n/2, k_{n/2}] = \binom{n}{n/2} \binom{d(n/2)}{k_{n/2}}^2 (k_{n/2})! \frac{N(d(n/2) - k_{n/2})^2}{N(dn)}.$$

Applying Stirling's formula several times we deduce that

$$P_0[n/2, k_{n/2}] \leq c(2^{4/d}(1-\eta)^{-(1-\eta)}(1+\eta)^{-(1+\eta)})^{n/4d},$$

for some constant $c > 0$. Using our assumption on η, concludes to proof. □

2.6 Literature

This chapter is based on [16], [17], [12], [23] and [3]. In particular, the section regarding regular graphs is inspired by [12] and [16]. The section on l^2-spaces is inspired by [23]. The last section concerning random graphs is [3], with great help from the lecture notes by [8].

Chapter 3

Groups

In this chapter we introduce a way to view groups as graphs, and thus, by the previous chapter, a way to view groups as metric spaces. We also discuss the notion of growth of a group in terms of balls of increasing size. In the end of the chapter we study the notion of unitary representations and cocycles. This will be useful in later chapters, where we want to study analytic and geometric properties of groups. Recall that we assume all groups to be discrete and finitely generated unless otherwise stated.

3.1 Cayley graphs

We start this section by associating a graph, called the Cayley graph, to a finitely generated discrete group. We will do this in a very natural way: each element of the group is represented as a vertex in this graph, and the edges are obtained by applying each of the generators to the element. One problem with this construction is that the graph seem to depend a lot on the chosen generating set, but as we shall see, these graphs, considered as metric spaces, are coarsely equivalent. However, before we introduce Cayley graphs, we need to introduce the notion of word length in a group.

Definition 3.1.1. Let $\Sigma \subseteq \Gamma$ be a finite (symmetric) generating subset of a group Γ. For each $g \in \Gamma$, we define the *word length* of g with respect to Σ as

$$l_\Sigma^\Gamma(g) = \min\{n \in \mathbb{N}_0 \mid g = s_1 s_2 \cdots s_n, s_1, \dots, s_n \in \Sigma\}.$$

We will sometimes write this as $|g|$, when the group and the generating subset are clear from the context. Note that $|g| = 0$ if and only if $g = e$, where e is the identity element of Γ. Furthermore, we may, in fact, always assume that Σ is symmetric, i.e., $\Sigma = \Sigma^{-1} = \{s \in \Sigma \mid s^{-1} \in \Sigma\}$, since if Σ is not symmetric, we consider $\Sigma' = \Sigma \cup \Sigma^{-1}$, which then will be a finite symmetric generating subset. We now show some properties of the word length.

Proposition 3.1.2. *Let $\Sigma \subseteq \Gamma$ be a finite generating subset. Then for all $g, h \in \Gamma$ we have*

$$|g| = |g^{-1}| \qquad \text{and} \qquad |gh| \leq |g| + |h|.$$

Proof. We start by proving that $|g^{-1}| \leq |g|$. If g is the identity of Γ, the statement is trivial. Hence we assume that $g \in \Gamma \setminus \{e\}$. Let $g = s_1 \cdots s_n$, for some $n \in \mathbb{N}$, where $s_i \in \Sigma$. Since $g^{-1} = (s_1 \cdots s_n)^{-1} = s_n^{-1} \cdots s_1^{-1}$ we see that $|g^{-1}| \leq n = |g|$, due to minimality in the definition of the word length. The other direction follows by noting that $g = (g^{-1})^{-1}$. To show the other property, let $g, h \in \Gamma \setminus e$ with $g = s_1 \cdots s_n$ and $h = t_1 \cdots t_m$ for some $n, m \in \mathbb{N}$, where $s_i, t_i \in \Sigma$. Now $gh = s_1 \cdots s_n t_1 \cdots t_m$, and thus $|gh| \leq n + m = |g| + |h|$. □

This proposition tells us that the word length induces a metric d on the group, given by

$$d(g, h) = l_\Sigma^\Gamma(g^{-1}h) = |g^{-1}h|,$$

called the *word metric*. The following proposition shows that this metric is left-invariant.

Proposition 3.1.3. *Let $\Sigma \subseteq \Gamma$ be a finite generating subset. For any $\gamma, g, h \in \Gamma$, we have*

$$d(\gamma g, \gamma h) = d(g, h).$$

Proof. This is seen by the following computation

$$d(\gamma g, \gamma h) = |(\gamma g)^{-1}(\gamma h)| = |g^{-1}\gamma^{-1}\gamma h| = |g^{-1}h| = d(g, h). \qquad \square$$

We now define the Cayley graph of a group.

Definition 3.1.4. Let Γ be a discrete group (not necessarily finitely generated), and $\Sigma \subseteq \Gamma$ a finite subset. The *Cayley* graph of Γ with respect to Σ is defined as $Cay(\Gamma; \Sigma) := (V, E)$, where V is the set of elements of Γ and $E = \{(g, s, gs) \in \Gamma \times \Sigma \times \Gamma \mid g \in \Gamma, s \in \Sigma\}$.

We note that the Cayley graph is $|\Sigma|$-regular, and if Σ generates Γ, then $Cay(\Gamma; \Sigma)$ is connected. Furthermore, the set of vertices V of $Cay(\Gamma; \Sigma)$ comes naturally equipped with the Γ-action of right (or left) translation, that is, for all $g \in \Gamma$ and $x \in V$, we have $(gf)(x) = f(xg)$, where $f : V \to \mathbb{R}$ is a map of the vertices. We saw in the previous chapter how to put a metric on a connected graph, so a natural question is whether the path metric and the word metric are equivalent in some sense. The following proposition shows that this is indeed the case.

Proposition 3.1.5. *Let Γ be a finitely generated group and $\Sigma \subseteq \Gamma$ a finite generating subset. Then the word metric d_w and the path metric d_p agree, i.e., $d_w(g, h) = d_p(g, h)$, for all $g, h \in \Gamma$.*

Proof. Pick $g, h \in \Gamma$. If $g = h$, then it is clear, since both terms are zero. Thus assume that $g \neq h$. We will start by showing that $d_w(g,h) \leq d_p(g,h)$. To do this, let $\pi = (e_1, \dots, e_n)$ be a path of minimal length connecting g and h (such a path always exists since $Cay(\Gamma; \Sigma)$ is connected). Now $e_1 = (g, s_1, gs_1)$, for some $s_1 \in \Sigma$, and if we write $g_1 = gs_1$, we can write $e_i = (g_i, s_i, gs_1 \cdots s_i)$, for $i = 1, \dots n$ with $gs_1 \cdots s_n = h$. This leads to the formula $g^{-1}h = s_1 \cdots s_n$, which implies that

$$d_w(g,h) = |g^{-1}h| = |s_1 \cdots s_n| \leq n = l(\pi) = d_p(g,h).$$

On the other hand, assume that $d_w(g,h) = m$, for some $m \in \mathbb{N}$. By definition, there exist $t_1, \dots, t_m \in \Sigma$ such that $g^{-1}h = t_1 \cdots t_m$, which is the same as $h = gt_1 \cdots t_m$. This string of generators induces a path, π', consisting of at most m elements connecting g and h. Hence we get

$$d_p(g,h) = l(\pi') \leq m = |g^{-1}h| = d_w(g,h),$$

proving the other inequality. $\square$

We now turn our attention to some common examples of Cayley graphs.

Example 3.1.6. One of the most intuitive examples of Cayley graphs is the Cayley graph of the group $\Gamma = \mathbb{Z}$, with generating set $\Sigma = \{-1, 1\}$. This is a finite symmetric generating subset of $\mathbb{Z}$, and the associated Cayley graph can be seen in Figure 3.1 below.

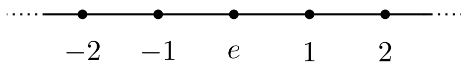

Figure 3.1: The Cayley graph of $\mathbb{Z}$.

Example 3.1.7. Another interesting example of a Cayley graph is for $\Gamma = \mathbb{F}_2$, the free group on two generators labelled a and b. The generating set is $\{a, b\}$, but since we want a symmetric generating set, we add the inverses of a and b to get $\Sigma = \{a, a^{-1}, b, b^{-1}\}$. The Cayley graph can be seen in Figure 3.2 below.

We now show that the generating subset is, in some sense, unimportant.

Theorem 3.1.8. *Let Γ be a finitely generated group, and let $\Sigma, \Sigma' \subseteq \Gamma$ be finite symmetric generating subsets. Then (Γ, d_Σ) and $(\Gamma, d_{\Sigma'})$ are coarsely equivalent (as metric spaces).*

Proof. To prove this, we show that the identity map $\mathrm{id} : (\Gamma, d_\Sigma) \to (\Gamma, d'_\Sigma)$ is a coarse equivalence in terms of Definition 1.2.8. Let d and d' denote the metrics on (Γ, d_Σ) and $(\Gamma, d_{\Sigma'})$, respectively. Note that for any fixed $R > 0$, the ball

$$B_d(e,R) := \{g \in \Gamma : d(e,g) \leq R\} = \{g \in \Gamma \mid g = s_1 \cdots s_n, s_i \in \Sigma, n \leq R\}$$

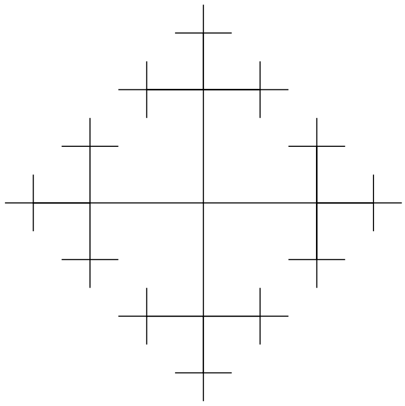

Figure 3.2: The Cayley graph of $\mathbb{F}_2$.

has size at most $|\Sigma|^R$ which is a finite number, and hence the image of this ball under the identity is a finite set. Furthermore, due to the metric d being left-invariant, the size of the ball stays fixed under left translations. This shows that the map is bornologous. Similarly, the pull-back of a finite ball $B_{d'}(e, S)$ is again finite, which shows that the map is proper. This means that the identity map is a coarse map, and it follows that the identity map $\mathrm{id}' : (\Gamma, d') \to (\Gamma, d)$ is also a coarse map. This finishes the proof, since composition of identities is close to the identity. $\square$

Note that this theorem tell us that, up to coarse equivalence, we may view a group as a metric space, and we also have the following useful corollary:

Corollary 3.1.9. *Let $\Lambda \subseteq \Gamma$ be a subgroup of a finitely generated group Γ. The inclusion map $\iota : \Lambda \hookrightarrow \Gamma$ is a coarse embedding.*

Since we can view groups as metric spaces, it makes good sense to introduce the notion of functions of positive (negative) type of groups in relation to kernels of metric spaces.

Definition 3.1.10. Let Γ be a group and $k : \Gamma \times \Gamma \to \mathbb{R}$ a kernel on Γ. We say that a function $f : \Gamma \to \mathbb{R}$ is of positive (negative) type on Γ if the map defined by $k(x, y) = f(x^{-1}y)$, for $x, y \in \Gamma$, is a kernel of positive (negative) type.

Recall that we can associate an ℓ^2-space to a discrete group Γ by

$$\ell^2(\Gamma) := \left\{ f : \Gamma \to \mathbb{C} \,\middle|\, \|f\|_2 = \left(\sum_{g \in \Gamma} |f(g)|^2 \right)^{1/2} < \infty \right\}.$$

Clearly $\ell^2(\Gamma)$ is a Hilbert space with orthonormal basis $\{\delta_g : g \in \Gamma\}$, where for $g \in \Gamma$,

$$\delta_g(s) = \begin{cases} 1 & g = s, \\ 0 & \text{else.} \end{cases}$$

3.2 Growth of groups

In this section we study growth of groups. The rate at which groups grows gives us (often) a lot of information about the group. For instance, a group which growth sufficiently slow must be amenable. We will return to this topic in the next chapter, but first we need to study growth of groups. Given two functions $f, g : \mathbb{R} \to \mathbb{R}$ with $f = O(g)$, we write $f \preceq g$. The idea is that if f is $O(g)$ then there exists a sufficiently large constant c such that f is dominated by cg, for all sufficiently large x. Furthermore, if f is $O(g)$ and g is $O(f)$, we say that the growth rate of f and g is equivalent, and we write $f \sim g$. Note that if $f \sim g$, there exist positive constants c_1, c_2 and $k \geq 0$ such that for all $x \geq k$,

$$c_1|f(x)| \leq |g(x)| \leq c_2|f(x)|.$$

Remark 3.2.1. It is immediate from the calculation above that $\sim$ is an equivalence relation.

We are now ready to define the growth of a group.

Definition 3.2.2. Let $\Sigma \subseteq \Gamma$ be a finite generating subset. We define the growth of Γ with respect to Σ by the function $\gamma : \mathbb{N}_0 \to \mathbb{N}$ given by

$$\gamma(n) = |\{g \in \Gamma : |g| \leq n\}| = |B(n)|,$$

where $B(n)$ denotes the ball of radius n (centered at the identity).

Note that the requirement that the ball is centered at the identity is only for convenience, due to Σ generating Γ. Thus, to study the growth of a group, we look at balls of increasing size, and count the elements in each of these balls. However, since we have defined growth in terms of a generating set, we wish to show that two generating sets have the same growth rate.

