Convergence Theorems

$(\mathcal{X}, \mathbb{E}, \mu)$ is a measure space.

The monotone convergence theorem: If $f_1 \leq f_2 \leq \ldots \in \mathcal{M}^+$, if

$$f_n(x) \nearrow f(x)$$

for $n \to \infty$ for all $x \in \mathcal{X}$ then $\int f_n \mathrm{d}\mu \nearrow \int f \mathrm{d}\mu$ for $n \to \infty$.

The dominated convergence theorem: If $f_1, f_2, \ldots \in \mathcal{M}^+$, if

$$f_n(x) \to f(x)$$

for $n \to \infty$ for all $x \in \mathcal{X}$ then if there is a $g \in \mathcal{M}^+$ with $\int g \mathrm{d}\mu < \infty$ such that $f_n \leq g$ then $\int f_n \mathrm{d}\mu \to \int f \mathrm{d}\mu$ for $n \to \infty$.

Example

Consider the measure space $(\mathbb{R}, \mathbb{B}, m)$ and the function

$$f_n(x) = \frac{n\sqrt{x}}{1+n^2x^2} \quad x > 0.$$

Then $f_n(x) \to 0$ for $n \to \infty$ for all $x \in (0, \infty)$ and

$$f_n(x) \le \frac{1}{2\sqrt{x}}.$$

Since $x \mapsto \frac{1}{2\sqrt{x}}$ is integrable over $(0,1)$ it follows from dominated convergence that

$$\int_{(0,1)} f_n \mathrm{d}\mu \to 0$$

for $n \to \infty$.

Convergence Theorems II

$(\mathcal{X}, \mathbb{E}, \mu)$ is a measure space.

The dominated convergence theorem: If $f_1, f_2, \ldots \in \mathcal{M}^+$, if

$$f_n(x) \to f(x)$$

for $n \to \infty$ for all $x \in \mathcal{X}$ then if there is a $g \in \mathcal{M}^+$ with $\int g \mathrm{d}\mu < \infty$ such that $f_n \leq g$ then $\int f_n \mathrm{d}\mu \to \int f \mathrm{d}\mu$ for $n \to \infty$.

Fatou's lemma: If $f_1, f_2, \ldots \in \mathcal{M}^+$, then

$$\liminf_{n\to\infty} \int f_n \mathrm{d}\mu \geq \int \liminf_{n\to\infty} f_n \mathrm{d}\mu.$$

The Lebesgue integral

$(\mathcal{X}, \mathbb{E}, \mu)$ is a measure space.

Definition: The integral of $f \in \mathcal{M}^+$ is defined as

$$\int f \mathrm{d}\mu = \sup\{I(s) \mid s \leq f,\ s \in \mathcal{S}^+\}.$$

$$f^+(x) = \max\{f(x), 0\} \quad f^-(x) = -\min\{f(x), 0\}$$

Definition: The function $f \in \mathcal{M}$ is integrable if

$$\int f^+ \mathrm{d}\mu < \infty \quad \text{and} \quad \int f^- \mathrm{d}\mu < \infty.$$

If $f \in \mathcal{M}$ is integrable the integral of f w.r.t. μ is

$$\int f \mathrm{d}\mu = \int f^+ \mathrm{d}\mu - \int f^- \mathrm{d}\mu.$$

The set of integrable functions is denoted $\mathcal{L} = \mathcal{L}(\mathcal{X}, \mathbb{E}, \mu) \subset \mathcal{M}(\mathcal{X}, \mathbb{E})$.

