Measures

Let $(\mathcal{X}, \mathbb{E})$ be a measurabel space.

Definition: A measure on $(\mathcal{X}, \mathbb{E})$ is a set-function $\mu : \mathbb{E} \to [0, \infty]$ satisfying

- $\mu(\emptyset) = 0$.
- μ is σ-additive, that is, for any sequence $A_1, A_2, \ldots \in \mathbb{E}$ of pairwise disjoint sets ($A_i \cap A_j = \emptyset$ for $i \neq j$) we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

The high-point is the existence of the Lebesgue measure, m, on the measurable space $(\mathbb{R}, \mathbb{B})$ – where $\mathbb{B}$ is the Borel-algebra. The Lebesgue measure is uniquely determined by the specification that

$$m((a,b)) = b - a \quad a < b.$$

Generated sigma-algebras

Lemma: With $(\mathbb{E}_i)_{i \in I}$ a family of σ-algebras on $\mathcal{X}$ then

$$\mathbb{G} = \bigcap_{i \in I} \mathbb{E}_i$$

is a σ-algebra – the “minimum” of the σ-algebras $\mathbb{E}_i$, $i \in I$.

Observation: The sigma-algebra generated by $\mathbb{D}$ is denoted $\sigma(\mathbb{D})$ and is defined as the smallest σ-algebra containing $\mathbb{D}$ – the “minimum” of all σ-algebras containing the paving $\mathbb{D}$.

Let $\mathbb{O}_k$ denote the paving of open sets in $\mathbb{R}^k$.
Definition: The Borel-algebra, $\mathbb{B}_k$, on $\mathbb{R}^k$ is the smallest σ-algebra containing the open sets, that is, $\mathbb{B}_k = \sigma(\mathbb{O}_k)$.

Example 1.19

If $\mathbb{D}$ is a paving on $\mathcal{X}$ define

$$\mathbb{D}' = \{D' \in \mathbb{P}(\mathcal{X}) \mid D' = D^c \text{ for } D \in \mathbb{D}\}.$$

Then $\sigma(\mathbb{D}) = \sigma(\mathbb{D}')$.

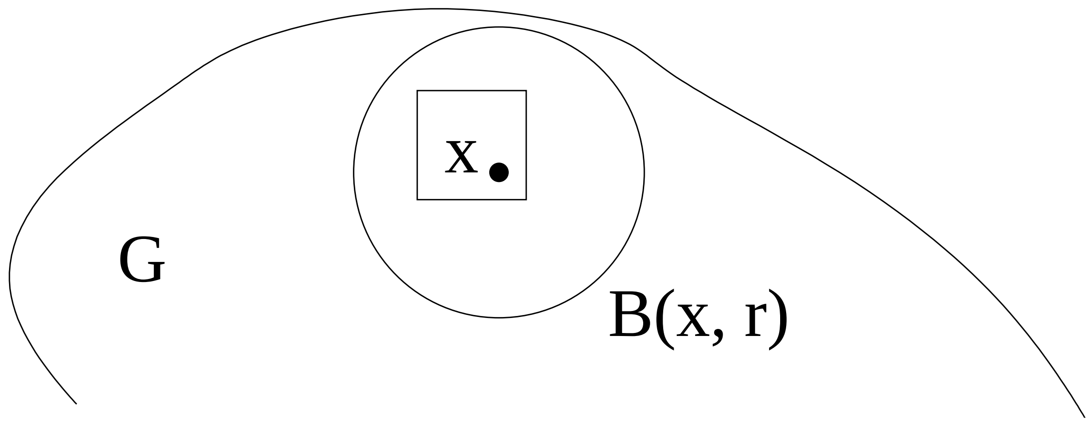
$\mathbb{O}'_k$ is the paving of closed sets in $\mathbb{R}^k$, then $\mathbb{B}_k = \sigma(\mathbb{O}'_k)$, that is, the Borel-algebra is also generated by the closed sets.

