Spectra of C* algebras, Extensions and $\mathbb{R}$-actions

Eberhard Kirchberg

Humboldt-Universität zu Berlin

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TOC

- Spectra of amenable C*-algebras.
- NC-Selection and semi-split Extensions.
- Study of coherent locally q-compact spaces.
- Application: Exotic line-action on Cuntz algebras.

Conventions and Notations

- Spaces $P, X, Y, \ldots$ are T_0 and *second countable*, algebras $A, B, \ldots$ are *separable*, ...

- ... *except* corona spaces $\beta(P) \setminus P$, multiplier algebras $\mathcal{M}(B)$, and ideals of corona algebras $\mathrm{Q}(B) := \mathcal{M}(B)/B$, the space $\mathrm{Prim}(\mathcal{M}(B))$, ...

- The isomorphisms $\mathcal{I}(A) \cong \mathbb{O}(\mathrm{Prim}(A)) \cong \mathcal{F}(\mathrm{Prim}(A))^{op}$ will be used frequently.

- $\mathbb{Q} := [0,1]^\infty$ denotes the Hilbert cube (with its coordinate-wise order).

- A T_0 space X is **sober** (or "point-complete") if each prime closed subset F of X is a the closure $\overline{\{x\}} = F$ of a singleton $\{x\}$. (Locally) "compact" means (locally) "quasi-compact" in case of T_0 spaces.

Spectra of amenable algebras (1)

Characterization of $\mathrm{Prim}(A)$ for amenable A (H.Harnisch, E.K., M.Rørdam):

Theorem 1. *A sober space X is homeomorphic to a primitive ideal space of an amenable C^*-algebra A, if and only if,*
there is a Polish l.c. space P and a continuous map $\pi\colon P \to X$ such that
*$\pi^{-1}\colon \mathbb{O}(X) \to \mathbb{O}(P)$ is injective (=: π is **pseudo-epimorphic**),*
and
*$(\bigcap_n \pi^{-1}(U_n))^\circ = \pi^{-1}((\bigcap_n U_n)^\circ)$ for each sequence $U_1, U_2, \ldots \in \mathbb{O}(X)$ (=: π is **pseudo-open**).*

The algebra $A \otimes \mathcal{O}_2 \otimes \mathbb{K}$ is uniquely determined by X up to (unitarily homotopic) isomorphisms.

Spectra of amenable algebras (2)

Notice: A continuous epimorphism $\pi\colon P \to X$ is not necessarily *pseudo-open*, e.g. $\sum_n \alpha_n 3^{-n} \mapsto \sum_n \alpha_n 2^{-n}$ is continuous epimorphism from the Cantor space $\{0,1\}^\infty$ onto $[0,1]$, but no pseudo-open continuous epimorphism from $\{0,1\}^\infty$ onto $[0,1]$ exists.

A map $\Psi\colon \mathbb{O}(X) \to \mathbb{O}(Y)$ is **lower semi-continuous** if $(\bigcap_n \Psi(U_n))^\circ = \Psi((\bigcap_n U_n)^\circ)$ for each sequence $U_1, U_2, \ldots \in \mathbb{O}(X)$.

(Thus, π is pseudo-open, if and only if, $\Psi := \pi^{-1}$ is lower semi-continuous.)

If one works with *closed sets*, then one has to replace intersections by unions and interiors by closures.

NC-Selection and Extensions (1)

Proposition 2. *If* $\Psi\colon \mathcal{I}(B) \to \mathcal{I}(A)$ *is a lower semi-continuous action of* $\mathrm{Prim}(B)$ *on* A *and* B *is stable, then there exists a* lower s.c. *action* $\mathcal{M}(\Psi)\colon \mathcal{I}(\mathcal{M}(B)) \to \mathcal{I}(A)$ *of* $\mathrm{Prim}(\mathcal{M}(B))$ *on* A*, that has the following properties (i)–(iii):*

(i) $\mathcal{M}(\Psi)$ *is* **monotone** *upper semi-continuous (*$:=$ $\sup$*'s of* upward directed *families of ideals will be respected).*

(ii) $\mathcal{M}(\Psi)(J_1) = \mathcal{M}(\Psi)(J_1)$ *if* $J_1 \cap \delta_\infty(\mathcal{M}(B)) = J_2 \cap \delta_\infty(\mathcal{M}(B))$.

(iii) $\mathcal{M}(\Psi)(\mathcal{M}(B,I)) = \Psi(I)$ *for all* $I \in \mathcal{I}(B)$.

