Towards the classification of outer actions of finite groups on Kirchberg algebras

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20 Sept. 2010

Classification of Amenable C*-Algebras

Banff international Research Station

20–24 September 2010

Introduction

Possible subtitle: Initiating the Elliott classification program for group actions.

This is work in progress. The intended main theorem has not yet been proved.

Caution: Even the results stated have not all been carefully checked. Don't quote them yet!

Rough outline

- Goal, results, and background.
  - The hoped for main theorem.
  - Previous results.
  - Examples.
  - A long term project: Even after this is done, there is a lot more to do.
- What has been done so far, and general description of the ideas.
  - The current intermediate result.
  - How to get from there to the end.
  - How to get to the current result.
  - Why only pointwise outer actions?
- Some further details.
  - The action on $\mathcal{O}_2$ and on $\mathcal{O}_\infty$.
  - Equivariant semiprojectivity.
  - What do we do with equivariant semiprojectivity?
  - Easy facts about equivariant semiprojectivity.
  - Equivariant semiprojectivity for G acting on $C(G)$ and for certain quasifree actions.

The goal

The intended main theorem is as follows. (Some items are described afterwards.)

Conjecture

Let G be a cyclic group of prime order. Let A and B be Kirchberg algebras (purely infinite simple separable nuclear C*-algebras) which are unital and satisfy the Universal Coefficient Theorem. Let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be pointwise outer actions of G which belong to a suitable bootstrap class (defined by Manuel Koehler). Suppose the extended K-theory of α (as defined by Koehler) is isomorphic to that of β. Then α and β are conjugate.

Conjugacy means that there exists an isomorphism $\varphi\colon A \to B$ such that $\beta_g = \varphi \circ \alpha_g \circ \varphi^{-1}$ for all $g \in G$.

Extended K-theory

Conjecture

Let G be a cyclic group of prime order. Let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be pointwise outer actions of G which belong to a suitable bootstrap class. Suppose the extended K-theory of α is isomorphic to that of β. Then α and β are conjugate.

The extended K-theory $EK^G(A)$ consists of three groups:

- $K_*(A)$.
- $K_*^G(A)$. (See the next slide.)
- With C being the mapping cone of the unital embedding of $\mathbb{C}$ in $C(G)$, the group $KK_G^*(C, A)$.

$EK^G(A)$ has additional structure, given by various operations, which must be preserved by isomorphisms.

If the actions are in Koehler's bootstrap class, then the algebras automatically satisfy the UCT.

A brief summary of equivariant K-theory

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a compact group G on a unital C*-algebra A. The ordinary K_0-group of A is made from finitely generated projective modules over A. (If we use right modules, the projection $p \in M_n(A)$ corresponds to the module pA^n.)

In a similar way, the equivariant K_0-group of A, written $K_0^G(A)$, is made from finitely generated projective modules over A which carry a compatible action of G. (It is a bit more complicated than just G-invariant projections in $M_\infty(A)$.)

One generalizes to nonunital algebras in the usual way, and one gets $K_1^G(A)$ by suspending, with the trivial action in the suspension direction.

The Green-Julg Theorem tells us that $K_*^G(A) \cong K_*(C^*(G, A, \alpha))$.

$K_*^G(A)$ is a module over the representation ring $R(G) = K_0^G(\mathbb{C})$, the Grothendieck group made from finite dimensional representations of G.

KK-theory also has an equivariant version. The groups are written $KK_G^*(A, B)$.

Pointwise outer actions

Conjecture

Let G be a cyclic group of prime order. Let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be pointwise outer actions of G which belong to a suitable bootstrap class. Suppose the extended K-theory of α is isomorphic to that of β. Then α and β are conjugate.

The action $\alpha\colon G \to \mathrm{Aut}(A)$ is called pointwise outer if for every $g \in G \setminus \{1\}$, the automorphism α_g is not inner. (Note that an action $\alpha\colon G \to \mathrm{Aut}(A)$ of a finite group may fail to be inner even when α_g is inner for all $g \in G$. There is an example with $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $A = M_2$.)

Previous result: Classification of Rokhlin actions

The following is Theorem 4.2 of M. Izumi, *Finite group actions on C*-algebras with the Rohlin property. II*, Adv. Math. **184**(2004), 119–160.

