Grothendieck Inequalities
—from classical to non-commutative

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In 1956 Grothendieck published the celebrated "Résumé de la théorie métrique des produits tensoriels topologiques", containing a general theory of tensor norms on tensor products of Banach spaces, describing several operations to generate new norms from known ones, and studying the duality theory between these norms.

The highlight of the paper, now referred to as "The Résumé" is a result that Grothendieck called "The fundamental theorem on the metric theory of tensor products", now called "Grothendieck's theorem".

Theorem (Grothendieck 1956):

Let K_1 and K_2 be compact spaces. Let $u : C(K_1) \times C(K_2) \to \mathbb{K}$ be a bounded bilinear form, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then there exist probability measures μ_1 and μ_2 on K_1 and K_2, respectively, such that

$$|u(f,g)| \le K_G^{\mathbb{K}} \|u\| \left(\int_{K_1} |f|^2 \, d\mu_1 \right)^{1/2} \left(\int_{K_2} |g|^2 \, d\mu_2 \right)^{1/2}$$

for all $f \in C(K_1)$ and $g \in C(K_2)$, where $K_G^{\mathbb{K}}$ is a universal constant.

Remarks about Grothendieck's constant $K_G^{\mathbb{K}}$:

- $\frac{1}{2} K_G^{\mathbb{R}} \le K_G^{\mathbb{C}} \le 2 K_G^{\mathbb{R}}$.

- $\frac{\pi}{2} \le K_G^{\mathbb{R}} \le \frac{\pi}{2\log(1+\sqrt{2})} = 1.782...$

  The left-hand side is due to Grothendieck. The right-hand side is due to Krivine (1977).

- $\frac{4}{\pi} \leq K_G^{\mathbb{C}} < 1.40491\,.$

The left-hand side is due to Grothendieck. The right-hand side is due to Haagerup (1987), who proved, more precisely, that

$$K_G^{\mathbb{C}} \leq \frac{8}{\pi(k_0+1)} < 1.40491\,,$$

where k_0 is the unique solution in the interval $[0,1]$ of the equation

$$\phi(k) = \frac{1}{8}\pi(k+1)\,,$$

where $\phi(k) := k \int_0^{\pi/2} \frac{\cos^2 t}{\sqrt{1-k^2\sin^2 t}}\,dt$, defined for $-1 \leq k \leq 1$.

The previously known upper bound was obtained by Pisier in 1976,

$$K_G^{\mathbb{C}} \leq e^{1-\gamma} \approx 1.52621\,,$$

where γ is Euler's constant $\gamma := \lim_{n\to\infty} (1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n)$.

Little Grothendieck Inequality:

Let $T : C(K) \to H$ be a bounded linear operator, where K is a compact space and H is a Hilbert space. Then there exists a probability measure μ on K such that

$$\|T(f)\| \leq \sqrt{K_G^{\mathbb{K}}}\, \|T\| \left(\int_K |f|^2 \, d\mu \right)^{1/2}, \quad f \in C(K)\,.$$

Proof: Define $u : C(K) \times C(K) \to \mathbb{C}$ by

$$u(f,g) :\ = \langle Tf, T\bar{g} \rangle_H\,, \quad f\,, g \in C(K)\,.$$

Then u is a bounded bilinear form, satisfying $\|u\| \leq \|T\|^2$. By Grothendieck's theorem there exist probability measures μ_1 and μ_2 on K such that for all $f\,, g \in C(K)\,,$

$$|u(f,g)| \leq K_G^{\mathbb{K}} \|u\| \left(\int_K |f|^2 \, d\mu_1 \right)^{1/2} \left(\int_K |g|^2 \, d\mu_2 \right)^{1/2}.$$

Set $\mu := \frac{1}{2}(\mu_1 + \mu_2)$. Then, for all $f \in C(K)\,,$

$$\begin{aligned} \|Tf\|^2 = u(f, \bar{f}) &\leq K_G^{\mathbb{K}} \|u\| \left(\int_K |f|^2 \, d\mu_1 \right)^{1/2} \left(\int_K |f|^2 \, d\mu_2 \right)^{1/2} \\ &\leq K_G^{\mathbb{K}} \|u\| \int_K |f|^2 \, d\mu \\ &\leq K_G^{\mathbb{K}} \|T\|^2 \int_K |f|^2 \, d\mu\,. \end{aligned}$$

□

The best constants in the Little Grothendieck Inequality are known, namely, $\sqrt{4/\pi}$ (in the complex case) and $\sqrt{\pi/2}$ (in the real case).