The Lebesgue integral

- $f = f^+ - f^-$, $|f| = f^+ + f^-$ and $f \in \mathcal{L}$ if and only if

$$\int |f| \mathrm{d}\mu < \infty.$$

- If $f \in \mathcal{L}$ and $c \in \mathbb{R}$ then $cf \in \mathcal{L}$ and

$$\int cf \mathrm{d}\mu = c \int f \mathrm{d}\mu.$$

- If $f, g \in \mathcal{L}$ then $f + g \in \mathcal{L}$ and

$$\int f + g \mathrm{d}\mu = \int f \mathrm{d}\mu + \int g \mathrm{d}\mu.$$

The Lebesgue integral

- If $f, g \in \mathcal{L}$ and $f \leq g$ then

$$\int f \mathrm{d}\mu \leq \int g \mathrm{d}\mu.$$

- If $f \in \mathcal{L}$ then

$$\left| \int f \mathrm{d}\mu \right| \leq \int |f| \mathrm{d}\mu.$$

- If $f, g \in \mathcal{M}$ then if $f = g$ μ-almost everywhere we have that $f \in \mathcal{L}$ if and only if $g \in \mathcal{L}$ in which case

$$\int f \mathrm{d}\mu = \int g \mathrm{d}\mu.$$

- Definition: If $A \in \mathbb{E}$, if $f \in \mathcal{M}$ and if $1_A f \in \mathcal{L}$ then

$$\int_A f \mathrm{d}\mu = \int 1_A f \mathrm{d}\mu.$$

Almost everywhere

Many statements and results in measure theory have formulations involving “almost everywhere” (or “almost surely” for probability measures).

Recall, a μ-nullset $G \subset \mathcal{X}$ is a set contained in a set $A \in \mathbb{E}$ with $\mu(A) = 0$.

We say that $f = g$ μ-almost everywhere (μ-a.e.) (næsten overalt, n.o.) for $f, g \in \mathcal{M}^+$ (or $f, g \in \mathcal{M}$) if the set

$$\{x \in \mathcal{X} \mid f(x) \neq g(x)\}$$

is a μ-nullset.

Almost everywhere

Lemma: If $f \in \mathcal{M}^+$ then $\int f \mathrm{d}\mu = 0$ if and only if $f = 0$ μ-a.e. Moreover, if $f, g \in \mathcal{M}^+$ then

$$\int f \mathrm{d}\mu = \int g \mathrm{d}\mu$$

if $f = g$ μ-a.e.

Theorem: If $f, g \in \mathcal{M}$ then if $f = g$ μ-a.e. we have that $f \in \mathcal{L}$ if and only if $g \in \mathcal{L}$ in which case

$$\int f \mathrm{d}\mu = \int g \mathrm{d}\mu.$$

Ultimative dominate convergence

The dominated convergence theorem – ultimate version: Let $f_1, f_2, \ldots \in \mathcal{M}(\mathcal{X}, \mathbb{E})$ and assume that

$$f_n \to f \quad \mu\text{-a.e.}$$

If there exists a function $g \in \mathcal{L}$ such that

$$|f_n| \leq g \quad \mu\text{-a.e.}$$

then $f, f_1, f_2, \ldots \in \mathcal{L}(\mathcal{X}, \mathbb{E}, \mu)$ and

$$\int f_n \mathrm{d}\mu \to \int f \mathrm{d}\mu$$

for $n \to \infty$.

Proof: Slightly different from the proof of Theorem 7.6 in the book – we give the high-lights on the following two slides.

Proof of Dom. Conv. Thm.

Define

$$\begin{aligned} A_n &= \{x \in \mathcal{X} \mid |f_n(x)| \leq g(x)\} \\ B &= \{x \in \mathcal{X} \mid f_n(x) \to f(x)\}. \end{aligned}$$

Their complements are μ-nullsets by assumption and so is A^c if

$$A = B \cap A_1 \cap A_2 \cap \ldots$$

Therefore there is a set $C \in \mathbb{E}$ with $\mu(C^c) = 0$ and with $C \subset A$.