Actually, the Borel-algebra is generated by virtually any sensible paving. Theorem 1.24 (and 1.22 for $k = 1$): The Borel-algebra, $\mathbb{B}_k$, is generated by the open boxes, $\mathbb{I}^k$, (the intervals for $k = 1$)

$$(a_1, b_1) \times \ldots \times (a_k, b_k).$$

In fact, $\mathbb{B}_k$, is generated by the open boxes with rational corners, $\mathbb{I}_0^k$.

Open sets are covered by $\mathbb{I}_0^k$



For any open set $G \subset \mathbb{R}^k$ and any point $x \in G$, there is an ordinary ball of radius $r > 0$ and center x such that $B(x, r) \subset G$.

There exists an $y \in \mathbb{Q}^k$ and an open box

$$x \in (y_1 - q, y_1 + q) \times \ldots \times (y_k - q, y_k + q) \subset B(x, r)$$

for some $q < r/(2\sqrt{k})$. Hence

$$G = \cup_{J \in \mathbb{I}_0^k, J \subset G} J.$$

Constructable sets

Definition: The enlargement of the paving $\mathbb{D}$ on $\mathcal{X}$ is

$$\mathbb{D}^\diamond = \{D \in \mathbb{P}(\mathcal{X}) \mid D = A^c \text{ or } D = \cup_n A_n \text{ for } A, A_1, A_2, \ldots \in \mathbb{D}\}.$$

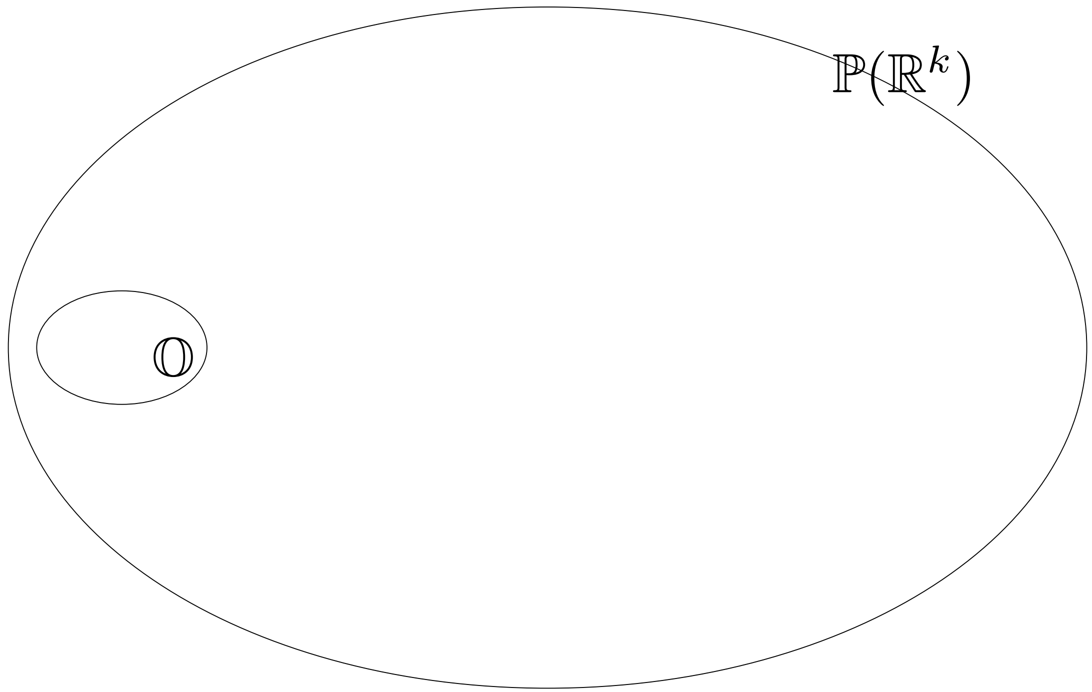
$\mathbb{D}^{\diamond n} = \left(\mathbb{D}^{\diamond(n-1)}\right)^\diamond$, but $\mathbb{D}^{\diamond n}$ does not need to be a σ-algebra.
Definition: The constructable sets are $\cup_{n=1}^\infty \mathbb{D}^{\diamond n}$.

