FOUR THINGS YOU MIGHT NOT KNOW ABOUT THE BLACK-SCHOLES FORMULA[1]

Abstract: I demonstrate four little-known properties of the Black-Scholes option pricing formula: (1) An easy way to find delta. (2) A quaint relation between call- and put-prices. (3) Why vega-hedging though non-sensical will help. (4) What happens if you take vega-hedging too far.

Introduction

The Black-Scholes formula is the mother of all option pricing formulas. It states that under perfect market conditions and Geometric Brownian motion dynamics, the only arbitrage-free time-*t* price of a strike-*K* expiry-*T* call-option is

$$Call(t) = BS^{call}(S(t), T-t, K, r, \sigma)$$

where *S(t)* is the time-*t* price of a dividend-free[2] stock, r is the risk-free rate, σ is volatility (i.e. the standard deviation of appropriately time-scaled returns), and the function BS^{call} is given by

$$BS^{call}(S,\tau,K,r,\sigma) = S\,N(d_1(S,\tau,K,r,\sigma)) - e^{-r\tau}K\,N(d_2(S,\tau,K,r,\sigma))$$

where *N* denotes the standard normal distribution function and

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[2] A “no dividends” assumption is not always without loss of generality; it may change “standard results”. For instance, with a positive dividend yield call-prices may decrease at long expiries. However, a dividend yield does not alter the results I present, so to ease the exposition, it has been left out.

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}},\ d_2 = d_1 - \sigma\sqrt{\tau}.$$

The Black-Scholes formula can be derived in a number of ways. Andreasen, Jensen and Poulsen (1998) is an account of some of them; Derman and Taleb (2005) is a recent (although debatable, see Ruffino and Treussard (2006)) addition.

In this note I show some less-know results related to the Black-Scholes formula. You may see them just as “cute”, “quaint” or “a nice exercise”. But they go deeper; as I briefly outline, they are special cases of more general results or techniques. Or poetically, they are the shadows cast from higher dimensions onto the walls of the Black-Scholes cave.

Deriving delta – correctly – without lengthy calculations

A central quantity for hedging and risk-management is the call-(or any other)option’s sensitivity to changes in the stock-price; its delta:

$$\Delta = \frac{\partial BS^{call}}{\partial S}.$$

A tempting way to a show this is to ignore/forget that *S* enters inside the *N*(…)-expressions which makes the differentiation very easy:

$$\Delta = N(d_1).$$

Rather un-pedagogically this happens to be the correct result. To derive it properly you must use the chain rule when differentiating. This gives two extra terms that cancel after tedious calculations. A simpler derivation that does not appear to be well-known is this: Remember that a function f (defined on some cone-shaped domain of $\Re^n$) is said to be

homogenous (of degree one) if $f(\alpha x) = \alpha f(x)$ for all $\alpha \in \Re_+$ and all x (in f's domain). Euler's Theorem (that is otherwise predominantly used in microeconomics) says that a differentiable function f is homogenous if an only if it has the form $f(x) = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}$.
Now observe that the Black-Scholes call-price is homogenous in stock-price and strike. Then Euler's Theorem tells us that the term that "multiplies S" in the formula is indeed the partial derivative with respect to S; the delta.

This homogeneity property (known in the financial engineering literature as "sticky moneyness regime") holds not just in the Black-Scholes model, but as discussed in Joshi (2003; chapter 15) in a more general class where the return distribution is independent of the current stock-price level. That involves affine jump-diffusions as well as some infinite intensity Levy-driven processes. Before we get carried away, Lee (2004) shows that although call-prices in these models can be written such that they look a lot like the Black-Scholes formula (from which delta is then recognized), that is far from the best representation for numerical calculations.

Not all models are homogeneous. The Bachelier-model (where S is arithmetic Brownian motion), the constant elasticity of variance model, the SABR stochastic volatility model as well as Dupire-Derman type local volatility models are inhomogeneous. Devising an empirical methodology that is powerful enough for testing (i.e. for rejecting) homogeneity remains an open problem.[3]

Put-call-duality

If you plug in $-\sigma$ into the call-price formula, you get minus the put-price:

[3] For instance, Cont and da Fonseca (2002) impose homogeneity from the outset in their empirical analysis.

$$
\begin{aligned}
& BS^{call}(S(t),\tau,K,r,-\sigma) = S\,N(-d_1) - e^{-r\tau}K\,N(-d_2) \\
& = S(1-N(d_1)) - e^{-r\tau}K(1-N(d_2)) \\
& = S - e^{-r\tau}K - BS^{call}(S(t),\tau,K,r,\sigma) \\
& = -BS^{put}(S(t),\tau,K,r,\sigma)
\end{aligned}
$$

where the first equality uses symmetry of the normal distribution and the third employs the put-call-parity. In the Black-Scholes model this put-call-duality is little more than quaint, but it is related to time-reversal that may among other things play a useful role in efficient calibration of local volatility models; see Andreasen, Jensen and Poulsen (1998, section 7) and Peskir and Shiryaev (2001).