We conclude that for all $x \in \mathcal{X}$

$$1_C(x) f_n^+(x) \to 1_C(x) f^+(x) \quad \text{and} \quad 1_C(x) f_n^-(x) \to 1_C(x) f^-(x)$$

for $n \to \infty$ and that

$$1_C(x) f_n^+(x) \leq 1_C(x) g(x) \quad \text{and} \quad 1_C(x) f_n^-(x) \leq 1_C(x) g(x).$$

Proof of Dom. Conv. Thm.

Since $g \in \mathcal{L}$ and since C^c is a μ-nullset we get that $f, f_1, f_2, \ldots \in \mathcal{L}$.

By Theorem 6.27 and since C^c is a μ-nullset it follows that

$$\int f_n^+ \mathrm{d}\mu = \int 1_C f_n^+ \mathrm{d}\mu \to \int 1_C f^+ \mathrm{d}\mu = \int f^+ \mathrm{d}\mu,$$

and likewise for f_n^-. In conclusion,

$$\int f_n \mathrm{d}\mu = \int f_n^+ \mathrm{d}\mu - \int f_n^- \mathrm{d}\mu \to \int f^+ \mathrm{d}\mu - \int f^- \mathrm{d}\mu = \int f \mathrm{d}\mu.$$

Integration w.r.t. m

For $f \in \mathcal{M}^+(\mathbb{R}, \mathbb{B})$ or $f \in \mathcal{L}(\mathbb{R}, \mathbb{B}, m)$ we introduce the notation

$$\int_a^b f(x)\mathrm{d}x = \int_{(a,b)} f\mathrm{d}m$$

for $a \leq b$ in $\mathbb{R}$.

Observe that since $\{a\}$ and $\{b\}$ are m-nullsets it does not matter if we integrate over $[a, b]$ or (a, b) or $(a, b]$ or $[a, b)$.

Integration w.r.t. m

Definition: The set of integrable functions on (a, b) with $a < b$ in $[-\infty, \infty]$ is denoted

$$\mathcal{L}(a, b)$$

A function $f \in \mathcal{M}(\mathbb{R}, \mathbb{B})$ is called locally integrable over (a, b) if $f \in \mathcal{L}(c, d)$

for all $c, d \in \mathbb{R}$ with $a < c < d < b$.

Riemann integrals

The proper (egentligt) Riemann integral of $f : \mathbb{R} \to \mathbb{R}$ over $[a, b]$, $a < b$ in $\mathbb{R}$, is denoted

$$\mathrm{RI}(f, a, b)$$

given that it exists.

From the definition

$$s \leq \mathrm{RI}(f, a, b) \leq S$$

with

$$s = \sum_i c_i(z_i - z_{i-1}) \quad S = \sum_i d_i(z_i - z_{i-1}),$$

$a = z_0 < z_1 < \ldots < z_{n-1} < z_n = b$ and

$$c_i = \inf\{f(x) \mid x \in [z_{i-1}, z_i]\} \quad d_i = \sup\{f(x) \mid x \in [z_{i-1}, z_i]\}.$$

Improper Riemann integrals

The improper (uegentligt) Riemann integral of $f : \mathbb{R} \to \mathbb{R}$ over (a, b), $a < b$ in $[-\infty, \infty]$, is defined by the limit

$$\mathrm{IRI}(f, a, b) = \lim_{\substack{c \to a,\, c > a \\ d \to b,\, d < b}} \mathsf{RI}(f, c, d)$$

given that f has a proper Riemann integral on $[c, d]$ for all $c, d \in \mathbb{R}$ with $a < c < d < b$ and given that the limits exist.

Riemann and Lebesgue

Theorem: If $f \in \mathcal{M}(\mathbb{R}, \mathbb{B})$ has a proper Riemann integral over the interval $[a, b]$ then $f \in \mathcal{L}(a, b)$ and

$$\int_{(a,b)} f \mathrm{d}m = \int_a^b f(x)\mathrm{d}x = \text{RI}(f, a, b).$$

Theorem: If $f \in \mathcal{M}(\mathbb{R}, \mathbb{B})$ has an improper Riemann integral over the interval (a, b) and if $f \in \mathcal{L}(a, b)$ then

$$\int_{(a,b)} f \mathrm{d}m = \int_a^b f(x)\mathrm{d}x = \text{IRI}(f, a, b).$$

There are nasty examples where $\text{IRI}(f, a, b)$ exists but where $f \notin \mathcal{L}(a, b)$.