Theorem

Let A be a unital UCT Kirchberg algebra, and let G be a finite group. Let $\alpha, \beta \colon G \to \mathrm{Aut}(A)$ be actions with the Rokhlin property. Then α is conjugate to β if and only if the actions of G they induce on $K_*(A)$ are equal.

Interpreted as a theorem about conjugacy of dynamical systems, the invariant involved includes A, equivalently, it includes $K_*(A)$ and $[1_A] \in K_0(A)$.

There are severe restrictions on the possible actions of G on $K_*(A)$.

The same result holds if "Kirchberg algebra" is replaced by "simple C*-algebra with tracial rank zero (in the sense of Lin)".

Previous result: Classification of actions of $\mathbb{Z}_2$ on $\mathcal{O}_2$

The following is essentially a restatement of part of Theorem 4.8 of M. Izumi, *Finite group actions on C*-algebras with the Rohlin property. I*, Duke Math. J. **122**(2004), 233–280.

Theorem

Let $\alpha, \beta \colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathcal{O}_2)$ be actions which are pointwise outer but strongly approximately inner. Then α is conjugate to β if and only if $K_*^{G,\alpha}(\mathcal{O}_2) \cong K_*^{G,\beta}(\mathcal{O}_2)$ via an isomorphism which sends $[1]$ to $[1]$.

$K_*(\mathcal{O}_2)$ isn't needed in the invariant, since it is zero.

An action $\alpha \colon G \to \mathrm{Aut}(A)$ of a finite abelian group G on a unital C*-algebra A is *strongly approximately inner* if for all $g \in G$, the automorphism α_g is the pointwise norm limit of inner automorphisms $\mathrm{Ad}(u_n)$ using α-invariant unitaries u_n.

$K_*^{G,\alpha}(A)$ is the equivariant K-theory of A with respect to the group action α.

Examples: Quasifree actions

Let $\rho \colon G \to L(\mathbb{C}^d)$ be a unitary representation of G. Write

$$\rho(g) = \begin{pmatrix} \rho_{1,1}(g) & \cdots & \rho_{1,d}(g) \\ \vdots & \ddots & \vdots \\ \rho_{d,1}(g) & \cdots & \rho_{d,d}(g) \end{pmatrix}$$

for $g \in G$. Then there exists a unique action $\beta^{\rho} \colon G \to \mathrm{Aut}(\mathcal{O}_d)$ such that

$$\beta^{\rho}_g(s_k) = \sum_{j=1}^{d} \rho_{j,k}(g) s_j$$

for $j = 1, 2, \ldots, d$. (This can be checked by a computation.)

Examples:

- For $G = \mathbb{Z}_n$, choose n-th roots of unity $\zeta_1, \zeta_2, \ldots, \zeta_d$ and let a generator of the group multiply s_j by ζ_j.
- Take $d = \mathrm{card}(G)$, and label the generators s_g for $g \in G$. Then define $\beta^G \colon G \to \mathrm{Aut}(\mathcal{O}_d)$ by $\beta^G_g(s_h) = s_{gh}$ for $g, h \in G$.

Examples: Quasifree actions (continued)

$\rho \colon G \to L(\mathbb{C}^d)$ is a unitary representation.

$$\rho(g) = \begin{pmatrix} \rho_{1,1}(g) & \cdots & \rho_{1,d}(g) \\ \vdots & \ddots & \vdots \\ \rho_{d,1}(g) & \cdots & \rho_{d,d}(g) \end{pmatrix} \quad \text{and} \quad \beta^{\rho}_g(s_k) = \sum_{j=1}^{d} \rho_{j,k}(g) s_j.$$

An analogous construction gives actions on $\mathcal{O}_{\infty}$. Although one doesn't need to, here we restrict to unitary representations $\rho \colon G \to L(l^2)$ which are direct sums of representations $\rho_n \colon G \to L(\mathbb{C}^{d(n)})$.

Example: Label the generators of $\mathcal{O}_{\infty}$ as $s_{g,j}$ for $g \in G$ and $j \in \mathbb{Z}_{>0}$. Define $\iota \colon G \to \mathrm{Aut}(\mathcal{O}_{\infty})$ by $\iota_g(s_{h,j}) = s_{gh,j}$ for $g \in G$ and $j \in \mathbb{Z}_{>0}$.