Theorem:

Any bounded linear operator $T : C(K_1) \to C(K_2)^*$ factors through a Hilbert space H,

$$\begin{array}{ccc} C(K_1) & \xrightarrow{\quad T \quad} & C(K_2)^* \\ & {\scriptstyle R}\searrow \qquad \nearrow{\scriptstyle S} & \\ & H & \end{array}$$

such that $\|R\|\|S\| \leq K_G^{\mathbb{K}} \|T\|$.

Proof: Follows from Grothendieck's theorem applied to the bilinear form $u : C(K_1) \times C(K_2) \to \mathbb{C}$ defined by

$$u(f,g) := (Tf)(g)\,, \quad f \in C(K_1)\,, g \in C(K_2)\,. \qquad \square$$

Remark: Grothendieck's theorem holds in the more general setting of locally compact topological spaces. As an interesting application, one can deduce the (known) fact that the Fourier transform $\mathcal{F} : L^1(\mathbb{R}) \to C_0(\mathbb{R})$ is not onto.

Indeed, suppose by contradiction that $\mathcal{F}$ were onto. Recall that $\mathcal{F}$ is a bounded linear operator, since $\|\mathcal{F}(f)\|_{C_0(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}\,,\ f \in L^1(\mathbb{R})$. Moreover, $\mathcal{F}$ is one-to-one (by the Riemann-Lebesgue Lemma).

$$\begin{array}{ccccc} C_0(\mathbb{R}) & \xrightarrow{\mathcal{F}^{-1}} & L^1(\mathbb{R}) & \xrightarrow{\ j\ } & C(K)^* \\ & {\scriptstyle R}\searrow & & \nearrow{\scriptstyle S} & \\ & & H & & \end{array}$$

$$L^1(\mathbb{R}) \cong j(L^1(\mathbb{R})) = S(R(C_0(\mathbb{R}))) \cong R(C_0(\mathbb{R})) \subseteq H\,.$$

The Résumé ends with a list of 6 problems that are linked together and revolve around the following questions:

- When does a bounded linear operator $u : X \to Y$ (X, Y Banach spaces) factor through a Hilbert space?
- For which Banach spaces X and Y does this happen for all such operators u?

The fourth problem in the Résumé was the C*-algebraic version of Grothendieck's theorem, as conjectured by Grothendieck himself.

Conjecture (Grothendieck):

Let A be a C*-algebra and $u : A \times A \to \mathbb{C}$ a bounded bilinear form. Then there exist $f, g \in S(A)$ such that for all $a, b \in A$,

$$|u(a,b)| \leq k\|u\| f(|a|^2)^{1/2} g(|b|^2)^{1/2},$$

where $|x| := \Big((x^*x + xx^*)/2\Big)^{1/2}$, all $x \in A$, k a universal constant.

Grothendieck Inequality (Haagerup 1985) (extension of Pisier's result from 1978):

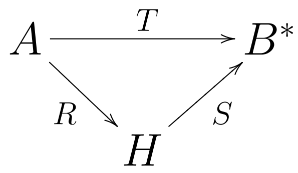
Let A and B be C*-algebras and let $u : A \times B \to \mathbb{C}$ be a bounded bilinear form. There exist $f_1, f_2 \in S(A)$ and $g_1, g_2 \in S(B)$ such that

$$|u(a,b)| \leq \|u\| \Big(f_1(aa^*) + f_2(a^*a)\Big)^{1/2} \Big(g_1(b^*b) + g_2(bb^*)\Big)^{1/2},$$

for all $a \in A$ and $b \in B$.

Corollary (Haagerup 1985):

Any bounded linear operator $T : A \to B^*$, where A and B are C*-algebras, factors through a Hilbert space H,



such that $\|R\|\|S\| \leq 2\|T\|$.

Little Grothendieck's Inequality (Haagerup 1985):

Let A be a C*-algebra and H a Hilbert space. If $T : A \to H$ is a bounded linear operator, then there exist $f_1, f_2 \in S(A)$ such that

$$\|Ta\| \leq \|T\| \Big(f_1(a^*a) + f_2(aa^*) \Big)^{1/2}, \quad a \in A.$$

Let A and B be C*-algebras and $u : A \times B \to \mathbb{C}$ a bounded bilinear form. There exists a unique bounded linear operator $\widetilde{u} : A \to B^*$ such that for all $a \in A$, $b \in B$,

$$u(a, b) = (\widetilde{u}(a))(b) .$$

The bilinear form u is called *jointly completely bounded* (j.c.b., for short) if $\widetilde{u} : A \to B^*$ is completely bounded, in which case we set