Bad news: Obviously $\cup_{n=1}^\infty \mathbb{I}_0^{\diamond n} \subset \mathbb{B}$ but in fact

$$\cup_{n=1}^\infty \mathbb{I}_0^{\diamond n} \neq \mathbb{B}.$$

Sets outside of the constructable sets are far outside of what we can possibly imagine!

The Borel-algebra on $\mathbb{R}^k$



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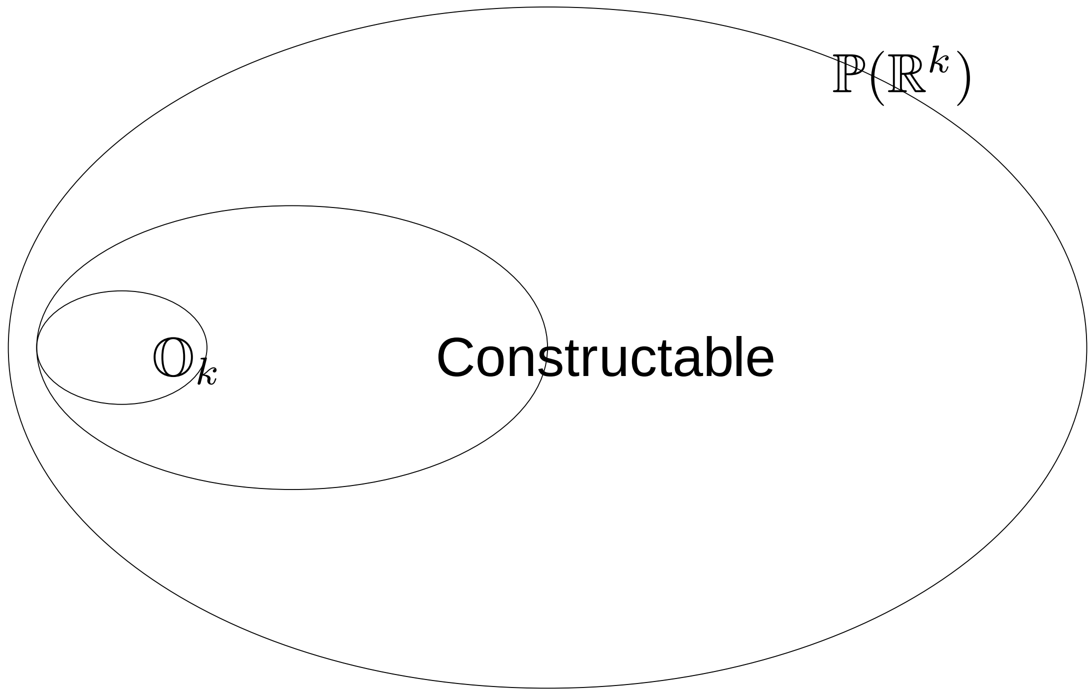

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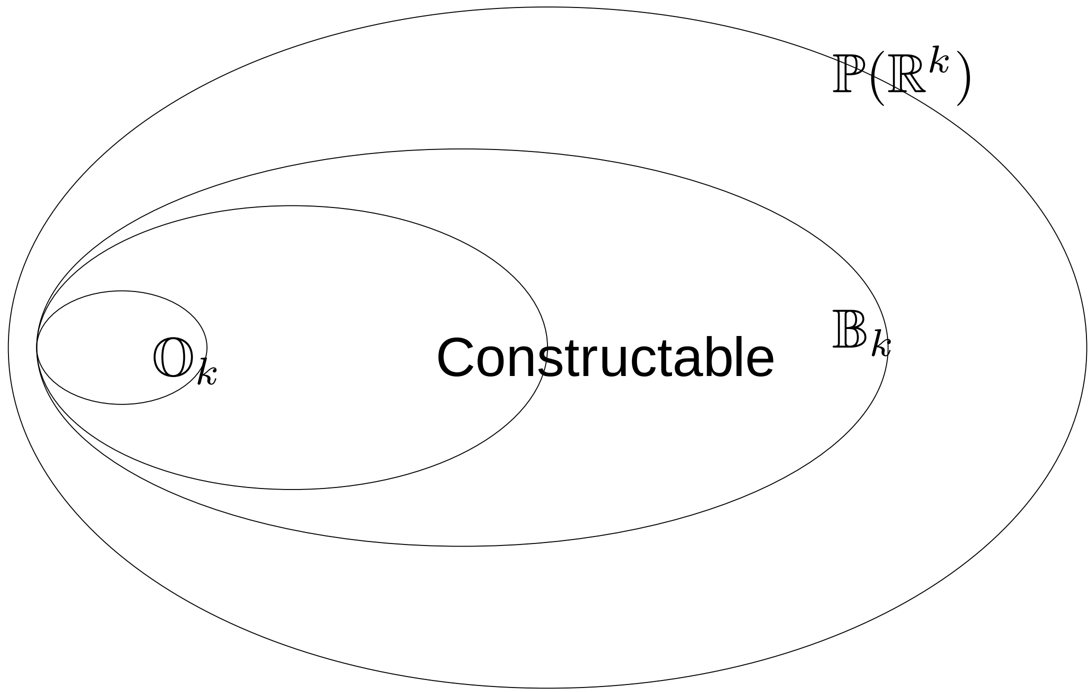