The "extension" $\mathcal{M}(\Psi)$ *of* Ψ *with (i)–(iii) is unique.*

NC-Selection and Extensions (2)

For strongly p.i. (not necessarily separable) B and exact A, there is a nuclear *-morphism $h\colon A \to B$ with $\Psi(J) = h^{-1}(h(A) \cap J)$, if and only if, Ψ is lower s.c. and monotone upper s.c. It yields the following theorem.

Theorem 3. [NC-selection] *Suppose that B is stable, $A \otimes \mathcal{O}_2$ contains a regular* exact *C*-algebra $C \subset A \otimes \mathcal{O}_2$, and that $\Psi\colon \mathcal{I}(B) \to \mathcal{I}(A)$ is a lower s.c. action of $\mathrm{Prim}(B)$ on A.*

*Then there is a *-morphism $h\colon A \to \mathcal{M}(B)$ such that $\delta_\infty \circ h$ is unitarily equivalent to h, $\Psi(J) = h^{-1}(h(A) \cap \mathcal{M}(B, J))$ and that*

$$[h]_J\colon A/\Psi(J) \to \mathcal{M}(B/J) \cong \mathcal{M}(B)/\mathcal{M}(B, J)$$

is weakly nuclear for all $J \in \mathcal{I}(B)$.

NC-Selection and Extensions (3)

Here, a subalgebra $C \subset D$ is **regular** if C separates the ideals of D and $C \cap (I + J) = (C \cap I) + (C \cap J)$ for all $I, J \in \mathcal{I}(D)$.

Theorem 3 applies to necessary and sufficient criteria for (ideal-system-) equivariant semi-splitness of extensions.

Let $\epsilon \colon B \to E$ a *-monomorphism onto a closed ideal of E and $\pi \colon E \to A$ an epimorphism such that $\epsilon(B)$ is the kernel of π. We denote by $\gamma \colon A \to \mathrm{Q}(B) = \mathcal{M}(B)/B$ the Busby invariant of the extension

$$0 \to B \xrightarrow{\epsilon} E \xrightarrow{\pi} A \to 0\,.$$

NC-Selection and Extensions (4)

Consider now general "actions" $\psi_B \colon S \to \mathcal{I}(B)$, $\psi_E \colon S \to \mathcal{I}(E)$, and $\psi_A \colon S \to \mathcal{I}(A)$, of a set S on B, E and A. We require that the extension E is ψ-equivariant:

(a) $\epsilon(\psi_B(s)) = \epsilon(B) \cap \psi_E(s) = \epsilon(B)\psi_E(s)$, and

(b) $\psi_A(s) = \pi(\psi_E(s))$ for all $s \in S$.

i.e., $0 \to \psi_B(s) \to \psi_E(s) \to \psi_A(s) \to 0$ is exact for each $s \in S$.

An action $\Phi \colon \mathcal{I}(A) \to \mathcal{I}(B)$ of $\mathrm{Prim}(A)$ on B is **upper semi-continuous** if Ψ preserves sup of families in $\mathcal{I}(A)$, i.e., $\Psi(I + J) = \Psi(I) + \Psi(J)$ and Ψ is monotone upper semi-continuous.

NC-Selection and Extensions (5)

Lemma 4. *There is a unique maximal upper semi-continuous map* $\Phi\colon \mathcal{I}(A) \to \mathcal{I}(B)$ *with the property that* $\Phi(\psi_A(s)) \subset \psi_B(s)$ *for all* $s \in S$*.*

Upper semi-continuous actions Φ have lower semi-continuous ($=$ inf preserving) adjoint maps $\Psi\colon \mathcal{I}(B) \to \mathcal{I}(A)$ such that (Ψ, Φ) build a Galois connection, $\mathrm{i.e.}$, $\Psi(J) \supset I$ iff $J \supset \Phi(I)$. The rule is: The *upper* adjoint is *lower* semi-continuous.

Applications of Theorem 3 to the adjoint Ψ of Φ in Lemma 4 implies the following necessary and sufficient criterion (ii):

NC-Selection and Extensions (6)

Theorem 5. *Let B, E, A, ϵ, π, γ, $\psi_Y \colon S \to \mathcal{I}(Y)$ (for $Y \in \{B, E, A\}$) be as above, and let $\Phi \colon \mathcal{I}(A) \to \mathcal{I}(B)$ the map given in Lemma 4.*

Suppose, in addition, that A is exact *and that B is weakly injective (i.e., has the* WEP *of Lance).*

Then the following properties (i) and (ii) of the extension are equivalent:

(i) The extension has an S-equivariant c.p. splitting map, i.e., there is a c.p. map $V \colon A \to E$ with $\pi \circ V = \mathrm{id}_A$ and $V(\psi_A(s)) \subset \psi_E(s)$ for all $s \in S$.