$$\|u\|_{\mathrm{jcb}} := \|\widetilde{u}\|_{\mathrm{cb}} .$$

Remark: It is easily checked that

$$\|u\|_{\mathrm{jcb}} = \sup_{n \in \mathbb{N}} \|u_n\| ,$$

where $u_n : M_n(A) \otimes M_n(B) \to M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$, $n \in \mathbb{N}$, is given by

$$u_n \left(\sum_{i=1}^{k} a_i \otimes c_i , \sum_{j=1}^{l} b_j \otimes d_j \right) := \sum_{i=1}^{k} \sum_{j=1}^{l} u(a_i, b_j) c_i \otimes d_j ,$$

for all $a_i \in A$, $b_j \in B$, $c_i , d_j \in M_n(\mathbb{C})$, $k , l \in \mathbb{N}$.

Moreover, for all C*-algebras C and D, and all $a_i \in A$, $b_j \in B$, $c_i \in C$, $d_j \in D$, $k , l \in \mathbb{N}$, one has

$$\left\| \sum_{i=1}^{k} \sum_{j=1}^{l} u(a_i , b_j) c_i \otimes d_j \right\|_{C \otimes_{\min} D} \leq \|u\|_{\mathrm{jcb}} \left\| \sum_{i=1}^{k} a_i \otimes c_i \right\|_{A \otimes_{\min} C} \left\| \sum_{j=1}^{l} b_j \otimes d_j \right\|_{B \otimes_{\min} D} .$$

Conjecture (Effros-Ruan 1991):

Let A and B be C*-algebras and let $u : A \times B \to \mathbb{C}$ be a jointly completely bounded bilinear form. Then there exist $f_1, f_2 \in S(A)$ and $g_1, g_2 \in S(B)$ such that for all $a \in A$ and $b \in B$,

$$|u(a,b)| \leq K\|u\|_{\mathrm{jcb}} \left(f_1(aa^*)^{1/2} g_1(b^*b)^{1/2} + f_2(a^*a)^{1/2} g_2(bb^*)^{1/2} \right) \quad (1)$$

where K is a universal constant.

Theorem (Pisier-Shlyakhtenko, Invent. Math. 2002):

Let $E \subseteq A$ and $F \subseteq B$ be *exact* operator spaces sitting in C*-algebras A and B. Let $u : E \times F \to \mathbb{C}$ be a j.c.b. bilinear form. Then there exist $f_1, f_2 \in S(A)$ and $g_1, g_2 \in S(B)$ such that the inequality (1) holds for all $a \in E$ and $b \in F$ with $K = 2\sqrt{2}\,\mathrm{ex}(E)\mathrm{ex}(F)$.

Theorem (Pisier-Shlyakhtenko, Invent. Math. 2002):

If either A or B is an *exact* C*-algebra and $u : A \times B \to \mathbb{C}$ is a j.c.b. bilinear form, then there exist $f_1, f_2 \in S(A)$ and $g_1, g_2 \in S(B)$ such that the inequality (1) holds for all $a \in A$ and $b \in B$ with $K = 2\sqrt{2}$.

Recall that an operator space E is called *exact* if there is $C \geq 1$ such that for every finite dimensional subspace $F \subseteq E$, there exists $n \in \mathbb{N}$ and a subspace $G \subseteq M_n(\mathbb{C})$ with $d_{\mathrm{cb}}(F, G) \leq C$. The infimum of all such constants C is denoted by $\mathrm{ex}(E)$.

Theorem (Kirchberg, Pisier): A C*-algebra is exact if and only if it is exact as an operator space. For any exact C*-algebra A, $\mathrm{ex}(A) = 1$.

Theorem (Haagerup-M., Invent. Math. 2008)

The Effros-Ruan conjecture holds for arbitrary C*-algebras A and B with $K = 1$, and this is the best possible constant.

Corollary A:

Let A and B be C*-algebras. Any completely bounded linear map $T : A \to B^*$ admits a factorization through $H_r \oplus K_c$, where H and K are Hilbert spaces,

$$\begin{array}{ccc} A & \xrightarrow{\quad T \quad} & B^* \\ & {\scriptstyle R}\searrow \quad \nearrow{\scriptstyle S} & \\ & H_r \oplus K_c & \end{array}$$

satisfying $\|R\|_{cb}\|S\|_{cb} \leq 2\|T\|_{cb}$.

Corollary B:

Let A be a C*-algebra. If $T : A \to OH$ is a completely bounded linear map, then there exist $f_1, f_2 \in S(A)$ such that for all $a \in A$,

$$\|T(a)\| \leq \sqrt{2}\|T\|_{cb} f_1(aa^*)^{1/4} f_2(a^*a)^{1/4}.$$

(Only an improvement of constant in the corresponding result by Pisier-Shlyakhtenko; they had this with constant $2^{9/4}$.)

