Classification of $\mathbb{Z}^N$-actions on simple C^*-algebras

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September 22, 2010
Classification of amenable C^*-algebras
Banff International Research Station

Goal

Goal (too ambitious)

Classify (strongly) outer actions of discrete amenable groups on simple classifiable C^*-algebras (Kirchberg algebras, UHF, AF, AT, AH algebras, the Jiang-Su alg. etc.) up to cocycle conjugacy.

Outer actions of finite groups on unital simple C^*-algebras do not always have the Rohlin property. In fact, if an action $\alpha : \Gamma \curvearrowright A$ of a finite group Γ on a unital simple C^*-algebra A has the Rohlin property, then $K_0(A)$ and $K_1(A)$ are completely cohomologically trivial as Γ-modules.

Goal (realistic)

Classify (strongly) outer actions of $\mathbb{Z}^N$ (or "poly-$\mathbb{Z}$" groups) on simple classifiable C^*-algebras up to cocycle conjugacy.

Cocycle conjugacy

Definition

Let $\alpha : \Gamma \curvearrowright A$ be an action of a countable discrete group Γ on a unital C^*-algebra A.
$(u_g)_{g\in\Gamma} \subset U(A)$ is called an α-cocycle if

$$u_g \alpha_g(u_h) = u_{gh} \quad \forall g, h \in \Gamma.$$

Two actions $\alpha, \beta : \Gamma \curvearrowright A$ are said to be cocycle conjugate, if

$$\exists \gamma \in \mathrm{Aut}(A), \quad \exists (u_g)_g \ \alpha\text{-cocycle}$$

$$\mathrm{Ad}\, u_g \circ \alpha_g = \gamma \circ \beta_g \circ \gamma^{-1} \quad \forall g \in \Gamma.$$

Strong cocycle conjugacy further requires

$$\exists v_n \in U(A), \quad \|u_g - v_n \alpha_g(v_n^*)\| \to 0 \quad \forall g \in \Gamma.$$

Strong outerness

An automorphism of the form $\mathrm{Ad}\, u$ is said to be inner.
An action $\alpha : \Gamma \curvearrowright A$ is said to be outer if α_g is not inner for every $g \in \Gamma \setminus \{e\}$.

Let $T(A)$ denote the set of tracial states and let π_τ be the GNS representation by $\tau \in T(A)$.
$\alpha \in \mathrm{Aut}(A)$ is said to be not weakly inner if the extension $\bar{\alpha}$ on $\pi_\tau(A)''$ is not inner for any $\tau \in T(A)^\alpha$, that is, there does not exist a unitary $U \in \pi_\tau(A)''$ such that $\bar{\alpha} = \mathrm{Ad}\, U$.
An action $\alpha : \Gamma \curvearrowright A$ is said to be strongly outer if α_g is not weakly inner for every $g \in \Gamma \setminus \{e\}$.

If $T(A) = \{\tau\}$, then

$$\alpha : \Gamma \curvearrowright A \text{ is strongly outer} \quad \Longleftrightarrow \quad \bar{\alpha} : \Gamma \curvearrowright \pi_\tau(A)'' \text{ is outer.}$$

$\mathbb{Z}$-actions on Kirchberg algebras

Complete classification is known for outer actions of $\mathbb{Z}$ on unital Kirchberg algebras.

Theorem (H. Nakamura 2000)

Let A be a unital Kirchberg algebra and let $\alpha : \mathbb{Z} \curvearrowright A$ be an outer action. Then α has the Rohlin property.

Theorem (H. Nakamura 2000)

Let A be a unital Kirchberg algebra. For two outer actions $\alpha, \beta : \mathbb{Z} \curvearrowright A$, the following are equivalent.

1. $KK(\alpha_1) = KK(\beta_1)$.
2. *α and β are cocycle conjugate via $\gamma \in \mathrm{Aut}(A)$ satisfying $KK(\gamma) = 1$.*

Actions of poly-$\mathbb{Z}$ groups on Kirchberg algebras

We say that a group Γ is poly-$\mathbb{Z}$ if there exists a normal series

$$\{e\} = \Gamma_0 \lhd \Gamma_1 \lhd \Gamma_2 \lhd \cdots \lhd \Gamma_m = \Gamma$$

such that $\Gamma_{i+1}/\Gamma_i \cong \mathbb{Z}$.