A simulation engine

The binomial model is often called the workhorse of finance. In that vein, the pseudo-code below may be termed a *simulation engine*. It mimics the behavior of a discrete delta-hedger.

```
V=BScall(S,T,K,r,sigma,greek=price)  # initial investment
a=BScall(S,T,K,r,sigma,greek=delta)  # stock position = delta
b=V-a*S                              # rest in bank; self-fin. Cond.
Loop over j = 1 to Npaths
       Loop over i=1 to Nhedgepoints
              eps=random.normal()  # simulate outcome of N(0,1)
              S=S*exp((mu-0.5*sigma^2)*dt+*sqrt(dt)*eps)
              V=a*S+b*exp(r*dt)
              a=BScall(S,T-i*dt, K,r,sigma, greek = delta)
              b=V-a*S
       next i
       hedge(j) = V
       error(j) = max(S-K,0) – V
next j
```

The first “Eureka”-experience is that the engine works. Exhibit1 clearly shows that as the hedge frequency increases the standard deviation of the hedge error (and hence the hedge error itself) goes to zero. The log/log-scaling and the fitted line of the form *constant* – 0.5 log(#hedge points) reveals that the order at which this happens is as (one over) the square root of the hedge frequency.[4]

Digressing slightly, I have found the simulation engine to be very useful in a teaching context. Among things that can readily be illustrated are:

- The hedge error goes to 0 irrespective of the drift in the simulations, but not if you use the wrong volatility (layman’s Girsanov).
- The discounted value-process of any self-financing trading strategy is a martingale under the risk-neutral probability measure. This means that if you simulate with $\mu = r$ the average discounted pay-off equals the initial investment. This is very good for detecting coding errors. (Try permuting the innermost lines in the loops.) This is also a good place to have students ponder the $\sigma^2 / 2$-Ito-term.
- Try different contacts; digital options are hard to hedge. (Precise statements are given in Gobet and Teman (2001).)

As discussed in Bjørk (2004, chapter 9), to hedge an out-of-the-money option (think of this as illiquid; strike K1) discretely you may want include a closer-to-the-money option (think liquid; strike K0) to make your portfolio gamma-neutral. This can be achieved by the following adjustment to the code:

[4] This has been known since Boyle and Emanuel (1980) and is thus not one of *the things* from the title.

```
V=a*S+b*exp(r*dt)+c*BScall(S,T-i*dt, K0,…,greek=price)
c=BScall(S,T-i*dt,K1,…,greek=gamma)/BScall(S,T-i*dt,K0,…,greek=gamma)   (*)
a= BScall(S,T-i*dt,K1,…,greek=delta)-c*BScall(S,T-i*dt,K0,…,greek=delta)
b=V-a*S-c* BScall(S,T-i*dt,K0,…, greek=price)
```

The effects are shown by the lower curve in Exhibit 2. We see a significant improvement in the hedge performance; the standard deviations of hedge errors are reduced by a factor of about 3. That a second-order correction (as gamma hedging can be thought of) improves the approximation is perfectly reasonable.

A trader might use exactly the same argument to make his portfolio vega-neutral, i.e. its value insensitive to changes in volatility. This is achieved be changing line (*) to

```
c=BS(S,T-i*dt,K0,…, greek=vega)/ BS(S,T-i*dt,K0,…, greek=vega)        (**)
```

Logically, this is non-sense. A key assumption in Black-Scholes model is that volatility is constant. But the trader may persist saying that his experience is that it improves hedge performance. And it does. To see why recall that in the Black-Scholes model

$$vega := \frac{\partial BS^{call}}{\partial \sigma} = S\phi(d_1)\sqrt{\tau}$$

$$gamma := \frac{\partial^2 BS^{call}}{\partial S^2} = \frac{\phi(d_1)}{S\sigma\sqrt{\tau}}$$

so the ratios of vega's and gamma's are the same, i.e. vega- and gamma-hedges from lines (*) and (**) are the same.