Corollary C:

Let E be an operator space such that E and its dual E^* embed completely isomorphically into preduals M_* and N_*, respectively, of von Neumann algebras M and N. Then E is cb-isomorphic to a quotient of a subspace of $H_r \oplus K_c$, for some Hilbert spaces H and K.

Corollary D:

Let E be an operator space, and let $E \subseteq A$ and $E^* \subseteq B$ be completely isometric embeddings into C*-algebras A and B such that both subspaces are cb-complemented. Then E is cb-isomorphic to $H_r \oplus K_c$, for some Hilbert spaces H and K.

(These are non-commutative analogues of the classical isomorphic characterization of a Hilbert space: If X is a Banach space such that both X and its dual X^* embed into L_1-spaces, then X is isomorphic to a Hilbert space. Corollaries C and D above are obtained by adjusting the proof of the corresponding results by Pisier-Shlyakhtenko.)

Corollary E:

Let A_0, A, B_0 and B be C*-algebras such that $A_0 \subseteq A$ and $B_0 \subseteq B$. Then any j.c.b. bilinear form $u_0 : A_0 \times B_0 \to \mathbb{C}$ extends to a bilinear form $u : A \times B \to \mathbb{C}$ such that

$$\|u\|_{\mathrm{jcb}} \leq 2\|u_0\|_{\mathrm{jcb}} .$$

Let $E \subseteq A$ and $F \subseteq B$ be operator spaces sitting in C*-algebras A, B. Given $u : E \times F \to \mathbb{C}$ a bounded bilinear form, let $\|u\|_{\mathrm{ER}} \in [0, \infty]$ be the infimum of all constants $\kappa \in [0, \infty]$ so that for all $a \in E$, $b \in F$,

$$|u(a,b)| \leq \kappa \left(f_1(aa^*)^{1/2} g_1(b^*b)^{1/2} + f_2(a^*a)^{1/2} g_2(bb^*)^{1/2} \right) ,$$

for some $f_1, f_2 \in S(A)$, $g_1, g_2 \in S(B)$.

Lemma:

If $\|u\|_{\mathrm{ER}} < \infty$, then the associated map $\widetilde{u} : E \to F^*$ admits a cb-factorization $\widetilde{u} = vw$ through $H_r \oplus K_c$ for some Hilbert spaces H and K, where $E \xrightarrow{v} H_r \oplus K_c \xrightarrow{w} F^*$, satisfying $\|v\|_{\mathrm{cb}} \|w\|_{\mathrm{cb}} \leq 2\|\widetilde{u}\|_{\mathrm{ER}}$.

Example:

Let E be an operator space which is not isomorphic to a Hilbert space, and let $u : E \times E^* \to \mathbb{C}$ be defined by

$$u(a,b) := b(a) , \quad a \in E, b \in E^* .$$

Then $\|u\|_{\mathrm{jcb}} = 1$ and $\|u\|_{\mathrm{ER}} = \infty$.

Proposition:

(i) If $u : A \times B \to \mathbb{C}$ is a bounded bilinear form, then

$$\|u\|_{\mathrm{ER}} \leq \|u\|_{\mathrm{jcb}} \leq 2\|u\|_{\mathrm{ER}} .$$

(ii) Let c_1, c_2 denote the best constants in the inequalities

$$c_1 \|u\|_{\mathrm{ER}} \leq \|u\|_{\mathrm{jcb}} \leq c_2 \|u\|_{\mathrm{ER}} ,$$

where A and B are arbitrary C*-algebras and $u : A \times B \to \mathbb{C}$ is a bounded bilinear form. Then $c_1 = 1$ and $c_2 = 2$.

Some preliminaries on Powers factors and Tomita-Takesaki theory

Let $0 < \lambda < 1$ be fixed, and let $(\mathcal{M}\,,\phi)$ be the Powers factor of type III_λ with product state ϕ, that is,

$$(\mathcal{M}\,,\phi) = \bigotimes_{n=1}^{\infty}(M_2(\mathbb{C})\,,\omega_\lambda)\,,$$

where $\phi = \bigotimes_{n=1}^{\infty}\omega_\lambda\,,\ \omega_\lambda(\,\cdot\,)\colon = \mathrm{Tr}(h_\lambda\,\cdot\,)$ and $h_\lambda\colon = \begin{pmatrix} \frac{\lambda}{1+\lambda} & 0 \\ 0 & \frac{1}{1+\lambda} \end{pmatrix}$.