Theorem (M. Izumi and M)

Let Γ be a poly-$\mathbb{Z}$ group and let A be either $\mathcal{O}_2$, $\mathcal{O}_\infty$ or $\mathcal{O}_\infty \otimes B$ with B being a UHF algebra of infinite type. Then there exists a unique strong cocycle conjugacy class of outer Γ-actions on A.

Why unique?

- For Γ as above, its classifying space $B\Gamma$ has the homotopy type of a finite CW complex.
- For A as above, the homotopy group $\pi_n(\mathrm{Aut}(A))$ is trivial for every $n \geq 0$ (M. Dadarlat 2007).

$\mathcal{O}_\infty$-splitting

In particular, for every poly $\mathbb{Z}$-group Γ and $A = \mathcal{O}_2, \mathcal{O}_\infty, \mathcal{O}_\infty \otimes B$ as in the previous slide, any outer action $\alpha : \Gamma \curvearrowright A$ is asymptotically representable, i.e. there exist continuous paths of unitaries $(v_g(t))_{g\in\Gamma, t\in[0,\infty)}$ in A such that

$$\|v_g(t)v_h(t) - v_{gh}(t)\| \to 0 \quad \forall g, h \in \Gamma,$$

$$\|\alpha_g(v_h(t)) - v_{ghg^{-1}}(t)\| \to 0 \quad \forall g, h \in \Gamma,$$

$$\|v_g(t)av_g(t)^* - \alpha_g(a)\| \to 0 \quad \forall g \in \Gamma, a \in A.$$

Theorem (M. Izumi and M)

Let Γ be a poly-$\mathbb{Z}$ group and let A be a unital Kirchberg algebra. Let $\alpha : \Gamma \curvearrowright A$ and $\sigma : \Gamma \curvearrowright \mathcal{O}_\infty$ be outer actions. Then (A, α) is strongly cocycle conjugate to $(A \otimes \mathcal{O}_\infty, \alpha \otimes \sigma)$. In particular, α has the Rohlin property.

Poly-$\mathbb{Z}$ groups of rank two

For a Γ-action α, the first classification invariant is $KK(\alpha_g)$. When two actions α and β satisfy $KK(\alpha_g) = KK(\beta_g)$, there exist homotopies $(\sigma_g(t))_{t\in[0,1]}$ in $\mathrm{Aut}(A \otimes \mathbb{K})$ connecting $\alpha_g \otimes \mathrm{id}_{\mathbb{K}}$ and $\beta_g \otimes \mathrm{id}_{\mathbb{K}}$. For each $g, h \in \Gamma$, $(\sigma_g(t) \circ \sigma_h(t) \circ \sigma_{gh}(t)^{-1})_{t\in[0,1]}$ is a loop in $\mathrm{Aut}(A \otimes \mathbb{K})_0$, which gives rise to a cohomology class

$$c(\alpha, \beta) \in H^2(\Gamma, \pi_1(\mathrm{Aut}(A \otimes \mathbb{K})_0)) \cong H^2(\Gamma, KK^1(A, A)).$$

Theorem (M. Izumi and M)

Let Γ be either $\mathbb{Z}^2$ or $\langle a, b \mid bab^{-1} = a^{-1}\rangle$. For outer actions α, β of Γ on a unital Kirchberg algebra A, the following are equivalent.