This is the quasifree action coming from the direct sum of infinitely many copies of the regular representation. One can compute its equivariant K-theory, getting $K_0^G(\mathcal{O}_{\infty}) \cong R(G)$ (recall that this is the representation ring of G), with $[1] \mapsto 1$, and $K_1^G(\mathcal{O}_{\infty}) = 0$.

Examples: Quasifree actions (continued)

Example: Label the generators of $\mathcal{O}_\infty$ as $s_{g,j}$ for $g \in G$ and $j \in \mathbb{Z}_{>0}$. Define $\iota \colon G \to \mathrm{Aut}(\mathcal{O}_\infty)$ by $\iota_g(s_{h,j}) = s_{gh,j}$ for $g \in G$ and $j \in \mathbb{Z}_{>0}$.

$K_0^G(\mathcal{O}_\infty) \cong R(G)$ with $[1] \mapsto 1$, and $K_1^G(\mathcal{O}_\infty) = 0$.

In fact, let $\rho \colon G \to L(l^2)$ be a direct sum of representations $\rho_n \colon G \to L(\mathbb{C}^{d(n)})$, and suppose (this is probably unnecessary) that ρ_1 contains the one dimensional trivial representation. Using the action β^ρ, one gets $K_0^G(\mathcal{O}_\infty) \cong R(G)$ with $[1] \mapsto 1$ and $K_1^G(\mathcal{O}_\infty) = 0$, just as with the action ι above.

If ρ is injective then β^ρ is pointwise outer. It probably has the same extended K-theory as ι. Is it conjugate to ι?

Example: The tensor flip

Define $\varphi \colon \mathcal{O}_\infty \otimes \mathcal{O}_\infty \to \mathcal{O}_\infty \otimes \mathcal{O}_\infty$ by $\varphi(a \otimes b) = b \otimes a$ for $a, b \in \mathcal{O}_\infty$. Using $\mathcal{O}_\infty \otimes \mathcal{O}_\infty \cong \mathcal{O}_\infty$, this defines an action of $\mathbb{Z}_2$ on $\mathcal{O}_\infty$, the tensor flip.

Is this action conjugate to the action ι above? (It is equivariantly strongly selfabsorbing. I don't know the equivariant K-theory, but I suspect it is $R(G)$.)

More generally, let S_n be the symmetric group, and let $g \mapsto \sigma_g \colon G \to S_n$ be an injective homomorphism. There is a pointwise outer action $\varphi^\sigma \colon G \to \mathrm{Aut}\big((\mathcal{O}_\infty)^{\otimes n}\big)$ such that

$$\varphi^\sigma_g(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = a_{\sigma_g(1)} \otimes a_{\sigma_g(2)} \otimes \cdots \otimes a_{\sigma_g(n)}$$

for $a_1, a_2, \ldots, a_n \in \mathcal{O}_\infty$.

Is is conjugate to $\iota \colon G \to \mathrm{Aut}(\mathcal{O}_\infty)$?

A long term project

If classification works for actions on Kirchberg algebras, one should also try it for actions on simple nuclear C*-algebras with tracial rank zero in the sense of H. Lin and which satisfy the UCT. (This will be much harder.) My suspicion is that one will need to assume the tracial Rokhlin property, not just pointwise outerness.

Fix $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and let A_θ be the usual irrational rotation algebra, with standard generators u and v. There are actions, coming from $\mathrm{SL}_2(\mathbb{Z})$, of $\mathbb{Z}_2$ (the "flip", $u \mapsto u^*$ and $v \mapsto v^*$), of $\mathbb{Z}_3$, of $\mathbb{Z}_4$ (the "noncommutative Fourier transform"), and of $\mathbb{Z}_6$ on A_θ.

It is known that all the crossed products are AF algebras. It is also known that the flip is a direct limit action: there is a direct system of circle algebras whose direct limit is A_θ, and actions of $\mathbb{Z}_2$ on the algebras in the system whose direct limit is the flip. In particular, the crossed product is the direct limit of the crossed products of the algebras in the system.

A long term project (continued)

The flip action of $\mathbb{Z}_2$ on A_θ is known to be a direct limit action. What about the other actions coming from $\mathrm{SL}_2(\mathbb{Z})$?