The modular automorphism group $(\sigma_t^\phi)_{t\in\mathbb{R}}$ of ϕ is given by

$$\sigma_t^\phi = \bigotimes_{n=1}^{\infty}\sigma_t^{\omega_\lambda}\,,$$

where for any matrix $x = [x_{ij}]_{1\le i,j\le 2} \in M_2(\mathbb{C})$ and any $t \in \mathbb{R}$,

$$\sigma_t^{\omega_\lambda}(x) = h_\lambda^{it} x h_\lambda^{-it} = \begin{pmatrix} x_{11} & \lambda^{it}x_{12} \\ \lambda^{-it}x_{21} & x_{22} \end{pmatrix}.$$

Therefore $\sigma_t^{\omega_\lambda}$ and σ_t^ϕ are periodic in $t \in \mathbb{R}$ with minimal period

$$t_0\colon = -\frac{2\pi}{\log\lambda}\,.$$

Let $\mathcal{M}_\phi$ denote the centralizer of ϕ, that is,

$$\mathcal{M}_\phi\colon = \{x \in \mathcal{M} : \sigma_t^\phi(x) = x\,, \forall t \in \mathbb{R}\}\,.$$

Theorem (Connes 1973):

The relative commutant of $\mathcal{M}_\phi$ in $\mathcal{M}$ is trivial, i.e.,

$$\mathcal{M}_\phi' \cap \mathcal{M} = \mathbb{C}\mathbf{1}\,.$$

Theorem (Haagerup 1989):

For all $x \in \mathcal{M}$,

$$\phi(x) \cdot \mathbf{1} \in \overline{\mathrm{conv}\{vxv^* : v \in \mathcal{U}(\mathcal{M}_\phi)\}}^{\|\cdot\|}\,,$$

where $\mathcal{U}(\mathcal{M}_\phi)$ denotes the unitary group on $\mathcal{M}_\phi$.

Corollary 1 (Strong version of the Dixmier averaging process):

There exists a net $\{\alpha_i\}_{i\in I} \subseteq \mathrm{conv}\{\mathrm{ad}(v) : v \in \mathcal{U}(\mathcal{M}_\phi)\}$ such that

$$\lim_{i\in I} \|\alpha_i(x) - \phi(x)\cdot \mathbf{1}\| = 0\,, \quad x \in \mathcal{M}\,.$$

We identify $\mathcal{M}$ with $\pi_\phi(\mathcal{M})$, where $(\pi_\phi\,, H_\phi\,, \xi_\phi)$ is the GNS representation of $\mathcal{M}$ associated to the state ϕ. Then

$$H_\phi :\, = \overline{\mathcal{M}\xi_\phi} = L^2(\mathcal{M}\,, \phi)\,.$$

By Tomita-Takesaki theory, the operator S_0 defined by

$$S_0(x\xi_\phi) :\, = x^*\xi_\phi\,, \quad x \in \mathcal{M}$$

is closable. Its closure $S :\, = \overline{S_0}$ has a unique polar decomposition

$$S = J\Delta^{1/2}\,,$$

where Δ is a positive self-adjoint unbounded operator on $L^2(\mathcal{M}\,, \phi)$ and J is a conjugate-linear involution.

Moreover, for all $t \in \mathbb{R}$,

$$\sigma_t^\phi(x) = \Delta^{it} x \Delta^{-it}, \quad x \in \mathcal{M}$$

and $J\mathcal{M}J = \mathcal{M}'$, where $\mathcal{M}'$ denotes the commutant of $\mathcal{M}$.

Theorem (Takesaki 1973):

For all $n \in \mathbb{Z}$, set

$$\begin{aligned} \mathcal{M}_n : &= \{x \in \mathcal{M} : \sigma_t^\phi(x) = \lambda^{int} x, \ \forall t \in \mathbb{R}\} \\ &= \{x \in \mathcal{M} : \phi(xy) = \lambda^n \phi(yx), \ \forall y \in \mathcal{M}\}. \end{aligned}$$

In particular, $\mathcal{M}_0 = \mathcal{M}_\phi$. Moreover, for all $n \in \mathbb{Z}$,

$$\mathcal{M}_n \neq \{0\}$$

and $\Delta(\eta) = \lambda^n \eta$, for every $\eta \in \overline{\mathcal{M}_n \xi_\phi}$. Furthermore,

$$L^2(\mathcal{M}, \phi) = \bigoplus_{n=-\infty}^{\infty} \overline{\mathcal{M}_n \xi_\phi}.$$

Corollary 2:

For every $n \in \mathbb{Z}$, there exists $c_n \in \mathcal{M}$ such that

$$\phi(c_n^* c_n) = \lambda^{-n/2}, \quad \phi(c_n c_n^*) = \lambda^{n/2}$$

and, moreover, $\langle c_n J c_n J \xi_\phi, \xi_\phi \rangle_{H_\phi} = 1$.