1. *α and β are cocycle conjugate via $\gamma \in \mathrm{Aut}(A)$ satisfying $KK(\gamma) = 1$.*
2. *$KK(\alpha_g) = KK(\beta_g)$ for all $g \in \Gamma$ and $c(\alpha, \beta) = 0$.*

$\mathbb{Z}$-actions on UHF algebras

Theorem (A. Kishimoto 1995)

Let A be a UHF algebra and let $\alpha : \mathbb{Z} \curvearrowright A$ be a strongly outer action. Then α has the Rohlin property, i.e. for any $m \in \mathbb{N}$, there exist sequences of projections $(e_n)_n, (f_n)_n$ in A such that

$$\sum_{i=0}^{m-1} \alpha^i(e_n) + \sum_{j=0}^{m} \alpha^j(f_n) \to 1,$$

$$[e_n, a] \to 0 \quad \textit{and} \quad [f_n, a] \to 0 \quad \forall a \in A.$$

Theorem (A. Kishimoto 1995)

Let A be a UHF algebra. All strongly outer $\mathbb{Z}$-actions on A are strongly cocycle conjugate to each other.

$\mathbb{Z}^2$-actions on UHF algebras

Theorem (H. Nakamura 1999)

Let A be a UHF algebra and let $\alpha : \mathbb{Z}^2 \curvearrowright A$ be a strongly outer action. Then α has the Rohlin property.

Since $\mathrm{Aut}(A)$ is path-connected, for $\alpha, \beta : \mathbb{Z}^2 \curvearrowright A$, we can define a loop $(\sigma_g(t) \circ \sigma_h(t) \circ \sigma_{gh}(t)^{-1})_{t \in [0,1]}$ in $\mathrm{Aut}(A)$ for each $g, h \in \mathbb{Z}^2$ as in the case of Kirchberg algebras, and obtain a cohomology class

$$c(\alpha, \beta) \in H^2(\mathbb{Z}^2, \pi_1(\mathrm{Aut}(A))) \cong \pi_1(\mathrm{Aut}(A)).$$

Theorem (T. Katsura and M 2008)

Let A be a UHF algebra. Two strongly outer actions $\alpha, \beta : \mathbb{Z}^2 \curvearrowright A$ are strongly cocycle conjugate if and only if $c(\alpha, \beta) = 0$.

The fundamental group $\pi_1(\mathrm{Aut}(A))$ is isomorphic to a (possibly infinite) direct product of finite cyclic groups (K. Thomsen 1987).

$\mathbb{Z}^N$-actions on UHF algebras of infinite type

A UHF algebra A is said to be of infinite type if $A \otimes A \cong A$.

Theorem (M)

Let A be a UHF algebra of infinite type and let $\alpha : \mathbb{Z}^N \curvearrowright A$ be a strongly outer action. Then α has the Rohlin property.

Theorem (M)

Let A be a UHF algebra of infinite type. Then, all strongly outer actions of $\mathbb{Z}^N$ on A are mutually strongly cocycle conjugate to each other.

When A is a UHF algebra of infinite type, it is known that the homotopy group $\pi_n(\mathrm{Aut}(A))$ is trivial for every $n \geq 0$ (K. Thomsen 1987).

$\mathbb{Z}$-actions on AH algebras

We can generalize Kishimoto's results for UHF algebras to certain AH algebras.

Theorem (M 2010)

Let A be a unital simple AH algebra with slow dimension growth, real rank zero and finitely many extremal tracial states. Let $\alpha : \mathbb{Z} \curvearrowright A$ be a strongly outer action. If α_k is approximately inner for some $k \in \mathbb{N}$, then α has the Rohlin property.

Theorem (M 2010)

Let A be a unital simple AH algebra with slow dimension growth and real rank zero. If two actions $\alpha, \beta : \mathbb{Z} \curvearrowright A$ have the Rohlin property and $\alpha_1 \circ \beta_{-1}$ is asymptotically inner, then α and β are cocycle conjugate.

Uniqueness of asymptotically representable actions

Let A be a unital simple AH algebra with real rank zero, slow dimension growth and finitely many extremal tracial states.

Theorem (M)

For an approximately representable action $\alpha : \mathbb{Z}^N \curvearrowright A$, the following are equivalent.

1. *α is strongly outer.*
2. *α has the Rohlin property.*

Theorem (M)

Let $\alpha, \beta : \mathbb{Z}^N \curvearrowright A$ be strongly outer, asymptotically representable actions. Then they are cocycle conjugate.