Sam Walters has a direct limit action of $\mathbb{Z}_4$ which he considers to be a model for the noncommutative Fourier transform but it is unknown whether the two actions are conjugate. One very long term goal is to prove an equivariant classification theorem that is good enough to prove such a conjugacy, if not with this particular model then with some related model.

This would require both a tracial rank zero version of equivariant classification and a UCT for more general groups.

Easier problems: Restrict the action to $\mathbb{Z}_2$, or tensor with $\mathrm{id}_{\mathcal{O}_\infty}$. If one does both, the resulting action would be covered by our main conjecture.

Methods

Recall the classification conjecture:

Conjecture

Let G be a cyclic group of prime order. Let A and B be unital UCT Kirchberg algebras. Let $\alpha \colon G \to \mathrm{Aut}(A)$ and $\beta \colon G \to \mathrm{Aut}(B)$ be pointwise outer actions of G in Koehler's bootstrap class. Suppose the extended K-theory of α (as defined by Koehler) is isomorphic to that of β. Then α and β are conjugate.

Three basic methods go into the work:

- Reduction to known results in the case in which there is no group.
- Imitating known arguments from the case in which there is no group.
- New arguments.

The current status

We don't have equivariant classification yet. We nearly have:

Conjecture

Let G be a finite group, let $\alpha \colon G \to \mathrm{Aut}(A)$ and $\beta \colon G \to \mathrm{Aut}(B)$ be pointwise outer actions on unital Kirchberg algebras, and let $\gamma \colon G \to \mathrm{Aut}(C)$ be any action on a unital C*-algebra C. Let $t \mapsto \varphi_t$ and $t \mapsto \psi_t$ be full equivariant asymptotic morphisms from A to $K \otimes B \otimes C$. Suppose φ and ψ are homotopic (as equivariant asymptotic morphisms). Then they are equivariantly asymptotically unitarily equivalent.

An asymptotic morphism $t \mapsto \varphi_t$ is equivariant if each φ_t is equivariant, that is, $\beta_g \circ \varphi_t = \varphi_t \circ \alpha_g$ for all $g \in G$ and all $t \in [0, \infty)$.

Equivariant asymptotic unitary equivalence means that there is a continuous path $t \mapsto u_t$ of G-invariant unitaries in B such that $\lim_{t \to \infty} \left(u_t \varphi_t(a) u_t^* - \psi_t(a) \right) = 0$ for all $a \in A$.

We can use the trivial action of G on K. (This is because we assume the action on B is pointwise outer.)

To get the rest of the way

Conjecture

Let G be a finite group, let $\alpha \colon G \to \mathrm{Aut}(A)$ and $\beta \colon G \to \mathrm{Aut}(B)$ be pointwise outer actions on unital Kirchberg algebras, and let $\gamma \colon G \to \mathrm{Aut}(C)$ be any action on a unital C*-algebra. Then homotopic full equivariant asymptotic morphisms from A to $K \otimes B \otimes C$ are equivariantly asymptotically unitarily equivalent.

What is needed to get from here to the goal:

- Show that if $\alpha \colon G \to \mathrm{Aut}(A)$, $\beta \colon G \to \mathrm{Aut}(B)$, and $\gamma \colon G \to \mathrm{Aut}(C)$ are as in the conjecture, then $KK_G^0(A, K \otimes B \otimes C)$ is the set of homotopy classes of full equivariant asymptotic morphisms from A to $K \otimes B \otimes C$.
- An equivariant approximate intertwining argument, to show that for pointwise outer actions on unital Kirchberg algebras, KK_G-equivalence implies equivariant isomorphism.
- The Universal Coefficient Theorem for actions of G.

To get the rest of the way (continued)

What is needed to get from homotopy implies equivariant asymptotic unitary equivalence to the goal:

- Show that if $\alpha \colon G \to \mathrm{Aut}(A)$, $\beta \colon G \to \mathrm{Aut}(B)$, and $\gamma \colon G \to \mathrm{Aut}(C)$ are as in the conjecture, then $KK_G^0(A, K \otimes B \otimes C)$ is the set of homotopy classes of full equivariant asymptotic morphisms from A to $K \otimes B \otimes C$.
  The point is that one does not have to suspend. The equivariant versions of the ingredients I used here in the nonequivariant case are mostly already known.
- An equivariant approximate intertwining argument, to show that for pointwise outer actions on unital Kirchberg algebras, KK_G-equivalence implies equivariant isomorphism.
  This should be standard.
- The Universal Coefficient Theorem for actions of G.
  Koehler has proved a Universal Coefficient Theorem for the case that G is cyclic of prime order. This is the only place we don't allow an arbitrary finite group.