Proposition 3.2.3. *Let $\Sigma, \Sigma' \subseteq \Gamma$ be two finite generating subsets of Γ. If γ, γ' are the associated growth functions, then $\gamma \sim \gamma'$.*

Proof. Set $c = \max\{l_{\Sigma'}(s) \mid s \in \Sigma\}$. Then $l_{\Sigma'}(g) \leq c l_\Sigma(g)$, for all $g \in \Gamma$. To see this, let $g \in \Gamma$ be fixed, and suppose that $l_\Sigma(g) = l_\Sigma(s_1 \cdots s_n) = n$, for some $s_i \in \Sigma$. Then

$$l_{\Sigma'}(g) = l_{\Sigma'}(s_1 \cdots s_n) \leq \sum_{i=1}^{n} l_{\Sigma'}(s_i) \leq cn.$$

Thus, for $g \in B_\Sigma(n)$ we have that $g \in B_{\Sigma'}(cn)$, which show that $\gamma \preceq \gamma'$. Similar calculations shows that $\gamma' \preceq \gamma$, and thus $\gamma \sim \gamma'$. $\square$

This shows that we can identify the growth rate of the group, regardless of the chosen generating subset. We will abuse notation a little, and write $\gamma(\Gamma)$ for the growth type of the group. We say that a group Γ has *exponential growth* if $\gamma(\Gamma) \sim \exp(n)$, for some $n \in \mathbb{N}$. We show below that any finitely generated group has at most exponential growth, which leads to notion of subexponential growth. That is, if $\gamma(\Gamma) \preceq \exp(n)$, but $\gamma(\Gamma) \not\sim \exp(n)$, we say that it has *subexponential growth*. Further, we say that a group has *polynomial growth* if $\gamma(\Gamma) \sim n^d$, for some integer $d \geq 0$. Note in particular that if Γ has constant growth, i.e., $\gamma(\Gamma) \sim 1$, then Γ has polynomial growth. The following proposition relates different types of growth.

Proposition 3.2.4. *Let Γ be a finitely generated group. Then*

(i) Γ *is finite if and only if* Γ *has constant growth.*

(ii) if Γ *has polynomial growth, then* Γ *has subexponential growth.*

(iii) Γ *has at most exponential growth.*

Proof. (i): Suppose Γ is finite. It is clear that $1 \preceq \gamma(\Gamma)$ since $\gamma(0) = 1$. On the other hand, we can pick $c = |\Gamma|$, which gives us $\gamma(n) = |B(n)| \leq c$, for all n. This shows that $\gamma(\Gamma) \preceq 1$. Assume now that $\gamma(\Gamma) \sim 1$. This means that there exists a constant c such that $\gamma(n) \leq c$, for all n, and this implies that $|\Gamma| \leq c < \infty$.
(ii): Let $d \geq 0$ be an integer such that $\gamma(\Gamma) \sim n^d$. It is well known that $\frac{n^d}{\exp(n)} \to 0$, as $n \to \infty$, which implies that there exists $c \in \mathbb{N}$ such that $n^d \leq c\exp(n)$. This shows that $\gamma(\Gamma) \preceq \exp(n)$. On the other hand, $\frac{\exp(n)}{n^d} \to \infty$, as $n \to \infty$, and thus there can not exist integers c, n_0 such that $\exp(n) \leq cn^d$, for all $n \geq n_0$.
(iii): Let $\Sigma \subseteq \Gamma$ be a symmetric finite generating subset. If Σ has only one element, then Γ has only two elements, namely e and g, and hence it has subexponential growth by *(ii)*. Suppose Σ has k elements. Note that Γ has maximal growth if there are no relations among the generators, i.e., Γ is the free group on k generators. The elements in the ball of radius 1 are the identity element and each of the generators, thus there are $1 + k$ elements. Each time we increase the size of the ball by one, we get $k - 1$ new elements for each element g with $|g| = n$. Now we can calculate the elements in a ball of radius n by the formula

$$\gamma(n) = 1 + k\sum_{i=0}^{n-1}(k-1)^i = \frac{k(k-1)^n - 2}{k-2}, \tag{3.1}$$

which shows that Γ grows exponentially. □

Remark 3.2.5. The above proposition tells us that for any finitely generated group Γ, the limit $\lim_{n\to\infty} \gamma(n)^{1/n}$ exists. Furthermore, if $\lim_{n\to\infty} \gamma(n)^{1/n} > 1$, then Γ has exponential growth, and if $\lim_{n\to\infty} \gamma(n)^{1/n} \leq 1$, then Γ has subexponential growth.

We now look at the growth type of some common groups.

Example 3.2.6. The growth type of the free group $\mathbb{F}_2$ on two generators is

$$\gamma(\mathbb{F}_2) \sim 3^n.$$

In particular it has exponential growth.

Proof. Recall that we want the generating subset to be symmetric which means that it contains 4 elements. Thus, by Equation 3.1, we see that

$$\gamma(n) = 1 + 4\sum_{i=0}^{n-1}(3)^i = \frac{4(3)^n - 2}{2} = 2 \cdot 3^n - 1 \sim 3^n.$$

□

Example 3.2.7. The growth type of the abelian group $\mathbb{Z}$ is

$$\gamma(\mathbb{Z}) \sim n,$$

in particular it has polynomial growth.

Proof. Let $\Sigma = \{-1, 1\}$ be the generating subset. Then the ball of radius n is

$$B_\Sigma(n) = \{g \in \mathbb{Z} \mid |g| \leq n\} = \{-n, -(n-1), \ldots, -1, 0, 1, \ldots, n-1, n\}.$$

Thus $\gamma(\mathbb{Z}) = |B_\Sigma(n)| = 2n + 1 \sim n$. □

We end this section by showing that any abelian group has polynomial growth. To do this, we start with a lemma:

Lemma 3.2.8. *Let Γ_1 and Γ_2 be two finitely generated groups, and let $\Gamma = \Gamma_1 \times \Gamma_2$ be their direct product. Then there exists a finite subset $\Sigma \subseteq \Gamma$ which generates Γ.*

Proof. Let $\Sigma_1 \subseteq \Gamma_1$ and $\Sigma_2 \subseteq \Gamma_2$ be finite generating subsets of Γ_1 and Γ_2, respectively. We show that the set

$$\Sigma = (\Sigma_1 \times \{e_2\}) \cup (\{e_1\} \times \Sigma_2)$$

is a finite generating subset of Γ. Clearly it is finite, since it is a union of finite sets. For $(g_1, g_2) \in \Gamma$, there exist $s_1, \ldots, s_n \in \Sigma_1$ and $r_1, \ldots, r_m \in \Sigma_2$ such that $g_1 = s_1 \cdots s_n$ and $g_2 = r_1 \cdots r_m$, respectively. Thus we have

$$(g_1, g_2) = (s_1 \cdots s_n, r_1 \cdots r_m) = (s_1, e_2) \cdots (s_n, e_2)(e_1, r_1) \cdots (e_1, r_m),$$

which is a product of elements in Σ. This shows that Σ generates Γ, as wanted. □

Note that this also shows us that for $(g_1, g_2) \in B_\Sigma^\Gamma(k)$ with $n + m \leq k$, we have $B_\Sigma^\Gamma \subseteq B_{\Sigma_1}^{\Gamma_1}(k) \times B_{\Sigma_2}^{\Gamma_2}(k)$.

Theorem 3.2.9. *Let $\Gamma = \Gamma_1 \times \Gamma_2$ be the direct product of two finitely generated groups. Then $\gamma(\Gamma) \sim \gamma(\Gamma_1)\gamma(\Gamma_2)$.*

Proof. Let $\Gamma = \Gamma_1 \times \Gamma_2$ and let $\Sigma \subseteq \Gamma$ be a finite generating subset. Given $(g_1, g_2) \in \Gamma$ with $g_1 = s_1 \cdots s_n$ and $g_2 = r_1 \cdots r_m$, there exists a ball $B_\Sigma^\Gamma(k)$ containing (g_1, g_2) such that $n + m \leq k$. Since $B_\Sigma^\Gamma(n) \subseteq B_{\Sigma_1}^{\Gamma_1}(n) \times B_{\Sigma_2}^{\Gamma_2}(n)$, it follows that $\gamma(\Gamma) \preceq \gamma(\Gamma_1)\gamma(\Gamma_2)$. Conversely, for $g_1 \in B_{\Sigma_1}^{\Gamma_1}(k)$ and $g_2 \in B_{\Sigma_2}^{\Gamma_2}(k)$, we have $(g_1, g_2) \in B_\Sigma^\Gamma(2k)$, showing that $B_{\Sigma_1}^{\Gamma_1}(k) \times B_{\Sigma_2}^{\Gamma_2}(k) \subseteq B_\Sigma^\Gamma(2k)$. From this it follows that $\gamma(\Gamma_1)\gamma(\Gamma_2) \preceq \gamma(\Gamma)$, so we conclude that $\gamma(\Gamma) \sim \gamma(\Gamma_1)\gamma(\Gamma_2)$. □

Corollary 3.2.10. *Any finitely generated abelian group has polynomial growth.*

Proof. If Γ is a finitely generated abelian group, it follows from the structure theorem for finitely generated abelian groups (cf. [13, Theorem 2.1]) that Γ is isomorphic to th product $\mathbb{Z}^d \times \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_n}$ for some integers $d, q_1, \ldots, q_n$. It follows that

$$\gamma(\Gamma) \sim \gamma(\mathbb{Z}^d)\gamma(\mathbb{Z}_{q_1}) \cdots \gamma(\mathbb{Z}_{q_n}) \sim \gamma(\mathbb{Z}^d) \sim \gamma(\mathbb{Z} \times \cdots \times \mathbb{Z}) \sim \gamma(\mathbb{Z}) \cdots \gamma(\mathbb{Z}) \sim n^d,$$

since $\gamma(\mathbb{Z}_{q_i}) \sim 1$ and $\gamma(\mathbb{Z}) = n$. □

If Γ has neither polynomial nor exponential growth, we say that Γ has *intermediate* growth. It was an open question for several decades, if groups with intermediate growth do exist. Grigorchuk showed in 1983 in [9] that they do exist. Unfortunately, we do not have the time to dive further into this subject.

3.3 Unitary representations and cocycles

In this section we introduce the notion of unitary representations of groups, and cocycles of such unitary representations. To do this we briefly recall the notion of bounded linear operators.

Definition 3.3.1. Let X and Y be normed vector spaces over the same field $\mathcal{F}$ (which for us will be $\mathbb{R}$ or $\mathbb{C}$). A map $T : X \to Y$ is called a linear operator, if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

for all $x, y \in X$ and $\alpha, \beta \in \mathcal{F}$.

Definition 3.3.2. A linear operator said to be bounded if

$$\|T\| := \sup\{\|Tx\| : \|x\| \leq 1\} < \infty$$

Note that a bounded linear operator is continuous and $T(0) = 0$. Denote by $B(H)$ the set of bounded linear operators on a Hilbert space H.

Definition 3.3.3. A *unitary operator* is a bounded linear operator $U : H \to H$ on a Hilbert space H satisfying $UU^* = I = U^*U$, where $I : H \to H$ is the identity operator, and U^* is the (Hilbert space) adjoint of U.

We denote by $\mathcal{U}(H)$ the set of unitary operators in $B(H)$. Note that $\mathcal{U}(H)$ is a group with respect to composition and identity element I. We say that two operators $T, S : H \to H$ are *unitarily equivalent* if there exists a unitary operator $U : H \to H$ such that $U^*TU = S$. We are now ready to introduce the notion of unitary representations.

Definition 3.3.4. Let Γ be a discrete group. A *unitary representation* of Γ on a Hilbert space H is a group homomorphism $\pi : \Gamma \to \mathcal{U}(H)$ such that $\pi(g) \in \mathcal{U}(H)$, for all $g \in \Gamma$.

Two important unitary representations of Γ on the associated Hilbert space $\ell^2(\Gamma)$ are the following:

1. The *left regular representation* $\lambda : \Gamma \to B(\ell^2(\Gamma))$, $\lambda_g(\delta_s) = \delta_{gs}$, for $g, s \in \Gamma$.

2. The *right regular representation* $\rho : \Gamma \to B(\ell^2(\Gamma))$, $\rho_g(\delta_s) = \delta_{sg^{-1}}$, for $g, s \in \Gamma$.

Note that λ and ρ are unitarily equivalent. This can be seen by considering the unitary operator $U : \ell^2(\Gamma) \to \ell^2(\Gamma)$ given by $U(\delta_g) = \delta_{g^{-1}}$, $g \in \Gamma$. Indeed, for any $g, s \in \Gamma$, we have

$$(U^*\rho_g U)(\delta_s) = (U^*\rho_g)(\delta_{s^{-1}}) = U^*(\delta_{s^{-1}g^{-1}}) = \delta_{(s^{-1}g^{-1})^{-1}} = \delta_{gs} = \lambda_g(\delta_s).$$

The above representations can be used to construct C^*-algebras associated to (discrete) groups. There is an interesting interplay between analytic properties of groups and approximation properties of their associated C^*-algebras. These are very interesting topics, but we will not pursue them here.