Since $\mathcal{M}$ is an injective factor, it follows (cf. Effros-Lance and Connes 1976) that the map $c \otimes d \mapsto cd$ ($c \in \mathcal{M}, d \in \mathcal{M}'$) extends uniquely to a C*-algebra isomorphism

$$C^*(\mathcal{M}, \mathcal{M}') \simeq \mathcal{M} \otimes_{\min} \mathcal{M}'.$$

Let $A\,, B$ be C*-algebras, and let

$$u : A \times B \to \mathbb{C}$$

be a jointly completely bounded bilinear form.

Lemma 3:

There is a bounded bilinear form $\widehat{u} : (A \otimes_{\min} \mathcal{M}) \times (B \otimes_{\min} \mathcal{M}') \to \mathbb{C}$ such that for all $a \in A$, $b \in B$, $c \in \mathcal{M}$, $d \in \mathcal{M}'$,

$$\widehat{u}(a \otimes c\,, b \otimes d) \colon = u(a\,, b)\langle cd\xi_\phi\,, \xi_\phi\rangle_{H_\phi}\,.$$

Moreover, $\|\widehat{u}\| \leq \|u\|_{\mathrm{jcb}}$.

Proof: Consider $a_1, \ldots, a_k \in A$, $b_1, \ldots, b_l \in B$, $c_1, \ldots, c_k \in \mathcal{M}$ and $d_1, \ldots, d_l \in \mathcal{M}'$, where $k, l \in \mathbb{N}$. Then

$$\begin{aligned}
&\left|\sum_{i=1}^{k}\sum_{j=1}^{l} u(a_i\,, b_j)\langle c_i d_j \xi_\phi\,, \xi_\phi\rangle_{H_\phi}\right| \\
&\qquad \leq \left\|\sum_{i=1}^{k}\sum_{j=1}^{l} u(a_i\,, b_j) c_i d_j\right\|_{\mathcal{B}(L^2(\mathcal{M}\,,\phi))} \\
&\qquad = \left\|\sum_{i=1}^{k}\sum_{j=1}^{l} u(a_i\,, b_j) c_i \otimes d_j\right\|_{\mathcal{M}\otimes_{\min}\mathcal{M}'} \\
&\qquad \leq \|u\|_{\mathrm{jcb}} \left\|\sum_{i=1}^{k} a_i \otimes c_i\right\|_{A\otimes_{\min}\mathcal{M}} \left\|\sum_{j=1}^{l} b_j \otimes d_j\right\|_{B\otimes_{\min}\mathcal{M}'}\,.
\end{aligned}$$

Corollary 4:

There exist $\widehat{f}_1\,,\widehat{f}_2\in S(A\otimes_{\min}\mathcal{M})$, $\widehat{g}_1\,,\widehat{g}_2\in S(B\otimes_{\min}\mathcal{M}')$ such that

$$|\widehat{u}(x,y)|\leq\|u\|_{\rm jcb}\Big(\widehat{f}_1(xx^*)+\widehat{f}_2(x^*x)\Big)^{1/2}\Big(\widehat{g}_1(y^*y)+\widehat{g}_2(yy^*)\Big)^{1/2},$$

for all $x\in A\otimes_{\min}\mathcal{M}$ and $y\in B\otimes_{\min}\mathcal{M}'$.

Lemma 5:

Let $v\in\mathcal{U}(\mathcal{M}_\phi)$ and set $v':=JvJ\in\mathcal{M}'$. Then

$$\widehat{u}\Big((\mathrm{Id}_A\otimes\mathrm{ad}(v))(x)\,,(\mathrm{Id}_B\otimes\mathrm{ad}(v'))(y)\Big)=\widehat{u}(x\,,y)\,,$$

for all $x\in A\otimes_{\min}\mathcal{M}$ and $y\in B\otimes_{\min}\mathcal{M}'$.

Proposition 6:

There exist $f_1\,,f_2\in S(A)$, $g_1\,,g_2\in S(B)$ and $\phi'\in S(\mathcal{M}')$ so that

$$|\widehat{u}(x,y)|\ \leq\ \|u\|_{\rm jcb}\Big[\Big((f_1\otimes\phi)(xx^*)+(f_2\otimes\phi)(x^*x)\Big)^{1/2}\cdot$$
$$\cdot\Big((g_1\otimes\phi')(y^*y)+(g_2\otimes\phi')(yy^*)\Big)^{1/2}\Big],$$

for all $x\in A\otimes_{\min}\mathcal{M}$ and $y\in B\otimes_{\min}\mathcal{M}'$.