Used to prove the conjecture

We want to show homotopy implies equivariant asymptotic unitary equivalence for suitable equivariant asymptotic morphisms. Here are some things that are used.

- Computation of equivariant K-theory for quasifree actions on Cuntz algebras. (Quasifree actions were defined above.)
- Equivariant semiprojectivity for certain quasifree actions on Cuntz algebras.
- A pointwise outer action of a finite group on a unital Kirchberg algebra has the tracial Rokhlin property.
- Equivariant analogs of Kirchberg's absorption theorems.

Used to prove the conjecture (continued)

Some things used to prove that homotopy implies equivariant asymptotic unitary equivalence for suitable equivariant asymptotic morphisms:

- Computation of equivariant K-theory for quasifree actions on Cuntz algebras.
  The equivariant K-theory is the same as the K-theory of the crossed product. In the cases we care about, this is the same as the K-theory of the fixed point algebra, and, in many of them, the K-theory of the fixed point algebra has been computed by Mann, Raeburn, and Sutherland. Unforunately, Mann, Raeburn, and Sutherland do not compute the module structure over the representation ring $R(G)$, but we need it.
- Equivariant semiprojectivity for certain quasifree actions on Cuntz algebras.
  This requires some new work. See the last part of the talk.

Used to prove the conjecture (continued)

Some things used to prove that homotopy implies equivariant asymptotic unitary equivalence for suitable equivariant asymptotic morphisms:

- A pointwise outer action of a finite group on a unital Kirchberg algebra has the tracial Rokhlin property.
  This follows easily from work of Nakamura. The statement is that if A is a UCT Kirchberg algebra then an action $\alpha \colon G \to \mathrm{Aut}(A)$ is pointwise outer if and only if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are nonzero mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:
  1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
  2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
- Equivariant analogs of Kirchberg's absorption theorems.
  We say a little more about these below.

Why pointwise outer actions?

Here are three main ingredients in the proof of classification without the group. The first two are Kirchberg's absorption theorems; in the nonequivariant case, the third is trivial.

Theorem

Let A be a simple separable unital nuclear C*-algebra. Then $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$.

Theorem

Let A be a Kirchberg algebra. Then $\mathcal{O}_\infty \otimes A \cong A$.

(In fact, there is an isomorphism from A to $\mathcal{O}_\infty \otimes A$ which is asymptotically unitarily equivalent to the map $a \mapsto 1 \otimes a$.)

Theorem

Let A be a purely infinite simple C*-algebra, and let $p \in A$ be a nonzero projection such that $[p] = 0$ in $K_0(A)$. Then there exists a unital homomorphism $\mathcal{O}_2 \to pAp$.

Why pointwise outer actions? (continued)

Three main ingredients for classification without the group:

1. $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$ for A simple separable unital nuclear.
2. $\mathcal{O}_\infty \otimes A \cong A$ for A a Kirchberg algebra.
3. If A is purely infinite and $p \in A$ is a nonzero projection such that $[p] = 0$ in $K_0(A)$, then there is a unital homomorphism $\mathcal{O}_2 \to pAp$.

We want equivariant versions of these. Suppose we allow arbitrary actions. Taking the trivial action on A in (2) forces one to use the trivial action on $\mathcal{O}_\infty$. Taking a nontrivial action on A in (1) forces one to use a nontrivial action on $\mathcal{O}_2$. These choices make (3) impossible when $A = \mathcal{O}_\infty$.

The right condition on the action is pointwise outerness.

It seems plausible that one can deal with more general actions, by including invariants coming from group cohomology and settling for cocycle conjugacy instead of conjugacy. This has been done for actions on II_1 factors, but here is left for a future project.

The actions on $\mathcal{O}_2$ and on $\mathcal{O}_\infty$

Recall Kirchberg's absorption theorem for $\mathcal{O}_2$: $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$ for A simple separable unital nuclear. We need an action $\zeta \colon G \to \mathrm{Aut}(\mathcal{O}_2)$ such that this isomorphism holds equivariantly whenever A is purely infinite simple and the action on A is pointwise outer. By Izumi, there is a unique action (up to conjugacy) of G on $\mathcal{O}_2$ with the Rokhlin property. Since a tensor product action has the Rokhlin property if one factor does, we had better choose this action for ζ.