One way of constructing unitary representation of a discrete group Γ is by using *positive definite functions* on Γ.

Definition 3.3.5. A function $\varphi : \Gamma \to \mathbb{C}$ is called *positive definite* if the matrix with entries given by $[\varphi(s^{-1}t)]_{s,t\in F}$ is a positive definite matrix for every finite set $F \subseteq \Gamma$.

Fix a positive definite function $\varphi : \Gamma \to \mathbb{C}$ and recall that $C_c(\Gamma)$ denotes the set of compactly supported functions on Γ. Note that since we only work with discrete groups, this is the same as finitely supported functions. Define $\langle \cdot, \cdot \rangle_\varphi : C_c(\Gamma) \times C_c(\Gamma) \to \mathbb{C}$ by

$$\langle f, g \rangle_\varphi = \sum_{s,t\in\Gamma} \varphi(s^{-1}t) f(s)\overline{g(t)},$$

for $f, g \in C_c(\Gamma)$. It is clear that $\langle \cdot, \cdot \rangle_\varphi$ is positive semidefinite, since φ is positive definite. Let $\ell^2_\varphi(\Gamma)$ be the Hilbert space completion of ${}^{C_c(\Gamma)}/_{\{f\in C_c(\Gamma) \mid \langle f, f\rangle_\varphi = 0\}}$. Furthermore, we write $\hat{f} = [f] \in \ell^2_\varphi(\Gamma)$, for all $f \in C_c(\Gamma)$.

Definition 3.3.6. Let $\varphi : \Gamma \to \mathbb{C}$ be positive definite. Define $\lambda^\varphi : \Gamma \to B(\ell^2_\varphi(\Gamma))$ by

$$\lambda^\varphi_s(\hat{f}) = \widehat{sf}, \quad s \in \Gamma,$$

where $(sf)(t) = f(s^{-1}t)$ for all $t \in \Gamma$.

Then λ^φ is a unitary representation satisfying $\lambda^\varphi_s \lambda^\varphi_t = \lambda^\varphi_{st}$, for all $s, t \in \Gamma$. Furthermore, the calculation

$$\|\lambda^\varphi_s(\hat{f})\|^2 = \sum_{x,y\in\Gamma} \varphi(x^{-1}y) f(s^{-1}x)\overline{f(s^{-1}y)} = \sum_{x',y'\in\Gamma} \varphi((x')^{-1}y') f(x')\overline{f(y')} = \|\hat{f}\|^2$$

shows that λ_s^φ is an isometry for all s. Moreover,

$$\langle \lambda_s^\varphi \hat{\delta}_e, \hat{\delta}_e \rangle_\varphi = \langle \hat{\delta}_s, \hat{\delta}_e \rangle = \varphi(s),$$

for $s \in \Gamma$, showing that we can recover φ from the inner product. From this we get the following connection between positive definite functions on Γ and unitary representations of the group.

Lemma 3.3.7. *A map $\varphi : \Gamma \to \mathbb{C}$ with $\varphi(e) = 1$ is positive definite if and only if there exists a unitary representation λ^φ of Γ on a Hilbert space H_φ and a unit vector $\xi_\varphi \in H_\varphi$, such that*

$$\varphi(g) = \langle \lambda^\varphi(g) \xi_\varphi, \xi_\varphi \rangle,$$

for all $g \in \Gamma$.

We now introduce the notion of cocycles of unitary representations.

Definition 3.3.8. A 1-cocycle $b : \Gamma \to H$ on Γ with coefficients in a unitary representation (π, H) of Γ is a function satisfying

$$b(st) = b(s) + \pi(s)b(t), \quad s, t \in \Gamma.$$

Note that $b(e) = 0$, by setting $s = t = e$.

Lemma 3.3.9. *If (π, H) is a unitary representation of Γ and $\xi \in H$, then*

$$b(s) = \xi - \pi(s)\xi, \quad s \in \Gamma$$

defines a 1-cocycle on Γ. Such a 1-cocycle is called a 1-coboundary.

Proof. Indeed, for all $s, t \in \Gamma$ we have

$$b(s) + \pi(s)b(t) = \xi - \pi(s)\xi + \pi(s)(\xi - \pi(t)\xi) = \xi - \pi(s)\pi(t)\xi = \xi - \pi(st)\xi = b(st). \quad \square$$

Let $\mathrm{AffIso}(H)$ denote the (sub)set of affine isometries of H satisfying $\varphi(\xi) = u(\xi) + \xi_0$, for $\xi \in H$ and some $u \in \mathcal{U}(H)$ and $\xi_0 \in H$. Note that any element $\varphi \in \mathrm{AffIso}(H)$ is an affine isometry of H, however the converse is not true in general. To see this, consider $H = \mathbb{C}$ and $\varphi(z) = \overline{z}$, $z \in \mathbb{C}$. Clearly, φ is an affine isometry of H, but $\varphi \notin \mathrm{AffIso}(H)$. On the other hand, if H is a real Hilbert space, then $\mathrm{AffIso}(H)$ is the set of all affine isometries of H.

Lemma 3.3.10. *The map $\theta : \Gamma \to \mathrm{AffIso}(H)$ is a group homomorphism of Γ into the group $\mathrm{AffIso}(H)$ if and only if*

$$\theta(s)\xi = \pi(s)\xi + b(s), \quad s \in \Gamma, \xi \in H,$$

for some unitary representation $\pi(\Gamma) \to B(H)$ and some 1-cocycle b on Γ with coefficients in (π, H).

Proof. Suppose $\theta : \Gamma \to \mathrm{AffIso}(H)$ is a group homomorphism. Then for all $s, t \in \Gamma$ and $\xi \in H$, we have

$$\theta(st)\xi = \pi(st)\xi + b(st).$$

Using that θ is a group homomorphism we get that

$$\pi(st)\xi + b(st) = \theta(st)\xi = \theta(s)\theta(t)\xi = \theta(s)(\pi(t)\xi + b(t)) = \pi(s)\pi(t)\xi + \pi(s)b(t) + b(s).$$

This shows that $\pi(st) = \pi(s)\pi(t)$ and $b(st) = \pi(s)b(t) + b(s)$, as wanted. The converse is proved similarly. $\square$

We end this section with a result showing that 1-cocycles on the group Γ give rise to positive definite functions on the group. For this we need the following result, known as *Schoenberg's theorem*, which we state without proof.

Theorem 3.3.11. *Let k be a conditionally negative definite kernel on Γ. Then, for all $\gamma > 0$, the kernel $\varphi_\gamma : \Gamma \times \Gamma \to \mathbb{R}$ defined by*

$$\varphi_\gamma(s, t) = e^{-\gamma k(s,t)},$$

for all $(s, t) \in \Gamma \times \Gamma$, is positive definite.

As a consequence, we deduce the following corollary, which will be useful in Chapter 4.

Corollary 3.3.12. *Let $b : \Gamma \to H$ be a 1-cocycle on Γ with coefficients in a unitary representation (π, H) of Γ. Then, for any $\gamma > 0$, the function $\varphi_\gamma^b : \Gamma \to \mathbb{C}$ defined by*

$$\varphi_\gamma^b(g) = e^{-\gamma \|b(g)\|^2},$$

for all $g \in \Gamma$, is positive definite.

Proof. For any $\gamma > 0$, we observe that $\sqrt{\gamma} b$ is also a 1-cocycle on Γ. By the definition of a 1-cocycle, it follows that for all $s, t \in \Gamma$,

$$\sqrt{\gamma}\|b(s) - b(t)\| = \sqrt{\gamma}\| - \pi(s)b(s^{-1}t)\| = \sqrt{\gamma}\|b(s^{-1}t)\|. \tag{3.2}$$

Hence the map $k_\gamma^b(s, t) = \gamma\|b(s^{-1}t)\|^2$, for $(s, t) \in \Gamma \times \Gamma$ is a conditionally negative definite kernel on Γ (see Theorem 1.5.3).
By Schoenberg's theorem, the map $(s, t) \mapsto e^{-k_\gamma^b(s,t)} = e^{-\gamma\|b(s^{-1}t)\|^2}$, is a positive definite kernel on Γ. Equivalently, by using 3.2, we see that φ_γ^b is a positive definite function on Γ, as wanted. $\square$

3.4 Literature

This chapter is written with inspiration from [6] and [5]. In particular the introduction to Cayley graphs and growth of groups are from [6], and the sections regarding cocycles and unitary representations are written with inspiration from [5].

Chapter 4

Analytic properties of groups

4.1 Amenability

In this section we investigate an important property of groups, called *amenability*. It was first introduced by John von Neumann in connection with the Banach-Tarski paradox. Amenability of a group ensures that no action of the group gives rise to this paradox. We will introduce amenability the way John von Neumann originally did it, which is in terms of the existence of a finite additive probability measure on the group. We will then discuss equivalent characterisations of amenability (and show that these are in fact equivalent). Last we will look at the examples from the previous chapter and decide whether or not these groups are amenable. We start with the definition of a finitely additive probability measure.

Definition 4.1.1. Let Ω be a set. A map $\mu : \mathcal{P}(\Omega) \to [0,1]$, where $\mathcal{P}(\Omega)$ denotes the power set of Ω, is a *finitely additive* probability measure on Ω provided $\mu(\Omega) = 1$, and for any disjoint subsets $A, B \subseteq \Omega$ we have $\mu(A \cup B) = \mu(A) + \mu(B)$.

We note that if Ω is finite, then the normalized counting measure, i.e., $\frac{1}{|\Omega|}\sum_{x\in\Omega} 1_{\{x\}}$, is a finitely additive probability measure. We are now ready to define amenability in terms of the existence of such a measure.

Definition 4.1.2. Let Γ be a (discrete) group. We say that Γ is *amenable* if there exists a finitely additive left-invariant measure $\mu : \mathcal{P}(\Gamma) \to [0,1]$ such that $\mu(\Gamma) = 1$.

Another common definition of amenability is in terms of the existence of a left-invariant mean on $\ell^\infty(\Gamma)$, where

$$\ell^\infty(\Gamma) := \{f : \Gamma \to \mathbb{C} \mid \|f\|_\infty = \sup_{x\in\Gamma} |f(x)| < \infty\}.$$

A mean on a discrete group Γ is a non-negative functional $m : \ell^\infty(\Gamma) \to \mathbb{C}$ satisfying $m(1_\Gamma) = 1$, where a non-negative functional m on $\ell^\infty(\Gamma)$ is a functional that satisfies $m(f) \geq 0$, whenever $0 \leq f \in \ell^\infty(\Gamma)$.

Theorem 4.1.3. *A discrete group Γ is amenable if and only if there exists a left-invariant mean $m : \ell^\infty(\Gamma) \to \mathbb{C}$.*

Proof. It is easy to see that any amenable group has an left-invariant mean so we only need to show the converse. Let $m : \ell^\infty(\Gamma) \to \mathbb{C}$ be a mean and define $\mu(A) := m(1_A)$, for all $A \subseteq \Gamma$. Then μ is a finitely additive probability measure since $\mu(\Gamma) = m(1_\Gamma) = 1$, and for disjoint $A, B \subseteq \Gamma$ we have $\mu(A \cup B) = m(1_{A\cup B}) = m(1_A) + m(1_B) = \mu(A) + \mu(B)$. Furthermore, m is left-invariant if and only if μ is left-invariant. To see this, observe that for all $g \in \Gamma$,

$$g\mu(A) = gm(1_A) = m(g^{-1}1_A) = m(1_{g^{-1}A}) = \mu(g^{-1}A) = g\mu(A). \qquad \square$$

We now introduce the Følner condition.

Definition 4.1.4. A discrete group Γ is said to satisfy the *Følner condition* if for every $\varepsilon > 0$ and all finite subsets $S \subseteq \Gamma$, there exists a finite subset $F \subseteq \Gamma$ such that

$$\max_{s\in S} \frac{|sF \oplus F|}{|F|} < \varepsilon,$$

where $A \oplus B$ is the symmetric difference. A net $(F_i)_{i\in I}$ of finite subsets of Γ is called a *Følner net* if

$$\frac{|sF_i \oplus F_i|}{|F_i|} \to 0,$$

for all $s \in \Gamma$.

Remark 4.1.5. The Følner condition can easily be translated into the notion of graphs as follows. Let Γ be a discrete group and $S \subseteq \Gamma$ a finite subset. If Γ satisfies the Følner condition, it means that for $\varepsilon > 0$, there exists a finite set F of vertices such that $\frac{|\partial F|}{|F|} < \varepsilon$. Comparing this to the definition of expanders, we see that a group satisfying the Følner condition is a somewhat bad expander. We will make this more precise later.

Definition 4.1.6. Let Γ be a discrete group. We say that Γ has an *approximate invariant mean* if for any finite set $S \subseteq \Gamma$ and any $\varepsilon > 0$, there exists $f \in \ell^1(\Gamma)_{1,+}$ such that

$$\|f - sf\| < \varepsilon,$$

for all $s \in S$.

We next give several equivalent characterisations of amenability. In order to do so, we need the following lemma:

Lemma 4.1.7. *For every mean m on $\ell^\infty(\Gamma)$, there exists a net $(\mu_i)_{i\in I}$ in $\ell^1(\Gamma)_{1,+}$ such that, for all $f \in \ell^\infty(\Gamma)$, we have*

$$\lim_i \left(\sum_{g\in\Gamma} f(g)\mu_i(g) \right) = m(f).$$

Proof. □

We are now ready to prove our main theorem of this section.