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- $\mu(\emptyset) = 0$.
- μ is σ-additive, that is, for any sequence $A_1, A_2, \ldots \in \mathbb{E}$ of pairwise disjoint sets ($A_i \cap A_j = \emptyset$ for $i \neq j$) we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

The Lebesgue measure, m_k, on the measurable space $(\mathbb{R}^k, \mathbb{B}_k)$ is uniquely specified by requiring that for for $a_i < b_i$, $i = 1, \ldots, k$,

$$m_k((a_1, b_1) \times \ldots \times (a_k, b_k)) = \prod_{i=1}^{k} (b_i - a_i).$$

Sigma-additivity means:

① If $\mu(A_n) = \infty$ for at least one n, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \infty.$$

② If $\mu(A_n) < \infty$ for all n while $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \infty$, then

$$\sum_{n=1}^{N} \mu(A_n) \to \infty \quad \text{for } N \to \infty.$$

③ If $\mu(A_n) < \infty$ for all n while $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) < \infty$, then

$$\sum_{n=1}^{N} \mu(A_n) \to \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \quad \text{for } N \to \infty.$$

Measures on finite sets

Let $\mathcal{X}$ be a finite set with the σ-algebra $\mathbb{P}(\mathcal{X})$.

Let $p : \mathcal{X} \to [0, \infty]$ be a function and define

$$\mu(A) = \sum_{x \in A} p(x) \qquad \text{for } A \subset \mathcal{X}$$

Then μ is finitely additive,

$$\mu(A \cup B) = \sum_{x \in A \cup B} p(x) = \sum_{x \in A} p(x) + \sum_{x \in B} p(x) = \mu(A) + \mu(B)$$

if $A \cap B = \emptyset$.

Since $\mathcal{X}$ is finite, finite additivity and σ-additivity is the same – any infinite sequence of disjoint sets are from some point necessarily empty.

Some simple measures

Definition: For an arbitray set $\mathcal{X}$ define on $\mathbb{P}(\mathcal{X})$ the one-point measure, ϵ_x, in $x \in \mathcal{X}$ by

$$\epsilon_x(A) = \begin{cases} 1 & \text{hvis } x \in A \\ 0 & \text{hvis } x \notin A \end{cases}$$

Then ϵ_x is a measure.

Definition: The empirical measure, $\epsilon_{x_1,\ldots,x_n}$, in $x_1, \ldots, x_n \in \mathcal{X}$ is defined as

$$\epsilon_{x_1,\ldots,x_n}(A) = \frac{1}{n} \sum_{i=1}^{n} \epsilon_{x_i}(A)$$

for $A \in \mathbb{P}(\mathcal{X})$. An empirical measure is a Probability measure since

$$\epsilon_{x_1,\ldots,x_n}(\mathcal{X}) = \frac{1}{n} \sum_{i=1}^{n} 1 = 1.$$

More examples

Definition: For an arbitrary set $\mathcal{X}$ we define the counting measure (tællemålet) τ on $\mathbb{P}(\mathcal{X})$ by

$$\tau(A) = \text{number of elements in } A$$

The Lebesgue measure, m_k, on the measurable space $(\mathbb{R}^k, \mathbb{B}_k)$ is uniquely specified by requiring that

$$m_k((a_1, b_1) \times \ldots \times (a_k, b_k)) = \prod_{i=1}^{k} (b_i - a_i)$$

for $a_i < b_i$, $i = 1, \ldots, k$, any finite real numbers.