Proof: For all $\alpha\,,\beta\geq 0$, $\sqrt{\alpha\beta}\leq(\alpha+\beta)/2$. By Corollary 4, it follows that for all $x\in A\otimes_{\min}\mathcal{M}$ and $y\in B\otimes_{\min}\mathcal{M}'$,

$$|\widehat{u}(x,y)|\leq\frac{1}{2}\|u\|_{\rm jcb}\left(\widehat{f}_1(xx^*)+\widehat{f}_2(x^*x)+\widehat{g}_1(y^*y)+\widehat{g}_2(yy^*)\right)\qquad(2)$$

Let $v \in \mathcal{U}(\mathcal{M}_\phi)$ and $v' := JvJ$. By Lemma 5 and inequality (2),

$$|\widehat{u}(x,y)| \leq \frac{1}{2}\|u\|_{\mathrm{jcb}}\Big[\widehat{f}_1((\mathrm{Id}_A \otimes \mathrm{ad}(v))(xx^*)) + \widehat{f}_2((\mathrm{Id}_A \otimes \mathrm{ad}(v))(x^*x)) + \widehat{g}_1((\mathrm{Id}_B \otimes \mathrm{ad}(v'))(y^*y)) + \widehat{g}_2((\mathrm{Id}_B \otimes \mathrm{ad}(v'))(yy^*))\Big] \quad (3)$$

Choose a net $(\alpha_i)_{i\in I} \subseteq \mathrm{conv}\{\mathrm{ad}(v) : v \in \mathcal{U}(\mathcal{M}_\phi)\}$ such that

$$\lim_{i\in I} \|\alpha_i(c) - \phi(c)\cdot \mathbf{1}\| = 0\,, \quad c \in \mathcal{M}\,.$$

For $i \in I$, set $\alpha_i'(d) = J\alpha_i(JdJ)J$, for all $d \in \mathcal{M}'$.

By convexity we can replace $\mathrm{ad}(v)$ and $\mathrm{ad}(v')$ in the inequality (3) by α_i and α_i', respectively, to get

$$|\widehat{u}(x,y)| \leq \frac{1}{2}\|u\|_{\mathrm{jcb}}\Big[\widehat{f}_1((\mathrm{Id}_A \otimes \alpha_i)(xx^*)) + \widehat{f}_2((\mathrm{Id}_A \otimes \alpha_i)(x^*x)) + + \widehat{g}_1((\mathrm{Id}_B \otimes \alpha_i')(y^*y)) + \widehat{g}_2((\mathrm{Id}_B \otimes \alpha_i')(yy^*))\Big]\,.$$

In the limit, this gives the inequality

$$|\widehat{u}(x,y)| \leq \frac{1}{2}\|u\|_{\mathrm{jcb}}\Big[(f_1 \otimes \phi)(xx^*) + (f_2 \otimes \phi)(x^*x) + + (g_1 \otimes \phi')(y^*y) + (g_2 \otimes \phi')(yy^*)\Big]\,, \quad (4)$$

where $f_i(a) := \widehat{f}_i(a \otimes \mathbf{1})$, $a \in A$, $g_i := \widehat{g}_i(b \otimes \mathbf{1})$, $b \in B$, for $i = 1\,,2$ and $\phi'(d) = \overline{\phi(JdJ)}$, for all $d \in \mathcal{M}'$.

Substituting x by $t^{1/2}x$ and y by $t^{-1/2}y$ in (4) for $t > 0$, we get

$$|\widehat{u}(x,y)| \leq \frac{1}{2}\|u\|_{\mathrm{jcb}}\Big[t\,((f_1 \otimes \phi)(xx^*) + (f_2 \otimes \phi)(x^*x)) + + t^{-1}\,((g_1 \otimes \phi')(y^*y) + (g_2 \otimes \phi')(yy^*))\Big] \quad (5)$$

Since for all $\alpha, \beta > 0$,

$$\inf_{t>0} (t\alpha + t^{-1}\beta) = 2\sqrt{\alpha\beta},$$

the conclusion follows by taking infimum over $t > 0$ in (5). □

Lemma 7:

For any $\alpha, \beta \geq 0$,

$$\inf_{n\in\mathbb{Z}} (\lambda^n\alpha + \lambda^{-n}\beta) \leq (\lambda^{1/2} + \lambda^{-1/2})\sqrt{\alpha\beta}.$$

Proof of the Effros-Ruan conjecture:

Let $0 < \lambda < 1$ and let $(\mathcal{M}, \phi)$ be the Powers factor of type III_λ with product state ϕ, as before. Set

$$C(\lambda) := \sqrt{\left(\lambda^{1/2} + \lambda^{-1/2}\right)/2}.$$

Let $u : A \times B \to \mathbb{C}$ be a jointly completely bounded bilinear form on C*-algebras A and B.