In stochastic volatility models vega-dependent hedge strategies are not just an improvement, they are a necessity, see Joshi (2003) or Cont and da Fonseca (2002). Furthermore, Ewald, Poulsen and Schenk-Hoppe (2006) show that for such hedges to be effective it is important to use the delta and especially vega from a genuine stochastic volatility model. This can be seen as a Lucas-critique; sensitivities from a static model can be misleading in a dynamic model.

Beware of Greeks

The last of the *four things* from the title is a caveat. Or a confession. I cheated when I produced the Exhibit 2 depicting the benefits of gamma-hedging over plain delta-hedging. If you use equation (*) directly, you get a picture like Exhibit 3. Nice at first, but if you hedge more often than about every other week, things go violently wrong. Daily hedging has a standard deviation of the order of magnitude of the number of atoms in the universe. The instability comes about because gamma goes to zero extremely quickly ("exponentially squared fast") when strike moves away from spot; and the closer we are to expiry, the worse things are. This means that even for strikes that are reasonably close, the denominator in line (*) can be much, much smaller than the numerator, and thus the option position in the hedge, c, becomes unreasonable large. A simple regularization is to truncate c at some level. This works; truncation a 10 was what produced Exhibit 2. Another solution is always to use the forward at-the-money option or the one with the highest (Black-Scholes model) gamma to hedge with. Note that is this means liquidating your whole hedge option portfolio at each trading day. Regularization is investigated in a more general hedging-with-options context Nalholm and Poulsen (2006) where the benefits of singular value decomposition are demonstrated.

Conclusion

I described four cases where general methods have interesting consequences in the Black-Scholes model. I'd be surprised if there aren't more.

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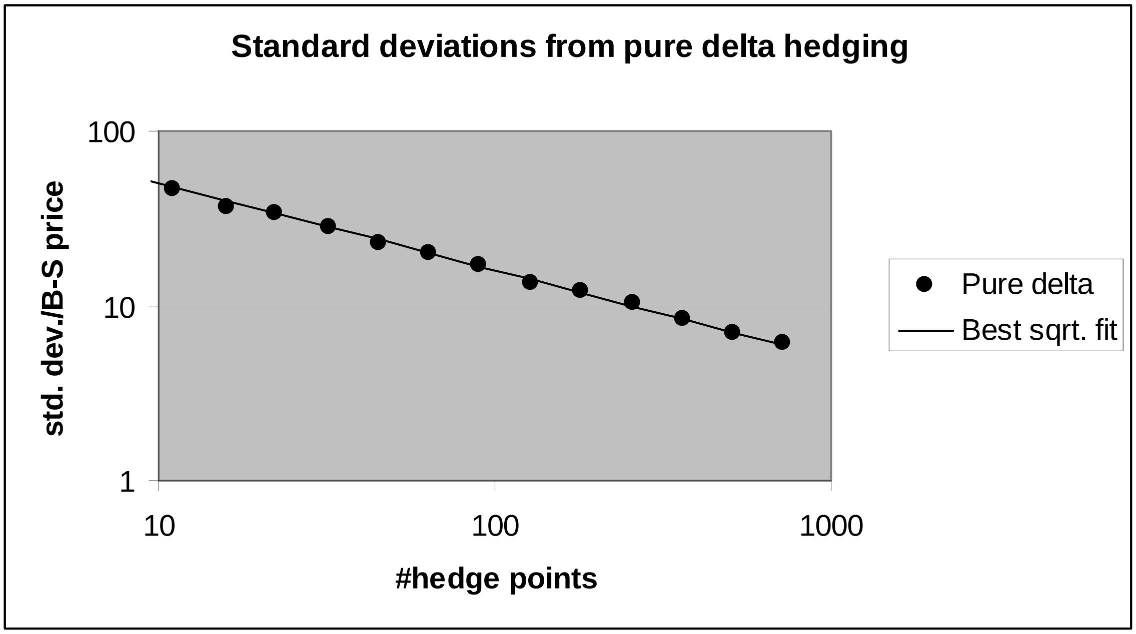

Exhibit 1: Standard deviation of the hedge error (as % of the initial option-price) for discrete delta-hedging in the Black-Scholes model. The option being hedged is a 1-year call with strike = 1.15*spot; the risk-free rate is 5%, the drift of the stock is 10% and its volatility is 20%.