(ii) The Busby invariant $\gamma \colon A \to \mathrm{Q}(B)$ is nuclear, and,

$$\pi_B(\mathcal{M}(B, \Phi(J))) \supset \gamma(J) \qquad \forall\, J \in \mathcal{I}(A)$$

Coherent Dini spaces (1)

Definition 6. *A map* $f\colon X \to [0,\infty)$ *is a* $\boldsymbol{Dini\ function}$ *if it is lower semi-continuous and* $\sup f(\bigcap_n F_n) = \inf_n\{\sup f(F_n)\}$ *for every decreasing sequence* $F_1 \supset F_2 \supset \cdots$ *of closed subsets of* X*.*

A sober T_0 *space* X *is a* $\boldsymbol{Dini\ space}$ *if the supports of the Dini functions build a base of the topology of* X*.*

The Dini functions f are exactly the functions that satisfy the (generalized) **Dini Lemma**: *Every upward directed net of l.s.c. functions converges* **uniformly** *to* f *if it converges point-wise to* f. If a T_0 space X is sober, then a function $f\colon X \to [0,1]$ is Dini, if and only if, f is lower semi-continuous and the restriction $f\colon X \setminus f^{-1}(0) \to (0,1]_{\mathrm{lsc}}$ is *proper*.

Coherent Dini spaces (2)

The class of Dini spaces X coincides with the class of sober locally compact T_0 spaces with a countable base of its topology.

A subset C of X is **saturated** if $C = \mathrm{Sat}(C)$, where $\mathrm{Sat}(C)$ means the intersection of all $U \in \mathbb{O}(X)$ with $U \supset C$.

Definition 7. *A sober T_0 space X is **coherent** if the intersection $C_1 \cap C_2$ of two* saturated *quasi-compact subsets $C_1, C_2 \subset X$ is again quasi-compact.*

Below, we consider some partial results concerning the open **Question:**
Is every (second-countable) *coherent* Dini space X homeomorphic to the primitive ideal spaces $\mathrm{Prim}(A)$ of some *amenable* C^*-algebra A ?

Let $\mathcal{F}(X)$ denote the lattice of closed subsets $F \subset X$.

Coherent Dini spaces (3)

Definition 8. *The topological space $\mathcal{F}(X)_{\mathrm{lsc}}$ is the set $\mathcal{F}(X)$ with the T_0 **order topology** that is* generated *by the complements*

$$\mathcal{F}(X) \setminus [\emptyset, F] = \{G \in \mathcal{F}(X)\,;\; G \cap U \neq \emptyset\} =: \mu_U$$

of the intervals $[\emptyset, F]$ for all $F \in \mathcal{F}(X)$ (where $U = X \setminus F$).

*The **Fell-Vietoris topology** on $\mathcal{F}(X)$ is the topology, that is* generated *by the sets μ_U ($U \in \mathbb{O}(X)$) and the sets $\mu_C := \{G \in \mathcal{F}(X)\,;\; G \cap C = \emptyset\}$ for all* quasi-compact *$C \subset X$.*

$\mathbb{O}(X) \cong \mathcal{F}(X)^{\mathrm{op}}$ defines the **Larson** topology on $\mathbb{O}(X)$. We denote by $\mathcal{F}(X)_H$ Fell-Vietoris topology.

The space $\mathcal{F}(X)_{\mathrm{lsc}}$ is a *coherent Dini* space, and the space $\mathcal{F}(X)_H$ is a *compact Polish* space.

Coherent Dini spaces (4)

The ordered Hilbert cube $\mathbb{Q}$ is nothing else $\mathcal{F}(Y)$ for $Y := X_0 \uplus X_0 \uplus \cdots$ where $X_0 := (0,1]_{\mathrm{lsc}}$. The Fell-Vietoris topology becomes the usual Hausdorff topology on $\mathbb{Q}$.