Theorem 4.1.8. *Let Γ be a discrete group. The following are equivalent:*

(1) Γ *is amenable.*

(2) Γ *has a left-invariant mean.*

(3) Γ *has an approximate invariant mean.*

(4) Γ *satisfies the Følner condition.*

(5) Γ *has a Følner net.*

(6) There exists a net of unit vectors $\xi_i \in \ell^2(\Gamma)$ such that for all $g \in \Gamma$,

$$\|\lambda_g \xi_i - \xi_i\| \to 0,$$

where λ is the left regular representation. That is, $(\xi_i)_{i\in I}$ is a net of almost invariant unit vectors for λ.

(7) There exists a net $(\varphi_i)_{i\in I}$ of finitely supported positive definite functions on Γ such that $\lim_i(\varphi_i(g)) = 1$, for all $g \in \Gamma$.

Proof. We have already showed $(1) \Leftrightarrow (2)$, so we proceed with $(2) \Rightarrow (3)$. To this end, let m be a left-invariant mean on $\ell^\infty(\Gamma)$. By Lemma 4.1.7 there exists a net $\mu_i \in \ell^1(\Gamma)_{1,+}$ such that μ_i converges to m in the weak *-topology. Given $g \in \Gamma$ and $f \in \ell^\infty(\Gamma)$ we have

$$(g\mu_i)(f) = \sum_{h\in\Gamma}(g\mu_i)(h)f(h) = \sum_{h\in\Gamma}\mu_i(g^{-1}h)f(h) = \sum_{s\in\Gamma}\mu_i(s)f(gs) = \mu_i(s^{-1}f),$$

which shows that $(g\mu_i)(f) \overset{w^*}{\to} m(s^{-1}f)$. Now using that m is left-invariant we see that $\mu_i - g\mu_i \overset{w^*}{\to} 0$ for all $g \in \Gamma$. Since $\mu_i - g\mu_i \in \ell^1(\Gamma)$ and $\ell^1(\Gamma)^*$ is isometrically isomorphic to $\ell^\infty(\Gamma)$, we see that $\mu_i - g\mu_i$ converges weakly in $\ell^1(\Gamma)$. Let now $S = \{s_1, \dots, s_n\} \subseteq \Gamma$ be a finite set. We have that the zero element belongs the weak closure of the convex set

$$(0, \dots, 0) \in \overline{\mathrm{conv}(\{(\mu_i - s_1\mu_i, \dots, \mu_i - s_n\mu_i) : i \in I\})}^{w},$$

and since the weak closure of a convex set in a Banach space is equal to the norm closure, we have

$$(0, \dots, 0) \in \overline{\mathrm{conv}\{(\mu_i - s_1\mu_i, \dots, \mu_i - s_n\mu_i) : i \in I\}}^{\|\cdot\|_1}.$$

Thus there exists a net $(\nu_i)_{i\in I}$ in the convex hull $\mathrm{conv}(\{\mu_i : i \in I\})$ satisfying

$$\|\nu_i - s\nu_i\|_1 \le \sum_{s\in S}\|\nu_i - s\nu_i\|_1 \to 0,$$

which means that for any $\varepsilon > 0$ there exists $i \in I$ such that $\|\nu_i - s\nu_i\|_1 < \varepsilon$ yielding the desired result.
$(3) \Rightarrow (4)$. Let $S \subseteq \Gamma$ be finite and $f \in \ell^1(\Gamma)_{1,+}$ with $\|f - sf\| < \delta$ for some $\delta > 0$. From Lemma 1.3.7 we know that there exists $\tilde{f} \in l^1(\Gamma)_{1,+}$ such that

$$\|\tilde{f} - f\| < \delta$$

for some $n \in \mathbb{N}$. Applying the triangle inequality three times we get

$$\|\tilde{f} - s\tilde{f}\|_1 \leq \|\tilde{f} - f\|_1 + \|f - sf\|_1 + \|sf - s\tilde{f}\|_1 \leq 3\delta,$$

for all $s \in S$. Define $A \subseteq \Gamma \times \mathbb{N}$ by

$$A = \{(g, i) \in \Gamma \times \mathbb{N} \mid \tfrac{i}{n} \leq \tilde{f}(g)\}.$$

Note that $|A| = n$ due to $\tilde{f}$ having norm 1. Thus we see that

$$\sum_{s\in S} \frac{|A \oplus As|}{|A|} = \sum_{s\in S} \|\tilde{f} - s\tilde{f}\| \leq |S|3\delta. \tag{4.1}$$

Let now $A^i = A \cap \Gamma \times \{i\}$ be the i'th level set of A. Then we see that

$$\sum_{s\in S} \frac{|A \oplus As|}{|A|} = \sum_{i\in\mathbb{N}} \sum_{s\in S} \frac{|A^i \oplus A^i s|}{|A|} = \sum_{i\in\mathbb{N}} \frac{|A^i|}{|A|} \sum_{s\in S} \frac{|A^i \oplus A^i s|}{|A^i|}.$$

To finish the proof, we assume, by contradiction, that

$$\sum_{s\in S} \frac{|A^i \oplus A^i s|}{|A^i|} > |S|3\delta$$

for all $i \in \mathbb{N}$. This implies by the above calculations that

$$\sum_{s\in S} \frac{|A \oplus As|}{|A|} = \sum_{i\in\mathbb{N}} \frac{|A^i|}{|A|} \sum_{s\in S} \frac{|A^i \oplus A^i s|}{|A^i|} > \sum_{i\in\mathbb{N}} \frac{|A^i|}{|A|}(|S|3\delta) = |S|3\delta,$$

since $\sum_{i\in\mathbb{N}} |A^i| = |A|$. But this contradicts our estimate in (4.1), so choosing $F = \{g \in \Gamma \mid (g, i) \in A^i\}$ satisfy the Følner condition. That is, given $\varepsilon > 0$ and some finite set $S \subseteq \Gamma$ we can choose $\delta = \frac{\varepsilon}{3|S|}$ to get the desired result.
$(4) \Rightarrow (5)$. Let $I = \{E \subseteq \Gamma \mid E \text{ finite}\} \times \mathbb{N}$. This is a partially ordered set with the order given by $(E, n) \preceq (E', n')$ if and only if $E \subseteq E'$ and $n \leq n'$. Since Γ satisfy the Følner condition for each $\alpha = (E, n) \in I$, we can find a finite subset $F_\alpha \subseteq \Gamma$ such that

$$\max_{s\in E} \frac{|sF_\alpha \oplus F_\alpha|}{|F_\alpha|} \leq \frac{1}{n}.$$

Thus $(F_\alpha)_{\alpha\in I}$ is a Følner net. Indeed, given $s \in \Gamma$ and $\varepsilon > 0$, set $\alpha_0 = (\{s\}, n)$, where $n \geq 1/\varepsilon$. Then for any $\alpha = (E, m)$ with $\alpha_0 \preceq \alpha$, we have $s \in E$ and $1/m \leq \varepsilon$ (by the definition of the partial order $\preceq$), and hence,

$$\frac{|sF_\alpha \oplus F_\alpha|}{|F_\alpha|} \leq \frac{1}{m} \leq \varepsilon.$$

This proves the claim.
(5) $\Rightarrow$ (6). Let $(F_i)_{i\in I}$ be a net of non-empty finite subsets of Γ satisfying

$$\frac{|sF_i \oplus F_i|}{|F_i|} \to 0,$$

for all $s \in \Gamma$. Consider $\xi_i = |F_i|^{-1/2} 1_{F_i}$. Then

$$\|\xi_i\|_2^2 = \sum_{g\in\Gamma} \left(\frac{1}{\sqrt{|F_i|}} 1_{F_i}(g)\right)^2 = \frac{|F_i|}{|F_i|} = 1,$$

and

$$\lambda_g \xi_i - \xi_i = g\xi_i - \xi_i = |F_i|^{-1/2}(1_{gF_i} - 1_{F_i}).$$

Thus, we have

$$\|\lambda_g \xi_i - \xi_i\|^2 = \sum_{s\in\Gamma} \left(\frac{1}{\sqrt{|F_i|}}(1_{gF_i} - 1_{F_i})(s)\right)^2 = \frac{|gF_i \oplus F_i|}{|F_i|} \to 0,$$

be assumption.
(6) $\Rightarrow$ (7). Consider $\varphi_i(g) = \langle \lambda_g \xi_i, \xi_i\rangle$ for $g \in \Gamma$. Note that $\varphi_i(e) = \|\xi\|^2 = 1$. Thus, by Lemma 3.3.7 we have that φ_i is positive definite. Furthermore, we can rewrite φ_i as $\varphi_i(g) = \langle \lambda_g \xi_i - \xi_i, \xi_i\rangle + \langle \xi_i, \xi_i\rangle = \langle \lambda_g \xi_i - \xi_i, \xi_i\rangle + 1$. Hence, for all $g \in \Gamma$, we have

$$|\varphi_i(g) - 1| = |\langle \lambda_g \xi_i - \xi_i, \xi_i\rangle \leq \|\lambda_g \xi_i - \xi_i\|\|\xi_i\| = \|\lambda_i \xi_i - \xi_i\| \to 0.$$

Unfortunately, φ_i is not (in general) finitely supported. However, since $\xi_i \in \ell^2(\Gamma)$, then for all $n \in \mathbb{N}$, there exists a finitely supported $\xi_{i,n} \in \ell^2(\Gamma)$ satisfying $\|\xi_i - \xi_{i,n}\| < 1/n$ with $\|xi_{i,n}\| = 1$. Defining $\varphi_{i,n} : \Gamma \to \mathbb{C}$ by $\varphi_{i,n}(g) = \langle \lambda_g \xi_{i,n}, \xi_{i,n}\rangle$, we get a finitely supported function which converges pointwise to φ_i as $n \to \infty$. To see that this is the case, observe that $\varphi_{i,n}(g)$ is non-zero only if both $\lambda_g \xi_{i,n}$ and $\xi_{i,n}$ are non-zero. This means that

$$\mathrm{supp}(\varphi_{i,n}) \subseteq \{g \in \Gamma \mid \exists x, y \in \mathrm{supp}(\xi_{i,n}) : gx = y\} = \{yx^{-1} \mid x, y \in \mathrm{supp}(\xi_{i,n})\},$$

and the support of $\xi_{i,n}$ is assumed to be finite. Let $P_1(\Gamma)$ be the set of all positive definite functions φ on Γ with $\varphi(e) = 1$ and $C_c(\Gamma)$ the set of finitely supported functions on Γ. Now $\varphi_{i,n} \in P_1(\Gamma)$ by Lemma 3.3.7, and $\varphi_{i,n} \in C_c(\Gamma)$ by construction, showing that $\varphi_{i,n}$ belongs to $P_1(\Gamma) \cap C_c(\Gamma)$. Due to $\varphi_{i,n}$ converges to φ_i pointwise, we get that $\varphi_i \in$

$\overline{P_1(\Gamma) \cap C_c(\Gamma)}$ and $1 \in \overline{P_1(\Gamma) \cap C_c(\Gamma)}$, where 1 is the constant function with value 1, and the closure is taken in the topology of pointwise convergence of functions. This shows us that there exists a net $(\psi_j)_{j \in J} \in P_1(\Gamma) \cap C_c(\Gamma)$ such that $\psi_j(g) \to 1$, for all $g \in \Gamma$ as wanted.
$(7) \Rightarrow (1)$. The proof of this implication requires results about completely positive maps and multipliers on C^*-algebras associated to the group Γ, which (unfortunately) is beyond the scope of this thesis. We refer the reader to [5] for the necessary background and the proof of Theorem 2.6.8 therein. □

Now we are ready to consider some of the most commonly known groups and determine if they are amenable or not. We start with the easiest example, which we almost have proved already.

Example 4.1.9. Let Γ be a finite group. Then Γ is amenable.

Proof. Take $F = \Gamma$. Then F satisfies the Følner condition. □

Example 4.1.10. Let $\Gamma = \mathbb{Z}$. Then Γ is amenable.

Proof. Let $F_n = \{-n, \dots, 1, 0, 1, \dots, n\}$. Then F_n is a Følner sequence. □

We now show a theorem connecting growth and amenability.

Theorem 4.1.11. *A finitely generated group Γ with subexponential growth is amenable.*

Proof. Let Γ be a finitely generated group with subexponential growth. We will use the sets $B(n)$ as Følner sets. From Remark 3.2.5, we know that $\lim |B(n)|^{1/n} = c \leq 1$, for some constant c. This means that for any $\varepsilon > 0$, there exists some $k_\varepsilon \in \mathbb{N}$ such that

$$\frac{|B(k_\varepsilon + 1)|}{|B(k_\varepsilon)|} < 1 + \varepsilon.$$

Let now $\Sigma \subseteq \Gamma$ be a finite generating subset of Γ, and define $n_i := k_{(1/i)}$ for each i. Then for each $s \in \Sigma$ we have

$$\frac{|sB(n_i) \oplus B(n_i)|}{|B(n_i)|} \leq \frac{2(|B(n_i + 1)| - |B(n_i)|)}{|B(n_i)|} < 2(1 + \tfrac{1}{i}) - 2 \to 0,$$

as $i \to \infty$. Thus, for every $g \in \Gamma$, we get

$$\frac{|gB(n_i) \oplus B(n_i)}{|B(n_i)|} \to 0,$$

as $i \to \infty$, since any $g \in \Gamma$ is a finite string of elements from Σ, i.e., g is of the form $g = s_1 \cdots s_n$ with $s_i \in \Sigma$. □

Corollary 4.1.12. *All abelian groups are amenable.*

Proof. Recalling Corollary 3.2.10 we know that any abelian group has polynomial growth. Next using Proposition 3.2.4, we see that any abelian group has subexponential growth. Last we invoke Theorem 4.1.11 above to conclude the proof. □

The last example is an example of group which is not amenable.