$\mathcal{Z}$-stability and classification

We denote the Jiang-Su algebra by $\mathcal{Z}$ and the universal UHF algebra by Q.
Let $\mathcal{C}$ be the set of all unital separable simple nuclear $\mathcal{Z}$-stable C^*-algebras A such that $A \otimes Q$ has tracial rank zero.

Theorem (W. Winter, H. Lin and Z. Niu 2008)

For $A, B \in \mathcal{C}$ satisfying the UCT, if there exists an ordered isomorphism $\varphi : K_*(A) \to K_*(B)$, then there exists an isomorphism $\Phi : A \to B$ inducing φ.

A unital simple ASH algebra A is $\mathcal{Z}$-stable if and only if A has slow dimension growth (A. Toms and W. Winter).
If A is a unital simple ASH algebra whose projections separate traces, then $A \otimes B$ has tracial rank zero for any UHF algebra B (W. Winter 2007).

$\mathcal{Z}$-stability of crossed products

Theorem (Y. Sato and M)

Suppose that $A \in \mathcal{C}$ satisfies the UCT and has finitely many extremal tracial states. Let $\alpha : \Gamma \curvearrowright A$ be a strongly outer action. Assume either of the following.

1. $\Gamma = \mathbb{Z}$.
2. $\Gamma = \mathbb{Z}^N$ *and* $T(A)^\alpha = T(A)$.
3. Γ *is a finite group.*

Then (A, α) is strongly cocycle conjugate to $(A \otimes \mathcal{Z}, \alpha \otimes \mathrm{id})$. In particular, $A \rtimes_\alpha \Gamma$ is $\mathcal{Z}$-stable.

Note that α may not have the Rohlin property, because A may not have rich projections. Instead we use the weak Rohlin property, in which projections are replaced with positive elements. By using this property, one can construct a unital embedding of $\mathcal{Z}$ into $(A^\infty \cap A')^\alpha$, which implies the conclusion.

Closedness of $\mathcal{C}$ under taking crossed products

Combining the theorem in the previous slide with the results of N. C. Phillips and A. Kishimoto, we get the following two theorems.

Theorem (Y. Sato and M)

Suppose that $A \in \mathcal{C}$ satisfies the UCT and has finitely many extremal tracial states. Let Γ be a finite group and let $\alpha : \Gamma \curvearrowright A$ be a strongly outer action. Then $A \rtimes_\alpha \Gamma$ belongs to $\mathcal{C}$.

Theorem (Y. Sato and M)

Suppose that $A \in \mathcal{C}$ satisfies the UCT and has a unique tracial state. Let $\alpha : \mathbb{Z} \curvearrowright A$ be a strongly outer action. If α_k induces the identity on $K_*(A) \otimes \mathbb{Q}$ for some $k \in \mathbb{N}$, then $A \rtimes_\alpha \mathbb{Z}$ belongs to $\mathcal{C}$.

$\mathbb{Z}$-actions and $\mathbb{Z}^2$-actions on $\mathcal{Z}$ (1/2)

Theorem (Y. Sato 2010)

All strongly outer $\mathbb{Z}$-actions on $\mathcal{Z}$ are strongly cocycle conjugate to each other.

Theorem (Y. Sato and M)

All strongly outer $\mathbb{Z}^2$-actions on $\mathcal{Z}$ are strongly cocycle conjugate to each other.

We sketch the proof.
Let $\alpha, \beta : \mathbb{Z}^2 \curvearrowright \mathcal{Z}$ be two strongly outer actions.
(1) By the theorem mentioned before, we may replace α, β with $\alpha \otimes \mathrm{id}, \beta \otimes \mathrm{id} : \mathbb{Z}^2 \curvearrowright \mathcal{Z} \otimes \mathcal{Z}$.
(2) We can find an 'almost' α-cocycle $(u_g)_{g \in \mathbb{Z}^2} \subset \mathcal{Z}$ such that $\mathrm{Ad}\, u_g \circ \alpha_g \approx \beta_g$ on a large finite subset of $\mathcal{Z}$.