Let $f_1, f_2 \in S(A)$ and $g_1, g_2 \in S(B)$ be states as in Proposition 6. We will prove that for all $a \in A$ and $b \in B$,

$$|u(a,b)| \leq C(\lambda)\|u\|_{\mathrm{jcb}} \left(f_1(aa^*)^{\frac{1}{2}} g_1(b^*b)^{\frac{1}{2}} + f_2(a^*a)^{\frac{1}{2}} g_2(bb^*)^{\frac{1}{2}}\right) \quad (6)$$

that is, the Effros-Ruan conjecture holds with constant $C(\lambda)$. Since $C(\lambda) \to 1$ as $\lambda \to 1$, by a simple compactness argument it follows that the conjecture also holds with constant 1.

To prove (6), let $n \in \mathbb{Z}$ and choose $c_n \in \mathcal{M}$ as in Corollary 2. Then

$$\phi(c_n^* c_n) = \lambda^{-n/2}\,, \quad \phi(c_n c_n^*) = \lambda^{n/2} \tag{7}$$

and $\langle c_n J c_n J \xi_\phi \,, \xi_\phi \rangle_{H_\phi} = 1$.

Then, for all $a \in A$ and $b \in B$, it follows by Proposition 6 that

$$\begin{aligned} |u(a,b)|^2 &= |\widehat{u}(a \otimes c_n \,, b \otimes J c_n J)|^2 \\ &\leq \|u\|_{\rm jcb}^2 \Big[\Big(f_1(aa^*)\phi(c_n c_n^*) + f_2(a^*a)\phi(c_n^* c_n) \Big) \cdot \\ &\qquad \cdot \Big(g_1(b^*b)\phi({c_n}^* c_n) + g_2(bb^*)\phi(c_n {c_n}^*) \Big) \Big] \end{aligned}$$

Using (7), it follows that

$$\begin{aligned} |u(a,b)|^2 &\leq \|u\|_{\rm jcb}^2 \Big[\Big(\lambda^{n/2} f_1(aa^*) + \lambda^{-n/2} f_2(a^*a) \Big) \cdot \\ &\qquad \cdot \Big(\lambda^{-n/2} g_1(b^*b) + \lambda^{n/2} g_2(bb^*) \Big) \Big] \\ &= \|u\|_{\rm jcb}^2 \Big[f_1(aa^*) g_1(b^*b) + f_2(a^*a) g_2(bb^*) + \\ &\qquad + \lambda^n f_1(aa^*) g_2(bb^*) + \lambda^{-n} f_2(a^*a) g_1(b^*b) \Big] . \end{aligned}$$

Since for any $\alpha \,, \beta \geq 0$,

$$\inf_{n \in \mathbb{Z}} (\lambda^n \alpha + \lambda^{-n} \beta) \leq (\lambda^{1/2} + \lambda^{-1/2}) \sqrt{\alpha\beta}\,,$$

and we have $\lambda^{1/2} + \lambda^{-1/2} = 2C(\lambda)^2$, we deduce that

$$
\begin{aligned}
|u(a,b)|^2 \;\leq\; & \|u\|_{\mathrm{jcb}}^2 \Big[f_1(aa^*)g_1(b^*b) + f_2(a^*a)g_2(bb^*) + \\
& \qquad +2C(\lambda)^2 f_1(a^*a)^{\frac{1}{2}} g_1(b^*b)^{\frac{1}{2}} f_2(aa^*)^{\frac{1}{2}} g_2(bb^*)^{\frac{1}{2}} \Big] \\
\leq\; & C(\lambda)^2 \|u\|_{\mathrm{jcb}}^2 \Big[f_1(aa^*)^{\frac{1}{2}} g_1(b^*b)^{\frac{1}{2}} + f_2(a^*a)^{\frac{1}{2}} g_2(bb^*)^{\frac{1}{2}} \Big]^2 ,
\end{aligned}
$$

wherein we have used the fact that $C(\lambda) > 1$.

The inequality (6) follows now by taking square roots, and the proof is complete. □