Example 4.1.13. The free group $\Gamma = \mathbb{F}_2$ on two generators is not amenable.

Proof. Let $\Sigma = \{a, a^{-1}, b, b^{-1}\}$ and let A, A^{-1}, B, B^{-1} denote the sets of all words starting with a, a^{-1}, b, b^{-1} respectively, i.e. $A = \{g \in \Gamma \mid g = ag' : g' \in \Gamma\}$. Suppose for contradiction that for any $\varepsilon > 0$, we can find a finite, non-empty set $F \subseteq \Gamma$ satisfying the Følner condition. Notice that the set $a \cdot (F \cap (A^{-1} \cup B \cup B^{-1})) = a \cdot (F \setminus A)$ is contained in $a \cdot F$ and in A. This implies that

$$|a(F \setminus A^{-1})| \leq |F \cap A| - \varepsilon|F|,$$

which leads to

$$|F| - |F \cap A^{-1}| \leq |F \cap A| - \varepsilon|F|.$$

This hold for each of the four permutations, and summing over all four, we get

$$4|F| - |F| \leq |F| + 4\varepsilon|F|,$$

and for $0 < \varepsilon < 1/2$ we obtain the contradiction. □

In fact, any group containing a subgroup isomorphic to $\mathbb{F}_2$ is non-amenable. This follows from the fact that amenability passes to subgroups. There are many nice permanence properties of amenability, but we will not pursue this further.

We end this section with a result relating amenability and property (A).

Theorem 4.1.14. *The Cayley graph associated to an amenable, finitely generated group Γ has property (A).*

Proof. Let $\varepsilon > 0$ and $R > 0$ be given and pick $\delta < \frac{\varepsilon}{\varepsilon+1}$. For $g \in \Gamma$ we choose $S \subseteq \Gamma$ to be

$$S = B(g, R) = \{h \in \Gamma \mid d(g, h) = |h^{-1}g| \leq R\}.$$

This is a finite set, and since Γ is assumed amenable, we know that there exists finite $F \subseteq \Gamma$ satisfying the Følner condition with respect to S and δ. This lets us define $A_g = gF \times \{1\}$. Thus for $g, h \in \Gamma$ with $d(g, h) \leq R$ we have

$$\frac{|A_g \oplus A_h|}{|A_g \cap A_h|} = \frac{|gF \oplus hF|}{|gF \cap hF|} = \frac{|h^{-1}gF \oplus F|}{|h^{-1}gF \cap F|} = \frac{|h^{-1}gF \oplus F|}{|F|} \cdot \frac{|F|}{|h^{-1}gF \cap F|}.$$

By assumption we have that $\frac{|h^{-1}gF \oplus F|}{|F|} < \delta$. For the other factor we note that

$$\frac{|h^{-1}gF \cap F|}{F} = \frac{|h^{-1}gF \cup F|}{|F|} - \frac{|h^{-1}gF \oplus F|}{|F|} \geq 1 - \delta,$$

where we have used that $A \cap B$ and $A \oplus B$ are disjoint and $A \cup B = (A \oplus B) \cup (A \cap B)$. Putting it all together we obtain

$$\frac{|A_g \oplus A_h|}{|A_g \cap A_h|} \leq \frac{\delta}{1-\delta} < \varepsilon. \qquad \square$$

A natural question is whether any group, for which the Cayley graph has property (A), is amenable. But the answer to this question is negative. To see this, we recall that any infinite tree has property (A), and the Cayley graph of $\mathbb{F}_2$ is indeed such a tree, but Example 4.1.13 shows that $\mathbb{F}_2$ is not amenable.

4.2 Haagerup's property (H)

We now introduce the *Haagerup property*, abbreviated property (H). This property is also known as *a-T-menability*, a terminology due to Gromov. Haagerup [11] and Gromov [10] independently introduced this property in different settings. The Haagerup property is a weaker notion than amenability. The definition of property (H) we use here is due to Alain Connes. First recall that given a discrete group Γ, we define

$$P_1(\Gamma) := \{\varphi : \Gamma \to \mathbb{C} \mid \varphi \text{ is positive definite and } \varphi(e) = 1\}.$$

Furthermore, we define $C_0(\Gamma)$ to be the set of functions $\varphi : \Gamma \to \mathbb{C}$ vanishing at infinity, i.e., for any $\varepsilon > 0$, the set $\{g \in \Gamma \mid |\varphi(g)| \geq \varepsilon\}$ is finite.

Definition 4.2.1. We say that a finitely generated group Γ has property (H) if there exists a net $(\varphi_i)_{i \in I} \subseteq C_0(\Gamma) \cap P_1(\Gamma)$ such that φ_i converges pointwise to 1 on Γ.

Notice how this definition resembles one of the properties of Γ being amenable. In fact, by Theorem 4.1.8, it follows that amenable groups have property (H). However, property (H) is a priori a weaker notion than amenability, since the net is allowed to exist in $C_0(\Gamma)$ and not just in $C_1(\Gamma)$. As we shall see later (see Example 4.2.5), property (H) turns out to be strictly weaker than amenability.

There are several other equivalent definitions of property (H). One which is commonly used is in terms of affine actions.

Definition 4.2.2. A 1-cocycle $b : \Gamma \to H$ is called *proper* if $\|b_g\| \to \infty$ as $|g| \to \infty$, i.e., the set $\{g \in \Gamma \mid \|b_g\| \leq R\}$ is finite, for all $R > 0$.

Note that if $\Sigma \subseteq \Gamma$ is a finite generating set of Γ, it suffices to require that the set $\{s \in \Sigma \mid \|b_s\| \leq R\}$ is finite, for all $R > 0$.

Definition 4.2.3. Let Γ be a finitely generated group. A group homomorphism $\theta : \Gamma \to \mathrm{AffIso}(\Gamma)$ is called *proper* if for all $\xi \in H$, $\|\theta(g)\xi\| \to \infty$, as $|g| \to \infty$.

Theorem 4.2.4. *Let Γ be a finitely generated group the following are equivalent:*

1. Γ *has property (H).*
2. Γ *admits a proper* 1*-cocycle.*
3. Γ *admits a proper isometric action on a Hilbert space.*

Proof. We will prove that $1 \Leftrightarrow 2$ and $2 \Leftrightarrow 3$.
$(1) \Rightarrow (2)$. Since Γ has property (H), there exists a sequence $(\varphi_n)_{n=1}^{\infty} \in C_0(\Gamma) \cap P_1(\Gamma)$ such that $\varphi_n \to 1$ pointwise. Let $\{s_n\}_{n=1}^{\infty}$ be an enumeration of Γ and choose $l_1 \in \mathbb{N}$ such that $|1 - \varphi_{l_1}(s_1)| < \frac{1}{2}$. Now choose $l_2 > l_1$ such that

$$|1 - \varphi_{l_2}(s_k)| < \tfrac{1}{4}, \quad k = 1, 2.$$

Continue this construction inductively by choosing $l_n > l_{n-1}$ such that

$$|1 - \varphi_{l_n}(s_k)| < \tfrac{1}{2^n}, \quad k = 1, 2, \ldots, n.$$

We define $\psi_n = \varphi_{l_n}$. Since $\psi_n \in P_1$ for all n we see that ψ_n is positive definite and $\psi_n(e) = 1$. Furthermore we have that

$$|1 - \psi_n(s_k)| < \tfrac{1}{2^n}, \quad k = 1, \ldots, n.$$

Now we use the *GNS construction* to obtain a triple (π_n, H_n, ξ_n), where H_n is a Hilbert space, $\pi_n : \Gamma \to B(H)$ is a unitary representation and $\xi_n \in H_n$ with $\|\xi_n\| = 1$ such that for all $g \in \Gamma$,

$$\psi_n(g) = \langle \pi_n(g)\xi_n, \xi_n \rangle,$$

Let $H = \bigoplus_{n=1}^{\infty} H_n$ and define $b : \Gamma \to H$ by $b(g) = (\xi_n - \pi_n(g)\xi_n)_{n=1}^{\infty} \in H$, for all $g \in \Gamma$. Further we set $\pi(g) = \bigoplus_{n=1}^{\infty} \pi_n(g)$. Note that for each n, $b_n(g) = \xi_n - \pi_n(g)\xi_n$ is a 1-coboundary with coefficients in (π_n, H_n). Thus, b becomes a 1-cocycle provided it is well defined, i.e., that $\sum_{n=1}^{\infty} \|b(g)\|^2 < \infty$, for all $g \in \Gamma$. To this end, we fix $g \in \Gamma$, and see that

$$\begin{aligned}\sum_{n=1}^{\infty} \|b_n(g)\|^2 &= \sum_{n=1}^{\infty} (\|\xi_n\|^2 + \|\pi_n(g)\xi_n\|^2 - 2\Re\langle \pi_n(g)\xi_n, \xi_n\rangle) \\ &= \sum_{n=1}^{\infty} (2 - 2\Re\psi_n(g)) \leq 2\sum_{n=1}^{\infty} |1 - \psi_n(g)|.\end{aligned}$$

Now we note that for $g = s_k$, we get

$$\sum_{n=k}^{\infty} |1 - \psi_n(s_k)| \leq \sum_{n=k}^{\infty} \frac{1}{2^n} < \infty.$$

From this we see that $\sum_{n=1}^{\infty} \|b(g)\|^2 < \infty$, for all $g \in \Gamma$, and hence b is well defined.
It remains to show that b is proper. To do this, it suffices to show that for all $N \in \mathbb{N}$, the set

$E_N := \{g \in \Gamma \mid \|b(g)\| \leq \sqrt{N}\}$ is finite. Applying the calculations above, we see that for any $g \in E_N$,

$$N \geq \|b(g)\|^2 \geq \sum_{n=1}^{N} \|b_n(g)\|^2 = 2\sum_{n=1}^{N} |1 - \Re\psi_n(g)\| \geq 2N - 2\sum_{n=1}^{N} \Re\psi_n(g).$$

This shows us that $E_N \subseteq \{g \in \Gamma \mid \sum_{n=1}^{N} \Re\psi_n(g) \geq \frac{N}{2}\}$. By construction of ψ_n, $\sum_{n=1}^{N} \Re\psi_n(g) \in C_0(\Gamma)$, so E_N must be finite, as wanted.
(1) $\Leftarrow$ (2). Suppose that $b : \Gamma \to H$ is a proper 1-cocycle. Then, by Corollary 3.3.12 (Schoenberg's theorem), we see that $e^{-\gamma\|b(g)\|} \in P_1(\Gamma) \cap C_0(\Gamma)$. Letting $\gamma \to 0$ we see that Γ has property (H).
(2) $\Rightarrow$ (3). Let $b : \Gamma \to H$ be a 1-cocycle for a unitary representation (π, H). By Lemma 3.3.10 we get that $\theta : \Gamma \to \mathrm{AffIso}$ given by

$$\theta(g)\xi = \pi(g)\xi + b(g),$$

defines a group homomorphism. Furthermore, we have that for all $\xi \in H$

$$\|\theta(g)\xi\| = \|\pi(g)\xi + b(g)\| \geq \|b(g)\| - \|pi(g)\xi\| = \|b(g)\| - \|\xi\|.$$

By assumption, $\|b(g)\| \to \infty$, as $|g| \to \infty$, and the assertion follows.
(2) $\Leftarrow$ (3). Let $\theta : \Gamma \to \mathrm{AffIso}$ be a group homomorphism. For $\xi \in H$ and $g \in \Gamma$ we get from Lemma 3.3.10 that

$$\theta(g)\xi = \pi(g)\xi + b(g),$$

for some unitary representation $\pi \to B(H)$ and some 1-cocycle $b : \Gamma \to H$ with coefficients in (π, H). Now using that θ is a proper isometric action, we see that for $\xi = 0_H$, the zero vector in H, we have $\|\theta(g)0_H\| \to \infty$ as $|g| \to \infty$, and this implies that

$$\|b(g)\| = \|\theta(g)0_H\| \to \infty,$$

as $|g| \to \infty$, proving that b is a proper 1-cocycle. □

As mentioned earlier, property (H) is a weaker notion than amenability. This can be seen by the following example, which was proven by Haagerup in [11]:

Example 4.2.5. The free group $\Gamma = \mathbb{F}_2$ on two generators has property (H).

More precisely, Haagerup showed that the map $s \mapsto e^{-\lambda|s|}$ belongs to $C_0(\Gamma) \cap P_1(\Gamma)$, for all $\lambda > 0$ (see [11, Lemma 1.2]). It was his result that prompted Alain Connes to introduce the *Haagerup property*.