If X is locally quasi-compact sober T$_0$ space, then a dense sequence $g_1, g_2, \ldots$ in the Dini functions g on X with $\sup g(X) = 1$ defines an order isomorphism $\iota\colon \mathcal{F} \to \mathbb{Q}$ onto a $\max$-closed subset $\iota(\mathcal{F})$ of $\mathbb{Q}$ with $\iota(\emptyset) = 0$, $\iota(X) = 1$ by

$$\iota(F) := (\sup g_1(F), \sup g_2(F), \ldots) \in \mathbb{Q}\,.$$

The image $\iota(\mathcal{F}(X))$ is closed in $\mathbb{Q}$ (with Hausdorff topology) and ι defines a homeomorphism from $\mathcal{F}(X)$ onto $\iota(\mathcal{F}(X))$ with respect to both topologies on $\mathcal{F}(X)$ and $\mathbb{Q}$.

Coherent Dini spaces (5)

In this way, $X \cong \eta(X) \subset \overline{\eta(X)}^H \backslash \{0\} \subset \mathcal{F}(X) \subset \mathbb{Q}$, considered as Polish spaces, with $X \ni x \mapsto \eta(x) := \overline{\{\, x \,\}} \in \mathcal{F}(X)$.

Theorem 9. *Let X a second countable locally (quasi-)compact sober T_0 space. Following properties (i)-(iv) of X are equivalent:*

(i) X is ***coherent****.*

(ii) The set $\mathcal{D}(X)$ of Dini functions on X is ***convex****.*

(iii) $\mathcal{D}(X)$ is $\min$***-closed****.*

(iv) $\mathcal{D}(X)$ is ***multiplicatively closed****.*

Coherent Dini spaces (6)

It is known that, *X is coherent, if and only if, the image $\eta(X) \cong X$ in $\mathcal{F}(X) \setminus \{\emptyset\}$ is* **closed** *in $\mathcal{F}(X) \setminus \{\emptyset\}$ with respect to the Fell-Vietoris topology on $\mathcal{F}(X)$.*

Lemma 10. *(I) Each closed subset $F \subset \mathbb{Q}_H$ is a coherent locally compact sober subspace F_{lsc} of $\mathbb{Q}_{\text{lsc}}$,* and *is the intersection of an decreasing sequence F_k of closed subspaces of $\mathbb{Q}_H$ that are continuously order-isomorphic to spaces $G_k \times \mathbb{Q}$ with $G_k \subset [0,1]^{n_k}$ a finite union of n_k-dimensional (small) cubes.*

(II) If $F = \bigcap_k F_k$ for a sequence $F_1 \supset F_2 \supset \cdots$ of closed subsets in $\mathbb{Q}_H$, and if each $(F_k)_{\text{lsc}} \subset \mathbb{Q}_{\text{lsc}}$ is the primitive ideal space of an amenable C-algebra, then F_{lsc} is the primitive ideal space of an amenable C*-algebra.*

Coherent Dini spaces (7)

Lemma 10 applies to $F := \eta(\mathcal{F}(X))$ for all Dini spaces X, and to $F := \{0\} \cup \eta(X)$ for all coherent Dini spaces X.

Corollary 11. *If there is a coherent sober l.c. space X that is not homeomorphic to the primitive ideal space of an amenable C*-algebra, then there is $n \in \mathbb{N}$ and a finite union Y of (Hausdorff-closed and small) cubes in $[0,1]^n$ such that Y with induced order-topology is not the primitive ideal space of any amenable C*-algebra.*

Theorem 12. [O.B. Ioffe, E.K.] *If $G \subset [0,1]^n$ is a finite union of (small) cubes, then the space G_{lsc} has a decomposition series $U_1 \subset U_2 \subset \cdots \subset U_k$, by open subsets $U_\ell \subset G_{\mathrm{lsc}}$ such that $U_{\ell+1} \setminus U_\ell$ is the primitive ideal space of an amenable C*-algebra.*

Now combine above results with the following conjecture.

Coherent Dini spaces (8)

Let X a Dini space and $U \subset X$ open.

Conjecture 13. *The space X is homeomorphic to the primitive ideal space of an amenable C*-algebra if U and $X \setminus U$ are homeomorphic to primitive ideal spaces of amenable C*-algebras.*

This Conjecture implies that Dini spaces are primitive ideal spaces of amenable C*-algebras — if they have decomposition series by open subsets $\{U_\alpha\}$ with coherent spaces $U_{\alpha+1} \setminus U_\alpha$.
A Dini space X is the primitive ideal space of an AF-algebra if U and $X \setminus U$ are primitive ideal spaces of AF-algebras.