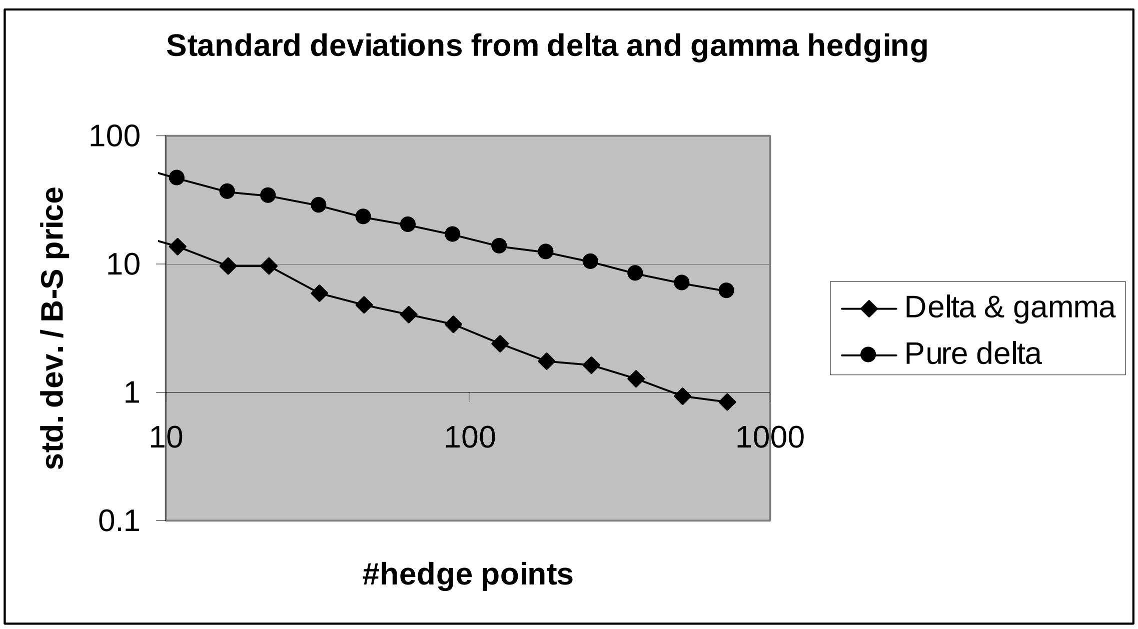

Exhibit 2: Standard deviation of hedge errors for delta and (delta,gamma)-hedging. The set-up is the same as in Exhibit 1. The gamma-hedge uses a forward at-the-money expiry-1call option.

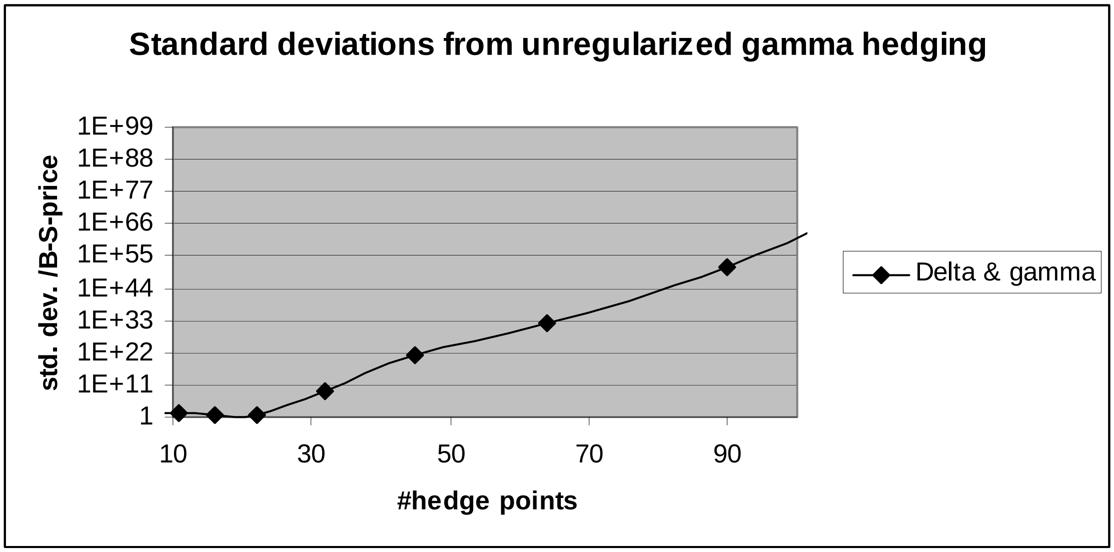

Exhibit 3: Hedge error standard deviations from gamma-hedging fixed strike hedge-option and no regularization. Note the units on the y-axis.