4.3 Kazhdan's property (T)

Property (T) is a another interesting property for groups, introduced by Kazhdan in [14]. It is in some sense the opposite of amenability and property (H) as we shall see.

We say that a net $(\xi_i)_{i\in I} \in H$ in some Hilbert space H is *almost* Γ*-invariant* if, for all $g \in \Gamma$, we have

$$\|\pi(g)\xi_i - \xi_i\| \to 0.$$

Definition 4.3.1. We say that a discrete group Γ has property (T) if for every unitary representation $\pi : \Gamma \to B(H)$ with almost invariant unit vectors $\xi_i \in H$, i.e., $\|\pi(g)\xi_i - \xi_i\| \to 0$ for all $g \in \Gamma$, there exists a non-zero $\xi \in H$ such that $\pi(g)\xi = \xi$, for all $g \in \Gamma$.

Lemma 4.3.2. *Let* Γ *be a discrete group. Then,*

1. *if* Γ *is amenable, then the left regular representation* λ *has almost* Γ*-invariant vectors.*

2. *if* Γ *is infinite, then* λ *has no non-zero* Γ*-invariant vectors.*

Proof. 1. This follows from Theorem 4.1.8.
2. Assume, for contradiction, that there exists a non-zero $\xi \in H$, such that ξ is Γ-invariant for λ. Then, for all $g \in \Gamma$, we have

$$\langle \xi, \delta_g\rangle = \langle \xi, \lambda(g)\delta_e\rangle = \langle \lambda(g^{-1})\xi, \delta_e\rangle = \langle \xi, \delta_e\rangle,$$

since ξ is Γ-invariant. This, however, implies that

$$\|\xi\|^2 = \sum_{g\in\Gamma} |\langle \xi, \delta_g\rangle| = \sum_{g\in\Gamma} |\langle \xi, \delta_e\rangle| < \infty,$$

which contradicts the fact that Γ is infinite. □

As a consequence we obtain the following:

Remark 4.3.3. If Γ is amenable and has property (T), then Γ is finite.

Next we introduce the notion of a *Kazhdan pair* for a discrete group Γ. For a subset $S \subseteq \Gamma$ and a constant $\varepsilon > 0$ we say that $\xi \in H$ is (S, ε)*-invariant* if ξ is non-zero and

$$\sup_{s\in S} \|\pi(s)\xi - \xi\| < \varepsilon\|\xi\|.$$

Definition 4.3.4. A pair (S, ε) where $S \subseteq \Gamma$ is a subset and $\varepsilon > 0$ is called a *Kazhdan pair* for Γ if any unitary representation of Γ with a non-zero (S, ε)-invariant vector has a non-zero Γ-invariant vector.

We say that Γ has property (T) with respect to a subset $S \subseteq \Gamma$ if there exists $\varepsilon > 0$ such that (S, ε) is a Kazhdan pair. The following proposition relates the two notions.

Proposition 4.3.5. *Let Γ be a discrete group. Then Γ has property (T) if and only if there exist finite $S \subseteq \Gamma$ and $\varepsilon > 0$ such that (S, ε) Kazhdan pair for Γ.*

Proof. We prove the first part by contraposition. Suppose that there exists no finite $S \subseteq \Gamma$ and $\varepsilon > 0$ such that (S, ε) is a Kazhdan pair for Γ. Then by negating the definition of a Kazhdan pair we see that for all finite sets $S \subseteq \Gamma$ and $\varepsilon > 0$ there exists a unitary representation (π, H) of Γ without any non-zero Γ-invariant vectors, but such that there exists a unit vector $\xi \in H$ with $\|\pi(s)\xi - \xi\| < \varepsilon$ for all $s \in S$. Define

$$I := \{(S, \varepsilon) \mid S \subseteq \Gamma \text{ is a finite set and } \varepsilon > 0\}.$$

We now turn I into a directed set by defining $(S_1, \varepsilon_1) \preceq (S_2, \varepsilon_2)$ if $S_1 \subseteq S_2$ and $\varepsilon_2 \leq \varepsilon_1$. By our assumption, we have that for all $i = (S_i, \varepsilon_i) \in I$ there exists a unitary representation (π_i, H_i) without any non-zero Γ-invariant vectors, but such that there exists unit vectors $\xi_i \in H_i$ with $\|\pi_i(s)\xi_i - \xi_i\| < \varepsilon_i$ for all $s \in S_i$. Let now $H = \bigoplus_{i \in I} H_i$ and define $\pi = \bigoplus_{i \in I} \pi_i : \Gamma \to B(H)$. Then $\xi_i \in H$ for all $i \in I$ and $\|\pi(g)\xi_i - \xi_i\| \to 0$ for all $g \in \Gamma$. This means that $(\xi_i)_{i \in I}$ is a net of almost Γ-invariant unit vectors. Thus, to finish the proof of the implication, we must show that there exists no non-zero Γ-invariant vectors, since then Γ does not have property (T). To this end assume, for contradiction, that there exists a non-zero Γ-invariant $\xi \in H$. Let $P_i : H \to H_i$ be the orthogonal projection and note that $\pi_i(g)P_i = P_i\pi(g)$ for all $g \in \Gamma$ and $i \in I$. This means that

$$\pi_i(g)P_i\xi = P_i\pi(g)\xi = P_i\xi,$$

since ξ is Γ-invariant. This, however, shows that $P_i\xi$ is Γ-invariant with respect to π_i for all $i \in I$, but π_i does *not* have any non-zero Γ-invariant vectors which implies that $P_i\xi$ must be zero. Rewriting ξ as $\xi = \sum_{i \in I} P_i\xi$, we conclude that $\xi = 0$ which contradicts our assumption of ξ being non-zero. This shows that Γ does not have property (T) as wanted.
Suppose now that there exists finite $S \subseteq \Gamma$ and $\varepsilon > 0$ such that (S, ε) is a Kazhdan pair for Γ. Let $(\xi_i)_{i \in I}$ be a net of almost Γ-invariant unit vectors for some given unitary representation (π, H), i.e., $\|\pi(g)\xi_i - \xi_i\| \to 0$ for all $g \in \Gamma$. Then for all $g \in \Gamma$ there exists $i_g \in I$ such that $\|\pi(g)\xi_i - \xi_i\| < \varepsilon$ for all $i \succeq i_g$. Due to I being directed and S finite, there exists $i_0 \in I$ (depending on S) such that $i_0 \succeq i_s$ for all $s \in S$. Setting $\xi_0 = \xi_{i_0}$ we have that $\|\pi(s)\xi_0 - \xi_0\| < \varepsilon$ for all $s \in S$ which means that ξ_0 is a (S, ε)-invariant vector. Now using that (S, ε) is a Kazhdan pair for Γ it follows that π has a non-zero Γ-invariant vector, and hence we see that Γ has property (T) as wanted. □

Remark 4.3.6. Using the notion of circumcenter of a bounded subset of a Hilbert space, one can prove the following very useful fact: for any group Γ, the pair $(\Gamma, \sqrt{2})$ is a Kazhdan pair. For a proof of this, see [5, Lemma 12.1.5]. Combining this with 4.3.5, we conclude that finite groups have property (T).

Another application of the Kazhdan pair is illustrated by the following result:

Proposition 4.3.7. *Suppose (S, ε) is a Kazhdan pair for Γ and that $\pi : \Gamma \to B(H)$ is a unitary representation of Γ with the property that there exists a non-zero $\xi \in H$ such that $\pi(s)\xi = \xi$ for all $s \in S$. Then $\pi(g)\xi = \xi$ for all $g \in \Gamma$, i.e., ξ is in fact Γ-invariant.*

Proof. Let H_0 be the set of all Γ-invariant vectors in H, and set $K = H_0^\perp$. Note that both H_0 and K are invariant under π, i.e., $\pi(g)H_0 \subseteq H_0$ and $\pi(g)K \subseteq K$ for all $g \in \Gamma$. Let π_1 be the restriction of π to K, i.e., $\pi_1 = \pi_{|K} : \Gamma \to B(K)$. Then, by construction, π_1 has no non-zero Γ-invariant vectors. This means that for all $\eta \in K$, there exists some $s \in S$ such that

$$\|\pi_1(s)\eta - \eta\| \geq \varepsilon\|\eta\|.$$

Now rewrite $\xi = \xi_0 + \eta$ for some $\xi_0 \in H_0$, $\eta \in K$. Then we see that for all $s \in S$,

$$\xi = \pi(s)\xi = \pi(s)\xi_0 + \pi(s)\eta = \xi_0 + \pi(s)\eta.$$

This implies that $\pi(s)\eta = \eta$ for all $s \in S$ and hence $\eta = 0$. Thus $\xi = \xi_0$ and hence ξ is Γ-invariant as wanted. □

We now introduce the notion of the *Kazhdan constant* associated to a finite subset of a group Γ with property (T). This concept will be useful for us, when we construct expanders in Section 5.4. We begin with the following more general definition:

Definition 4.3.8. Let Γ be a discrete group and $S \subseteq \Gamma$ a finite set. We define the *Kazhdan constant* k_S by

$$k_S := \inf_{(\pi,H)} \inf_{\substack{\xi \in H, \\ \|\xi\|=1}} \max_{s \in S}(1 - \Re\langle\pi(s)\xi, \xi\rangle),$$

where the infimum is taken over all unitary representations (π, H) having *no* non-zero Γ-invariant vectors.

Note that $k_S \geq 0$. This leads us to our next result.

Lemma 4.3.9. *Let Γ be a discrete group and $S \subseteq \Gamma$ a finite set.*

- *If (S, ε) is a Kazhdan pair for some $\varepsilon > 0$, then $k_S \geq \varepsilon^2/2$.*
- *For any unitary representation (π, H) of Γ with no non-zero Γ-invariant vectors, and for any unit vector $\xi \in H$, there exists $s \in S$ such that*

$$1 - \Re\langle\pi(s)\xi, \xi\rangle \geq k_S.$$

Proof. Both statements follows from the estimate

$$\|\pi(s)\xi - \xi\|^2 = \langle\pi(s)\xi - \xi, \pi(s)\xi - \xi\rangle = 2(1 - \Re\langle\pi(s)\xi, \xi\rangle).$$ □

Remark 4.3.10. If Γ has property (T) with respect to a finite set $S \subseteq \Gamma$, then $k_S > 0$ and, moreover, $(S, \sqrt{2k_S})$ is a Kazhdan pair.

Our next result shows us that any discrete group with property (T) is finitely generated.

Theorem 4.3.11. *Any discrete group with property (T) is finitely generated.*

Proof. Let (S, ε) be a Kazhdan pair for Γ. Since Γ has property (T), Proposition 4.3.5 tells us that such a pair exists with $S \subseteq \Gamma$ finite. We wish to show that S is in fact a generating set for Γ. Denote by Γ_0 the set generated by S, and consider $\Gamma/\Gamma_0 = \{g\Gamma_0 : g \in \Gamma\}$. Let $\pi : \Gamma \to B(\ell^2(\Gamma/\Gamma_0))$ be the unitary representation defined by

$$\pi(g)\delta_{s\Gamma_0} = \delta_{gs\Gamma_0},$$

for $g, s \in \Gamma$. Consider the vector $\xi = \delta_{\Gamma_0} \in \ell^2(\Gamma/\Gamma_0)$ and note that for all $s \in S$,

$$\pi(s)\xi = \pi(s)\delta_{\Gamma_0} = \delta_{s\Gamma_0} = \delta_{\Gamma_0} = \xi,$$

since $s \in S \subseteq \Gamma_0$. Using Theorem 4.3.7 we see that $\pi(g)\xi = \xi$ for all $g \in \Gamma$. This implies that $g\Gamma_0 = \Gamma_0$ for every $g \in \Gamma$ which shows that $\Gamma_0 = \Gamma$ as wanted. □

Lemma 4.3.12. *Let $N \lhd \Gamma$ be a normal subgroup of Γ. If Γ has property (T), then so does the quotient Γ/N.*

Proof. Let $(\xi_i)_{i \in I} \in H$ be a net of almost Γ/N-invariant vectors for some unitary representation $(\tilde{\pi}, H)$ of Γ/N. Our goal is to show that there exists $\xi \in H$ such that $\tilde{\pi}(s)\xi = \xi$, for all $s \in \Gamma/N$. Define $\pi : \Gamma \to B(H)$ by $\pi = \tilde{\pi} \circ q$, where $q : \Gamma \to \Gamma/N$ is the quotient map. Then π is a unitary representation (since π is the composition of group homomorphisms) of Γ, and $(\xi_i)_{i \in I}$ is a net of almost invariant Γ vectors. Using that Γ has property (T), we know that there exists $\xi \in H$ such that $\pi(g)\xi = \xi$ for all $g \in \Gamma$. Thus, for any $s \in \Gamma/N$, we can choose a representative $g \in \Gamma$ such that $q(g) = s$. From this, we see that

$$\xi = \pi(g)\xi = \tilde{\pi}(q(g))\xi = \tilde{\pi}(s)\xi$$

for all $s \in \Gamma/N$, showing that Γ/N has property (T). □

In order to establish connections between property (T) and property (H), we state (without proof) the following important equivalent definition of property (T) in terms of positive definite functions on the group (for a proof, see [5, Theorem 12.1.7]).