Proposition 14. *Conjecture 13 reduces, in the case where X is* coherent, *to the case, where $X \setminus U = \{p\}$ is a singleton and $U \cong \mathrm{Prim}(B)$, and where B is an inductive limit of algebras $B_n \cong \mathrm{C}_0(\Gamma_n \setminus \{g_n\}) \otimes M_{k_n}$ for connected pointed graphs (Γ_n, g_n).*

Exotic $\mathbb{R}$-actions (1)

Theorem 15. [N.Ch. Phillips, E.K.] *Suppose that A is an amenable C^*-algebra, G an amenable l.c. group, and that G acts minimally by $\alpha\colon G \to \mathrm{Homeo}(\mathrm{Prim}(A))$ on $\mathrm{Prim}(A)$. Then there exists a continuous group-action $\beta\colon G \to \mathrm{Aut}(B)$ on the C^*-algebra $B := A \otimes \mathcal{O}_2 \otimes \mathbb{K}$ that implements α, and has crossed product $B \rtimes_\beta G \cong \mathcal{O}_2 \otimes \mathbb{K}$.*

A part of the proof is an G-equivariant improvement of Theorem 1. Then the spectra of the actions will be enriched by tensoring (infinitely often if necessary) with the natural action of G on $\mathcal{O}_\infty \cong \mathcal{O}(\mathrm{L}_2(G))$.

Definition 16. [N.C.Phillips compactification] *Let $\Xi(P)$ denote the prime T_0 space $P \cup \{\infty\}$ with topology given by the system of open subsets*

$$\mathbb{O}(\Xi(P)) = \{\emptyset, \Xi(P) \setminus C\,;\; C \subset P,\ \textit{compact in}\, P\,\}\,.$$

Exotic $\mathbb{R}$-actions (2)

Theorem 17. [N.Ch. Phillips, E.K.] *There exists an amenable C*-algebra A with $\mathrm{Prim}(A) \cong \Xi(P)$.*

If we apply the above theorems to $\Xi(G)$, we get:

Corollary 18. *Every non-compact amenable l.c. group G has a* co-action *$\widehat{\beta}$ on $\mathcal{O}_2 \otimes \mathbb{K}$ such that $B := (\mathcal{O}_2 \otimes \mathbb{K}) \rtimes \widehat{G}$ is prime and the (dual) action β of G on B is minimal and toplogically free.*

If $G := \mathbb{R} = \widehat{G}$, there is also an action $\widehat{\beta}$ of $\mathbb{R} = \widehat{\mathbb{R}}$ on $\mathcal{O}_2$ itself with this property.

General extensions (1)

The existence problem for extensions reduces in case of non-coherent X to the case where $U \cong \mathrm{Prim}(B)$ with $B \cong B \otimes \mathcal{O}_2 \otimes \mathbb{K}$ is an inductive limit of algebras $B_n \cong \mathrm{C}_0(\Gamma_n \setminus \{g_n\}) \otimes M_{k_n}$ for connected pointed graphs (Γ_n, g_n), and where $F := X \setminus U$ is homeomorphic to $(0,1]_{\mathrm{lsc}}$.
This is equivalent to the below formulated question:

Given sequences of positive contractions $T_1, T_2, \ldots \in \mathcal{M}(B)_+$ and isometries $V_n \in \mathcal{M}(B)$ with $T_{n+1} = V_n^* T_n V_n$.
Let $\gamma(J) := \lim_n \|T_n + \mathcal{M}(B,J)\|$, and suppose that, for each $J \in \mathcal{I}(B)$ and $n \in \mathbb{N}$, there is $b := b_{n,J} \in B$ such that

$$(\delta_\infty(T_n) - \gamma(J))_+ - \delta_\infty(b) \in \mathcal{M}(B,J)\,,$$

$$\text{i.e.,} \quad \delta_\infty(\mathcal{M}(\pi_J)(T_n) - \gamma(J)_+) \in \delta_\infty(B/J).$$

General extensions (2)

Question 19. *Does there exist a contraction* $S \in \mathcal{M}(B)_+$ *such that*

$$\|\mathcal{M}(\pi_J)(S)\| = \|S + \mathcal{M}(B,J) + B\| = \gamma(J)$$

for each $J \in \mathcal{I}(B)$.

If the answer is positive, then the element $\pi_B(S) \in \mathrm{Q}(B)$ defines the desired Busby invariant of the desired extension.