The equivariant absorption theorem for $\mathcal{O}_2$ follows immediately.

There is no action of G on $\mathcal{O}_\infty$ which has the Rokhlin property. Instead, we use the action $\iota \colon G \to \mathrm{Aut}(\mathcal{O}_\infty)$ above. Recall that it is the quasifree action coming from the direct sum of infinitely many copies of the regular representation.

The equivariant absorption theorem for $\mathcal{O}_\infty$ requires a considerable amount of work, but that is a subject for a different talk.

Equivariant semiprojectivity

To keep down repetition, for a topological group G we define a G-algebra to be a C*-algebra together with a continuous action of G.

Definition

Let G be a topological group, and let (G, A, α) be a unital G-algebra. We say that (G, A, α) is *equivariantly semiprojective* if whenever (G, C, γ) is a G-algebra, $J_0 \subset J_1 \subset \cdots$ are G-invariant ideals in C, $J = \overline{\bigcup_{n=0}^{\infty} J_n}$, and $\varphi \colon A \to C/J$ is a unital equivariant homomorphism, then there exists n and a unital equivariant homomorphism $\psi \colon A \to C/J_n$ such that the composition

$$A \xrightarrow{\psi} C/J_n \longrightarrow C/J$$

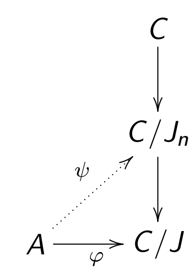
is equal to φ.

When no confusion can arise, we say that A is equivariantly semiprojective, or that α is equivariantly semiprojective.

(Diagram on next slide.)

Equivariant semiprojectivity

(G, A, α) is equivariantly semiprojective if whenever (G, C, γ) is a G-algebra, $J_0 \subset J_1 \subset \cdots$ are G-invariant ideals in C, $J = \overline{\bigcup_{n=0}^{\infty} J_n}$, and $\varphi \colon A \to C/J$ is unital equivariant, then there exists n and a unital equivariant $\psi \colon A \to C/J_n$ such that the following diagram commutes:



Probably equivariant semiprojectivity is only interesting when G is compact, or perhaps even finite.

At least under good conditions, one has the equivariant analog of the usual relation between semiprojectivity and stable relations.

What do we do with equivariant semiprojectivity?

One easy consequence, which we use, is:

Theorem

Let G be a finite group. Let A be a unital Kirchberg algebra, let D be any unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(D)$ be actions of G on A and D. Equip $\mathcal{O}_\infty$ with the action ι above (the quasifree action coming from the direct sum of infinitely many copies of the regular representation). Then any unital equivariant asymptotic morphism from $\mathcal{O}_\infty$ to $A \otimes D$ is asymptotically equal to a continuous path of unital equivariant homomorphisms.

We also need:

Theorem

Let A, α, D, β, and ι be as in the previous theorem, and suppose α is pointwise outer. Then any two unital equivariant asymptotic morphisms from $\mathcal{O}_\infty$ to $A \otimes D$ are equivariantly asymptotically unitarily equivalent.

What do we do with equivariant semiprojectivity?

Theorem

Let A, α, D, β, and ι be as in the previous theorem, and suppose α is pointwise outer. Then any two unital equivariant asymptotic morphisms from $\mathcal{O}_\infty$ to $A \otimes D$ are equivariantly asymptotically unitarily equivalent.

The theorem should seem plausible: the equivariant K-theory of an equivariant homomorphism (or equivariant asymptotic morphism) from $\mathcal{O}_\infty$ is entirely determined by what it does to $[1]$.

The theorem is used to prove a factorization result (getting equivariant asymptotic unitary equivalence rather than equivariant approximate unitary equivalence):

Corollary

Let A be any unital C*-algebra such that $\mathcal{O}_\infty \otimes A \cong A$. Then there exists an isomorphism $\xi\colon \mathcal{O}_\infty \otimes A \to A$ such that the homomorphism $a \mapsto \xi(1 \otimes a)$ is equivariantly asymptotically unitarily equivalent to id_A.