Theorem 4.3.13. *Let Γ be a discrete countable group. Then Γ has property (T) if and only if any sequence of positive definite functions on Γ that converges pointwise to the constant function 1, converges uniformly on Γ.*

Combining this with the definition of property (H), we obtain the following result.

Corollary 4.3.14. *A countable discrete group Γ having both property (H) and property (T) is finite.*

We end this section by discussing some concrete examples.

Example 4.3.15. The group $\Gamma = \mathbb{Z}^n$, $n \geq 1$, does not have property (T).

Proof. We know from Corollary 4.1.12 that $\mathbb{Z}^n$ is amenable, and clearly, $\mathbb{Z}^n$ is not finite for $n \geq 1$. Thus, it follows from Remark 4.3.3 that $\mathbb{Z}^n$ does not have property (T). □

Example 4.3.16. The group $\Gamma = \mathbb{F}_n$, $n \geq 2$, is non-amenable, but does not have property (T).

Proof. It is a well-known fact that $\mathbb{Z}^n$ is a quotient of $\mathbb{F}_n$, and combining this with the result of Lemma 4.3.12, we conclude that $\mathbb{F}_n$ cannot have property (T). Another way to prove that $\mathbb{F}_n$ does not have property (T), is to recall that $\mathbb{F}_n$ has property (H), but is infinite. $\square$

It is an important result of Kazhdan [14] that the group $SL_n(\mathbb{Z})$, $n \geq 3$ has property (T). These are all residually finite groups. We refer the reader to [22] for a nice proof of Kazhdan's result. These groups will play an important role in connection with a concrete construction of expanders. We discuss this construction in section 5.4. Note also that the case $n = 2$ is different, since the group $SL_2(\mathbb{Z})$ does not have property (T).

4.4 Literature

This chapter is written with inspiration from [5].

Chapter 5

Box spaces

5.1 Introduction to box spaces

In this chapter we introduce the notion of *box spaces*. These are of great interest for us, since they give us ways of constructing spaces without property (A). We first show that a box space of a finitely generated group Γ has property (A) if and only if Γ is amenable. The other important result we will show is that box spaces of residually finite groups with property (T) are in fact families of expanders. This result has a lot of interest to us, since it gives us a way to explicit construct a family of expanders. We start with some notation. Let Γ be a finitely generated, residually finite group, i.e. the exists a finite (symmetric) subset $\Sigma \subseteq \Gamma$ generating Γ, and for each non-trivial $g \in \Gamma$ there exists a homomorphism h to a finite group with $h(g) \neq e$. Note that this means that the intersection of all finite index normal subgroups of a residually finite group Γ is trivial. Let $(N_i)_{i\geq 1}$ be a decreasing sequence of finite index normal subgroups with trivial intersection, i.e., $N_i \subseteq N_j$ if $i \geq j$ and with $\bigcap_{i=1}^{\infty} N_i = \{e\}$. We call such a sequence a filtration of Γ.

Definition 5.1.1. The box space $\square_{(N_i)}\Gamma$ of a residually finite group Γ with respect to the filtration (N_i), is the coarse disjoint union $\bigsqcup \Gamma/N_i$.

The coarse structure of the box space $\square_{(N_i)}\Gamma$ is independent of the generating set Σ, but it may depend on the sequence (N_i). Thus we have to be careful when we are dealing with box spaces. Given a box space of a group Γ, it can be thought of as a family of finite Cayley graphs which converges to the Cayley graph of Γ. This can be justified by the following result which shows that the geometry of the finite quotients approximates, in some sense, the geometry of the group.

Proposition 5.1.2. *Let $\square_{(N_i)}\Gamma$ be the box space of a residually finite group Γ. Then for all $R > 0$, there exist isometries $\pi_i : B_\Gamma(e, R) \to B_{\Gamma/N_i}(e, R)$, for all $i \geq i_0$, for some $i_0 \in \mathbb{N}$.*

Proof. Let Γ be a residually finite group and let $N_i \subseteq \Gamma$ be a filtration. Each N_i is the kernel of the quotient homomorphism π_i, and π_i is an isomorphisms when restricted to Γ/N_i. Since the intersection of the subgroups is trivial, the kernels satisfy $\lim_{i=1}^{\infty} \ker(\pi_i) = \{e\}$. Let

$R > 0$ be given and consider the finite ball $B_\Gamma(e, R)$. For each $g_1, g_2, \ldots, g_n \in B_\Gamma(e, R)$ we can pick $i_1, i_2, \ldots, i_n \in \mathbb{N}$ such that $\pi_i(g_i) \notin \ker(\pi_i)$. For $i_0 = \max\{i_1, i_2, \ldots, i_n\}$ we see that $\pi_{i_0}(B_\Gamma(e, R)) \simeq B_{\Gamma/N_i}(e, R)$, and thus $\pi_i : B_\Gamma(e, R) \to B_{\Gamma/N_i}(e, R)$ is an isometry for all $i \geq i_0$. $\square$

Note that for any finite subset $S \subseteq \Gamma$ there exists an $R > 0$ such that $S \subseteq B_\Gamma(e, R)$. This implies that for all $i \geq i_0$ the quotient maps are isometries when restricted to S.

5.2 Property (A)

In this section we prove that a box space of a group has property (A) if and only if the group is amenable. Before we can do that, we need a few results.

Theorem 5.2.1. *Let Γ be an amenable group, and let $\varepsilon, R > 0$ be given. Then for any $S > 0$ the following are equivalent:*

1. *There exists a map $\xi : \Gamma \to \ell^1(\Gamma)_{1,+}$ such that $\|\xi_x - \xi_y\| < \varepsilon$ when $d(x, y) \leq R$ and $\operatorname{supp}(\xi_x) \subseteq B(x, S)$.*
2. *There exists a function $f \in \ell^1(\Gamma)_{1,+}$ such that $\|f - gf\| < \varepsilon$ when $|g| \leq R$ and $\operatorname{supp}(f) \subseteq B(e, S)$.*

This theorem tells us that given constants ε and R, then any $S > 0$ satisfies the conditions for the Higson-Roe function ξ and the approximate invariant mean f simultaneously.

Proof. Suppose there exists $\xi : \Gamma \to \ell^1(\Gamma)_{1,+}$ satisfying the above. Since Γ is amenable, there exists a left-invariant mean $m : \ell^\infty(\Gamma) \to \mathbb{R}$. We define $f : \Gamma \to \mathbb{R}$ by

$$f(g) = m(\xi_x(xg)),$$

for all $g \in \Gamma$. This is a well defined function, since $\xi_x(xg) \leq 1$, for all $x, g \in \Gamma$, and $\xi_x(xg) \in \ell^\infty(\Gamma)$ as a function of x. First we note that if

$$d(x, xg) = |g^{-1}x^{-1}x| = |g^{-1}| = |g| > S,$$

then $\xi_x(xg) = 0$, for all $x \in \Gamma$, and thus $f(g) = 0$ whenever $|g| > S$. This implies that

$$\|f\| = \sum_{g \in \Gamma} f(g) = \sum_{g \in B(e,S)} m(\xi_x(xg)) = m(1_\Gamma) = 1,$$

which shows that $f \in \ell^1(\Gamma)_{1,+}$. We now have to show that f satisfies $\|f - hf\| < \varepsilon$, when $|h| \leq R$. To this end, let $h \in \Gamma$ satisfy $|h| \leq R$, and note that

$$m(\xi_x(xh^{-1}g)) = m(\xi_{xh}((xh)h^{-1}g)) = m(\xi_{xh}(xg)).$$

Thus we get

$$\begin{aligned}
\|f - hf\| &= \sum_{g\in\Gamma} |f(g) - f(h^{-1}g)| \\
&= \sum_{g\in B(e,S)\cup hB(e,S)} |m(\xi_x(xg)) - m(\xi_x(xh^{-1}g))| \\
&= \sum_{g\in B(e,S)\cup hB(e,S)} |m(\xi_x(xg) - \xi_{xh}(xg))| \\
&\leq m\left(\sum_{g\in B(e,S)\cup B(h,S)} |\xi_x(xg) - \xi_{xh}(xg)|\right) \\
&< m(\varepsilon 1_\Gamma) = \varepsilon.
\end{aligned}$$

On the other hand, suppose that $f \in \ell^1(\Gamma)_{1,+}$ satisfies the above. Set $\xi_g = g \cdot f$ for all $g \in \Gamma$. Then $\xi : \Gamma \to \ell^1(\Gamma)_{1,+}$ and we see that $\operatorname{supp}(\xi_x) \subseteq B(x,S)$ and

$$\|\xi_x - \xi_y\| = \|xf - yf\| = \|f - x^{-1}yf\| < \varepsilon,$$

whenever $|x^{-1}y| = d(x,y) \leq R$ as wanted. □

We are now ready to show the result relating amenability, box spaces and property (A).

Theorem 5.2.2. *Let $\square_{(N_i)}\Gamma$ be the box space of a residually finite group Γ. Then $\square_{(N_i)}\Gamma$ has property (A) if and only if Γ is amenable.*

Proof. First assume that Γ is amenable, and let $\varepsilon, R > 0$ given. Then there exists an approximate invariant mean $f \in l^1(\Gamma)_{1,+}$ such that

$$\|f - gf\|_1 < \varepsilon,$$

for all $|g| \leq R$, and there exists $S > 0$ such that $\operatorname{supp}(f) \subseteq B(e,S)$. By Proposition 5.1.2 there exists $i_0 \in \mathbb{N}$ such that $B_\Gamma(e, 2(S+R))$ is isometric to $B_{\Gamma/N_i}(e, 2(S+R))$ for all $i \geq i_0$. Let $\varphi_i : B_{\Gamma/N_i}(e, 2(S+R)) \to B_\Gamma(e, 2(S+R))$ be this isometry. For all $i \in \mathbb{N}$, we define $f_i \in \ell^1(\Gamma/N_i)_{1,+}$ by

$$f_i(s) = \begin{cases} f(\varphi_i(s)) & s \in B_{\Gamma/N_i}(e, 2(S+R), i \geq i_0, \\ 0 & s \notin B_{\Gamma/N_i}(e, 2(S+R)), i \geq i_0, \\ \frac{1}{|\Gamma/N_i|} & i < i_0. \end{cases}$$

It is easy to see that f_i is an approximate invariant mean on Γ/N_i for the given ε and R, and applying Theorem 5.2.1 and Theorem 1.3.8 we conclude that $\square_{(N_i)}\Gamma$ has property (A).
Suppose now that $\square_{(N_i)}\Gamma$ has property (A). This means that for given $\varepsilon, R > 0$, there exists $\xi : \square_{(N_i)}\Gamma \to \ell^1(\square_{(N_i)}\Gamma)_{1,+}$ such that for any $S > 0$, we have

$$\|\xi_x - \xi_y\| < \varepsilon,$$

whenever $d(x,y) \leq R$. From Lemma 1.3.9 we get that for each $i \in \mathbb{N}$, there exists a Higson-Roe function ξ satisfying the conditions for the same ε, R and S. Since each Γ/N_i is finite, we know from Example 4.1.9 that each Γ/N_i is amenable, and applying Theorem 5.2.1, we get the existence of functions $f_i \in \ell^1(\Gamma/N_i)_{1,+}$ satisfying $\|f - gf\| < \varepsilon$, whenever $|g| \leq R$ and with $\mathrm{supp}(f_i) \subseteq B(e,S)$, for all $i \in \mathbb{N}$. From Proposition 5.1.2 we know that there exists $i_0 \in \mathbb{N}$ such that the ball $B_{\Gamma/N_i}(e, 2(S+R))$ is isometric to $B_\Gamma(e, 2(S+R))$ via an isometry φ_i, for all $i \geq i_0$. We define

$$f(g) = \begin{cases} f_i(\varphi^{-1}(g)) & g \in B_G(e, 2(S+R)), \\ 0 & g \notin B_G(e, 2(S+R)). \end{cases}$$

Then f is an approximate invariant mean, as wanted. □

5.3 Property (H)

In this section we show that if the box space of a group coarsely embeds into a Hilbert space, then the group has property (H).