Riemann-like rules

- Convention: If $b < a$ then $\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x$.
- Insertion rule: If f is locally integrable on (a, b) and $c_1, c_2, c_3 \in (a, b)$ then
$$\int_{c_1}^{c_3} f(x)\mathrm{d}x = \int_{c_1}^{c_2} f(x)\mathrm{d}x + \int_{c_2}^{c_3} f(x)\mathrm{d}x.$$
- If $f \in \mathcal{M}(\mathbb{R}, \mathbb{B})$ is continuous on (a, b) and F is an antiderivative of f on (a, b), that is, F is differentiable on (a, b) with $F'(x) = f(x)$ for $x \in (a, b)$ then
$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a), \quad x_1, x_2 \in (a, b).$$
- Integration by parts works (Example 7.22).

Antiderivatives

Theorem 7.18: If $f \in \mathcal{M}(\mathbb{R}, \mathbb{B})$ is locally integrable over (a, b) then $F : (a, b) \to \mathrm{R}$ defined by

$$F(x) = \int_{x_0}^{x} f(x)\mathrm{d}x \quad x \in (a, b)$$

for any fixed $x_0 \in (a, b)$ is a continuous function. If f is continuous in $x \in (a, b)$ then F is differentiable in x with $F'(x) = f(x)$.

Example

Consider the $\mathcal{M}^+(\mathbb{R},\mathbb{B})$-function

$$x \mapsto x^{-\alpha}, \quad x > 0$$

(implicit convention, the function is 0 for $x \leq 0$ and the global function is measurable by Lemma 4.11).

$$\int_1^\infty x^{-\alpha} \mathrm{d}x = \begin{cases} \frac{1}{\alpha-1} & \text{if } \alpha > 1 \\ \infty & \text{if } \alpha \leq 1 \end{cases}$$

$$\int_0^1 x^{-\alpha} \mathrm{d}x = \begin{cases} \frac{1}{1-\alpha} & \text{if } \alpha < 1 \\ \infty & \text{if } \alpha \geq 1 \end{cases}$$

Functions given as integrals

Let $(\mathcal{X}, \mathbb{E}, \mu)$ be a measure space, $\mathcal{Y}$ any set and $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ a function whose section function (snitfunktion)

$$x \mapsto f(x, y)$$

is either in $\mathcal{M}^+$ or in $\mathcal{L}$ for all $y \in \mathcal{Y}$ then we can define a function $\phi : \mathcal{Y} \to \mathbb{R}^*$ by

$$\phi(y) = \int f(x, y) \mathrm{d}\mu.$$

The Γ- and B-functions

Definition: The x-section of $f(x,\lambda) = x^{\lambda-1}e^{-x}$ for $x > 0$ and $\lambda > 0$ are in $\mathcal{M}^+$ and we define the Γ-function by

$$\Gamma(\lambda) = \int_0^\infty x^{\lambda-1}e^{-x}\mathrm{d}x.$$

The integral is finite for all $\lambda > 0$.

Definition: The x-section of $f(x,\lambda,\mu) = x^{\lambda-1}(1-x)^{\mu-1}$ for $x \in (0,1)$ and $\lambda, \mu > 0$ are in $\mathcal{M}^+$ and we define the B-function (Beta-function) by

$$B(\lambda,\mu) = \int_0^1 x^{\lambda-1}(1-x)^{\mu-1}\mathrm{d}x.$$

The integral is finite for all $\lambda, \mu > 0$.