$\mathbb{Z}$-actions and $\mathbb{Z}^2$-actions on $\mathcal{Z}$ (2/2)

(3) $Z = \{f : [0,1] \to M_{2^\infty} \otimes M_{3^\infty} \mid f(0) \in M_{2^\infty},\ f(1) \in M_{3^\infty}\}$ is a unital subalgebra of $\mathcal{Z}$ (M. Rørdam and W. Winter 2010).
(4) For the $\mathbb{Z}^2$-action $\alpha \otimes \mathrm{id}$ on the UHF algebra $\mathcal{Z} \otimes M_{2^\infty}$, the cohomology vanishing theorem is known. (T. Katsura and M 2008). Hence there exists a unitary $v_0 \in \mathcal{Z} \otimes M_{2^\infty}$ such that $u_g \otimes 1 \approx v_0(\alpha_g \otimes \mathrm{id})(v_0^*)$.
(5) In the same way, we obtain a unitary $v_1 \in \mathcal{Z} \otimes M_{3^\infty}$ such that $u_g \otimes 1 \approx v_1(\alpha_g \otimes \mathrm{id})(v_1^*)$.
(6) By modifying v_0, v_1 a little bit, we can find a path of unitaries $(v_t)_{t\in[0,1]} \subset \mathcal{Z} \otimes M_{2^\infty} \otimes M_{3^\infty}$ connecting v_0 and v_1, without breaking the condition $u_g \otimes 1 \approx v_t(\alpha_g \otimes \mathrm{id})(v_t^*)$.
(7) $v = (v_t)_t$ is regarded as an element of Z. Therefore the 'almost' $(\alpha \otimes \mathrm{id})$-cocycle $(u_g \otimes 1)_{g\in\mathbb{Z}^2}$ is approximated by $(\alpha \otimes \mathrm{id})$-coboundaries in $\mathcal{Z} \otimes Z$.
(8) Evans-Kishimoto intertwining argument completes the proof.

Open problems

- Classify outer actions of $\mathbb{Z}^N$ $(N \geq 3)$ or poly-$\mathbb{Z}$ groups on unital Kirchberg algebras.
- Classify strongly outer actions of $\mathbb{Z}^N$ on a general UHF algebra when $N \geq 3$.
- Show the uniqueness of strongly outer actions of poly-$\mathbb{Z}$ groups on a UHF algebra of infinite type.
- Show the uniqueness of strongly outer $\mathbb{Z}^N$-actions on the Jiang-Su algebra $\mathcal{Z}$ when $N \geq 3$.
- Classify strongly outer $\mathbb{Z}^2$-actions on a unital simple AF algebra (as much as possible).
- Classify strongly outer $\mathbb{Z}$-actions on $A \in \mathcal{C}$ satisfying the UCT (as much as possible).

References

- M. Izumi and H. Matui, *$\mathbb{Z}^2$-actions on Kirchberg algebras*, Adv. Math. 224 (2010), 355–400.
- T. Katsura and H. Matui, *Classification of uniformly outer actions of $\mathbb{Z}^2$ on UHF algebras*, Adv. Math. 218 (2008), 940–968.
- H. Matui, *$\mathbb{Z}$-actions on AH algebras and $\mathbb{Z}^2$-actions on AF algebras*, Comm. Math. Phys. 297 (2010), 529–551.
- H. Matui, *$\mathbb{Z}^N$-actions on UHF algebras of infinite type*, to appear in J. Reine Angew. Math.
- H. Matui and Y. Sato, *$\mathcal{Z}$-stability of crossed products by strongly outer actions*, preprint. arXiv:0912.4804
- H. Nakamura, *Aperiodic automorphisms of nuclear purely infinite simple C^*-algebras*, Ergodic Theory Dynam. Systems 20 (2000), 1749–1765.
- Y. Sato, *The Rohlin property for automorphisms of the Jiang-Su algebra*, J. Funct. Anal. 259 (2010), 453–476.