Theorem 5.3.1. *Let Γ be a finitely generated, residually finite group. If the box space $\square_{(N_i)}\Gamma$ coarsely embeds into a Hilbert space, then Γ has property (H).*

Proof. Let $(N_i)_{i\geq 1} \subseteq \Gamma$ be a filtration of Γ and set $X_i = \Gamma/N_i$. Let d be the metric on $\square_{(N_i)}\Gamma$ induced by d_i, the metric on each component X_i. By assumption, $\square_{(N_i)}\Gamma$ coarsely embeds into a Hilbert space, and thus by Theorem 1.5.4 we know that there exists a symmetric, normalized kernel of negative type k on $\square_{(N_i)}\Gamma$. This kernel induces negative type kernels $k_i : X_i \times X_i \to \mathbb{R}$ and uniformly non-decreasing functions $\rho_-, \rho_+ : \mathbb{R}^+ \to \mathbb{R}^+$ as in Definition 1.2.10 such that

$$\rho_-(d_i(x,y)) \leq k_i(x,y) \leq \rho_+(d_i(x,y)),$$

for all i simultaneously. We now define averaging functions f_i on each X_i by

$$f_i(x^{-1}y) = \frac{1}{|X_i|}\sum_{g\in X_i} k_i(gx, gy).$$

Clearly, each f_i is a negative type function satisfying the same estimates

$$\rho_-(d_i(x,y)) \leq f_i(x^{-1}y) \leq \rho_+(d(_i(x,y)).$$

Let $\hat{f}_i = f_i \circ \pi_i$ be the lift of f_i to Γ, where $\pi_i : \Gamma \to X_i$ is the quotient map. Recall from Proposition 5.1.2 that for any finite subset $S \subseteq \Gamma$, there exists $i_0 \in \mathbb{N}$ such that each π_i is an isometry, for all $i \geq i_0$, when restricted to S. Thus for $x, y \in S$ we have that $\hat{f}_i(x^{-1}y) \leq \rho_+(d(x,y))$, for almost all i. Thus we can pick a subsequence such that $\hat{f}_i$ converges pointwise to a function $\hat{f} : \Gamma \to \mathbb{R}$. Since each $\hat{f}_i$ is of negative type, the pointwise limit $\hat{f}$ is also of negative type, and thus $\hat{f}$ satisfies

$$\rho_-(d(x,y)) \leq \hat{f}(x^{-1}y).$$

By Theorem 4.2.4, this shows that Γ has property (H). □

Remark 5.3.2. Note that the converse of this theorem is not true. To see this, we assume for contradiction that the converse is true. Let H be a quotient of some group Γ, i.e., there exists a surjective homomorphism $\varphi : \Gamma \to H$. Then the box space $\square_{(N_i)}H$ coarsely embeds into $\square_{(N_i)}\Gamma$. Due to Haagerup's result [11], we know that the free group on n generators has property (H) for all n. Thus, if we set $\Gamma = \mathbb{F}_n$, we deduce that the box space of any finitely generated group can be coarsely embedded into some Hilbert space. However, in the next section we show that the box space of groups with property (T), which are finitely generated by Theorem 4.3.11, give rise to a family of expander graphs. Combining this with Theorem 2.4.4, we see that there exists finitely generated groups with property (H), but such that the box space do not coarsely embed into any Hilbert space.

5.4 Property (T)

In this section, we investigate the box spaces of residually finite groups with property (T). As mentioned before, examples of such groups include $SL_n(\mathbb{Z})$, for $n \geq 3$. In particular we show that box spaces of residually finite groups with property (T) are expanders. In order to prove this, we first need a result by Alon and Milman [1]:

Theorem 5.4.1 (Alon-Milman)**.** *If Γ has property (T) with respect to a finite symmetric generating subset Σ then for every $N \lhd \Gamma$ of finite index, the Cayley graph $G = Cay(\Gamma/N; \Sigma)$ is a (n, d, ε)-expander, where $n = |\Gamma/N|$, $d = |\Sigma|$ and $0 < \varepsilon = k_\Sigma$ is the Kazhdan constant.*

Proof. Let $S \subseteq \Gamma/N$. We may assume $|S| \leq \frac{n}{2}$ by picking $\overline{S}$ instead of S if necessary. Let $|\Sigma| = d$ and recall that $Cay(\Gamma/N; \Sigma)$ is a finite, connected d-regular graph and that the set of vertices comes naturally equipped with the Γ action of right translation. Let $f \in \ell^2_0(V)$ be given by

$$f(x) = 1_S(x)|\overline{S}| - 1_{\overline{S}}(x)|S|.$$

Recall here that $\ell^2(V)$ is a real Hilbert space and that $\ell^2_0(V)$ is the orthogonal complement to the kernel of $\triangle$, which is a unitary representation of Γ with no Γ-invariant vectors. The weighted ℓ^2-norm of f is then

$$\|f\|_2^2 = \frac{1}{d}\sum_{x\in V} d_x f(x)^2 = \frac{1}{d}d(|\overline{S}|^2|S| + |S|^2|\overline{S}|) = |S||\overline{S}|(|S| + |\overline{S}|) = |S||\overline{S}|n.$$

Now for $s \in \Sigma$ we have

$$|(sf)(x) - f(x)| = \begin{cases} |S| + |\overline{S}| & x \in S, xs \in \overline{S} \text{ or } xs \in S, x \in \overline{S}, \\ 0 & x, xs \in S \text{ or } x, xs \in \overline{S}. \end{cases}$$

If we rewrite ∂S as $\bigcup_{s\in\Sigma} \partial_s S$, where $\partial_s S = \{(x, y) \in \partial S \mid y = xs \vee y = xs^{-1}\}$. This gives us

$$|(sf)(x) - f(x)| = \begin{cases} |S| + |\overline{S}| & s \in \partial_s S, \\ 0 & \text{else}, \end{cases}$$

which leads to

$$\|sf - f\|_2^2 = (|S| + |\overline{S}|)^2 |\partial_s S| = n^2 |\partial_s S|,$$

so that $|\partial_s S| = \frac{\|sf-f\|^2}{n^2}$. Using that Γ has property (T) we know from Lemma 4.3.9 that there exists some $s \in \Sigma$ such that $\|sf - f\|_2^2 \geq 2\varepsilon \|f\|_2^2$, and thus

$$\frac{|\partial S|}{|S|} \geq \frac{|\partial_s S|}{|S|} = \frac{\|sf - f\|_2^2}{|S| n^2} \geq \frac{2\varepsilon \|f\|_2^2}{n^2 |S|} = \frac{2\varepsilon |S| |\overline{S}| n}{n^2 |S|} = \varepsilon \frac{2|\overline{S}|}{n} \geq \varepsilon,$$

since $2|\overline{S}| \geq n$. As S was arbitrary, it will in particular hold for S satisfying the definition of the Cheeger constant, which completes the proof. □

We get the following corollary:

Corollary 5.4.2. *Given a discrete residually finite group Γ with property (T), any box space of Γ is a family of expanders.*

The reader may note from the proof of Theorem 5.4.1 that property (T) is, in fact, too strong a requirement for the box spaces to be expanders. A weaker condition, introduced by Lubotzky and Zimmer in [18], called *property (τ)*, turns out to be sufficient. We will study this property in the next section.

5.5 Property (τ)

As mentioned above, property (τ) is a weaker notion than property (T). We begin this section with the definition of property (τ), cf. [16].

Definition 5.5.1. Let Γ be a finitely generated discrete group, $\Sigma \subseteq \Gamma$ a (symmetric) generating subset, and $\mathcal{L} = \{N_i\}_{i \in I}$ a collection of finite index normal subgroups. We say that Γ has property (τ) (with respect to $\mathcal{L}$), if for every unitary representation $\pi : \Gamma \to B(H)$, with $N_i \subseteq \ker(\pi)$ for some $i \in I$, without any non-zero Γ-fixed vectors, and for every $0 \neq \xi \in H$, there exists $s \in \Sigma$ such that

$$\|\pi(s)\xi - \xi\| > \|\xi\| \varepsilon,$$

for some $\varepsilon > 0$. If $\mathcal{L}$ is the collection of *all* finite index normal subgroups, we say that Γ has property (τ).

With this definition, we have the following important theorem (for a proof see [16, Theorem 4.3.2]):

Theorem 5.5.2. *Let Γ be a finitely generated discrete group, $\Sigma \subseteq \Gamma$ a symmetric generating subset, and $\mathcal{L} = \{N_i\}_{i \in I}$ a collection of finite index normal subgroups. Then Γ has property (τ) with respect to $\mathcal{L}$ if and only if $\Box_{\mathcal{L}} \Gamma$ is a family of expander graphs.*

We do not prove this theorem here. Instead, we give some intuition on how property (τ) relates to property (T). First note that if Γ has property (T), then clearly Γ also has property (τ). However, a group Γ has property (T) if for *all* unitary representations, the trivial representation is isolated, whereas for property (τ), we only require the trivial representation being isolated with respect to the unitary representations which factors through the quotients Γ/N_i. Notice that this is exactly how we constructed the expander graphs from groups with property (T). This justifies the claim that property (T) is a priori too strong a property when constructing expanders.

We now give an example of a group which does not have property (T), but does have property (τ) with respect to a certain family of finite index normal subgroups.

Example 5.5.3. Let $\Gamma = SL_2(\mathbb{Z})$ and let $\Gamma(m) = \ker(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/m\mathbb{Z})$ be the congruence subgroup, where $m \in \mathbb{N}$. It is known that Γ neither has property (T), nor property (τ), however, Γ has property (τ) with respect to the family of congruence subgroups $\Gamma(p)$, where p is a prime, as shown in [21]. Thus, if we let $a = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}$ be the generators of $SL_2(\mathbb{Z})$, we deduce from Theorem 5.5.2 that the Cayley graphs $Cay(SL_2(\mathbb{Z}/p\mathbb{Z}); \{a, b\})$ with respect to these generators form a family of expanders.

For an overview of recent developments concerning property (τ) and applications to expander graphs, we refer the reader to the lecture notes by Breuillard [4].

5.6 Literature

This chapter is written with inspiration from [20],[25],[16],[17],[19],[15] and [23]. In particular, the introduction to box spaces has benefited from [15]. The section regarding property (A) is mainly based on [19]. The section on property (H) has in particular used [25] and [20]. The section on property (T) is mainly based on [23] and [16]. The last section, regarding property (τ) has used [16] and [17].

Bibliography

[1] N. Alon and V. D. Milman. λ_1, isoperimetric inequalities for graphs, and superconcentrators. *J. Comtin. Theory Ser. B*, 38(1):73–88, 1985.

[2] M. E. B. Bekka, P.-A. Cherix, and A. Valette. Proper affine isometric actions of amenable groups. In *Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993)*, volume 227 of *London Math. Soc. Lecture Note Ser.*, pages 1–4. Cambridge Univ. Press, Cambridge, 1995.

[3] Béla Bollobás. The isoperimetric number of random regular graphs. *European J. Comtin.*, 9(3):241–244, 1988.

[4] Emmanuel Breuillard. Expander graphs, property (τ) and approximate groups. In *Geometric group theory*, volume 21 of *IAS/Park City Math. Ser.*, pages 325–377. Amer. Math. Soc., Providence, RI, 2014.

[5] Nathanial P. Brown and Narutaka Ozawa. *C^*-algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[6] Tullio Ceccherini-Silberstein and Michel Coornaert. *Cellular automata and groups*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2010.

[7] Jozef Dodziuk. Difference equations, isoperimetric inequality and transience of certain random walks. *Trans. Amer. Math. Soc.*, 284(2):787–794, 1984.

[8] David Ellis. The expansion of random regular graphs. `https://www.dpmms.cam.ac.uk/~dce27/randomreggraphs3.pdf`, 2011.

[9] R. I. Grigorchuk. On the Milnor problem of group growth. *Dokl. Akad. Nauk SSSR*, 271(1):30–33, 1983.

[10] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[11] Uffe Haagerup. An example of a nonnuclear C^*-algebra, which has the metric approximation property. *Invent. Math.*, 50(3):279–293, 1978/79.

[12] Shlomo Hoory, Nathan Linial, and Avi Wigderson. Expander graphs and their applications. *Bull. Amer. Math. Soc. (N.S.)*, 43(4):439–561 (electronic), 2006.

[13] Thomas W. Hungerford. *Algebra*, volume 73 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1980. Reprint of the 1974 original.

[14] D. A. Kazhdan. On the connection of the dual space of a group with the structure of its closed subgroups. *Funkcional. Anal. i Priložen.*, 1:71–74, 1967.

[15] Anastasia Khukhro. Coarse geometry and groups. http://eprints.soton.ac.uk/341780/1.hasCoversheetVersion/ThesisSubmission.pdf_Khukhro.pdf, April 2012.

[16] Alexander Lubotzky. *Discrete groups, expanding graphs and invariant measures*. Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2010. With an appendix by Jonathan D. Rogawski, Reprint of the 1994 edition.

[17] Alexander Lubotzky. Expander graphs in pure and applied mathematics. *Bull. Amer. Math. Soc. (N.S.)*, 49(1):113–162, 2012.

[18] Alexander Lubotzky and Robert J. Zimmer. Variants of Kazhdan's property for subgroups of semisimple groups. *Israel J. Math.*, 66(1-3):289–299, 1989.

[19] Piotr W. Nowak and Guoliang Yu. *Large scale geometry*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2012.

[20] John Roe. *Lectures on coarse geometry*, volume 31 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2003.

[21] Atle Selberg. On the estimation of Fourier coefficients of modular forms. In *Proc. Sympos. Pure Math., Vol. VIII*, pages 1–15. Amer. Math. Soc., Providence, R.I., 1965.

[22] Yehuda Shalom. Bounded generation and Kazhdan's property (T). *Inst. Hautes Études Sci. Publ. Math.*, (90):145–168 (2001), 1999.

[23] Lior Silberman. Expanding graphs and property (T). http://www.math.ubc.ca/~lior/work/expanders_T.pdf, 2002.

[24] R. Michael Tanner. Explicit concentrators from generalized N-gons. *SIAM J. Algebraic Discrete Methods*, 5(3):287–293, 1984.

[25] Rufus Willett. Some notes on property A. In *Limits of graphs in group theory and computer science*, pages 191–281. EPFL Press, Lausanne, 2009.

[26] Guoliang Yu. The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Invent. Math.*, 139(1):201–240, 2000.