Existence of m_k: A little hard and lengthy – not this course.

Uniqueness of m_k: Proved next week.

Lebesgue measure

Lemma: For all $x \in \mathbb{R}^k$

$$m_k(\{x\}) = 0$$

Proof for $k = 1$: Observe that

$$\{x\} = \bigcap_{n=1}^{\infty} \left(x - \frac{1}{n}, x + \frac{1}{n} \right).$$

Then downward continuity implies that

$$m(\{x\}) = \lim_{n \to \infty} m\left(\left(x - \frac{1}{n}, x + \frac{1}{n} \right) \right) = \lim_{n \to \infty} \frac{2}{n} = 0$$

Consequence: All intervals

$$(a, b) \qquad [a, b) \qquad (a, b] \qquad [a, b]$$

have the same Lebesgue measure.

Nullsets

Consider a measure μ on $(\mathcal{X}, \mathbb{E})$.

Definition: A set $A \subset \mathcal{X}$ is a μ-nullset if there is a $B \in \mathbb{E}$ such that

$$A \subset B \ , \qquad \mu(B) = 0$$

Definition: A set $A \subset \mathcal{X}$ is μ-almost everywhere if A^c is a μ-nullset.

Union of nullsets

Lemma: Let μ be a measure on $(\mathcal{X}, \mathbb{E})$. If $A_1, A_2, \ldots$ is a sequence of μ-nullsets the $\bigcup_{n=1}^{\infty} A_n$ is a μ-nullset.

Proof: For any A_n we have $B_n \in \mathbb{E}$ such that $A_n \subset B_n$ and $\mu(B_n) = 0$. Hence

$$\bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} B_n.$$

According to Boole's inequality

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} \mu(B_n) = 0.$$

Lebesgue measure

$\mathbb{Q}^k$ is a m_k-nullset.

Proof: Let $\mathbb{Q}^k = \{x_1, x_2, \ldots, \}$. Using Boole's inequality

$$m_k(\mathbb{Q}^k) = m_k\left(\bigcup_{n=1}^{\infty}\{x_n\}\right) \leq \sum_{n=1}^{\infty} m_k(\{x_n\}) = 0.$$

All lower-dimensional sets in $\mathbb{R}^k$ have Lebesgue measure 0.

Closed and open boxes have the same m_k-measure;

$$m_k([a_1, b_1] \times \ldots \times [a_k, b_k]) = \prod_{i=1}^{k}(b_i - a_i)$$

and so do all semi-closed boxes.

Cantors set

- Divide a closed interval – the unit interval $[0, 1]$, say – into three equal parts.
- Remove the central open part.
- Repeat the procedure on the remaining two intervals.
- Continue forever.

The Cantor set $\mathcal{C}$ is what is left when we are done.

Cantors set

Cantors set

Cantors set

Cantors set

Cantors set

Cantors set

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Cantors set

In the n'th step we have a collection of 2^{n-1} disjoint, closed intervals each of length $3^{-(n-1)}$.

Let C_n denote the union of these intervals, then $\mathcal{C} = \bigcap_{n=1}^{\infty} C_n$.

Remark: Since C_n is closed, $\mathcal{C}$ is closed. Especially, $\mathcal{C} \in \mathbb{B}$.

Lemma: $m(\mathcal{C}) = 0$.

Proof:

$$m(\mathcal{C}) \leq m(C_n) = 2^{n-1} \left(\frac{1}{3}\right)^{n-1} = \left(\frac{2}{3}\right)^{n-1} \to 0 \qquad \text{for } \; n \to \infty$$

Remark: There exists a bijective map from Cantors set to the set of 0-1 sequences – hence the Cantor set is uncountable.