What do we do with equivariant semiprojectivity?

Theorem

Let A, α, D, β, and ι be as in the previous theorem, and suppose α is pointwise outer. Then any two unital equivariant asymptotic morphisms from $\mathcal{O}_\infty$ to $A \otimes D$ are equivariantly asymptotically unitarily equivalent.

Very sketchy outline:

- Reduce to continuous paths of unital equivariant homomorphisms.
- Reduce to the case $[1] = 0$ in $K_0^G(A \otimes D)$. (Requires work.)
- Set $d = \mathrm{card}(G)$. For large m, restrict to $E_{md} \subset \mathcal{O}_\infty$, then extend to $\mathcal{O}_{(m+1)d}$. (The extension uses $[1] = 0$ in $K_0^G(A \otimes D)$.)
- Known results without G give uniqueness for asymptotic morphisms from $\mathcal{O}_{(m+1)d}$ to $C([0,1]) \otimes A \otimes D$.
- The action on $\mathcal{O}_{(m+1)d}$ has the Rokhlin property, so can get equivariant uniqueness from the case without G. (Needs work.)
- Piece together results over $[0,1]$, $[1,2]$, etc. with ever larger m. Equivariant semiprojectivity is needed because they don't fit together exactly.

Easy facts about equivariant semiprojectivity

Lemma

Let G be a compact group, let A be a unital C*-algebra, and let $\iota\colon G \to \mathrm{Aut}(A)$ be the trivial action of G on A. If A is semiprojective, then (G, A, ι) is equivariantly semiprojective.

Proof.

The range of φ is contained in the fixed point algebra $(C/J)^G = C^G/J^G$. Moreover, one checks that $J^G = \overline{\bigcup_{n=0}^{\infty} J_n^G}$.

Now use ordinary semiprojectivity to find n and a lifting of φ to $\psi\colon A \to C^G/J_n^G$. □

Proposition

A finite direct sum of equivariantly semiprojective unital G-algebras is equivariantly semiprojective.

The proof is easy.

G acting on $C(G)$ by translation

Theorem

Let G be a finite group. Let G act on $C(G)$ by the translation action, $\tau_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$. Then $(G, C(G), \tau)$ is equivariantly semiprojective.

(This implies that in the Rokhlin property for an action α of a finite group, the Rokhlin projections e_g can be chosen to satisfy $\alpha_g(e_h) = e_{gh}$ for $g, h \in G$, rather than merely $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for $g, h \in G$.)

We describe the first step:

Lemma

Let G be a finite cyclic group. Let G act on $C(G)$ by the translation action, $\tau_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$. Then $(G, C(G), \tau)$ is equivariantly semiprojective.

G acting on $C(G)$ by translation (continued)

Lemma

Let G be a finite cyclic group. Let $\tau\colon G \to \mathrm{Aut}(C(G))$ be translation. Then $(G, C(G), \tau)$ is equivariantly semiprojective.

Idea of the proof:

Let C, J_n, and J be as before (so G acts on everything and $J = \overline{\bigcup_{n=0}^{\infty} J_n}$), and let $\varphi\colon A \to C/J$ be unital and equivariant.

Take $G = \mathbb{Z}/d\mathbb{Z}$, and identify G with the subgroup

$$\left\{1,\ e^{2\pi i/d},\ e^{4\pi i/d},\ \ldots,\ e^{2(d-1)\pi i/d}\right\}$$

of the circle S^1.

Let u be the inclusion of G in S^1, which we regard as a unitary in $C(G)$. Then u generates $C(G)$ as a C*-algebra, subject to the relations that u is unitary and $u^d = 1$. Moreover, $\tau_\lambda(u) = \lambda^{-1}u$ for $\lambda \in G$.

G acting on $C(G)$ by translation (continued)

$G = \left\{1,\ e^{2\pi i/d},\ e^{4\pi i/d},\ \ldots,\ e^{2(d-1)\pi i/d}\right\}$, and $u \in C(G)$ is $u(\zeta) = \zeta$.

Relations: u is unitary, $u^d = 1$, and $\tau_\lambda(u) = \lambda^{-1}u$ for $\lambda \in G$.

$\varphi\colon A \to C/J$ is unital and equivariant, and $J = \overline{\bigcup_{n=0}^{\infty} J_n}$.

Let $\gamma_n\colon G \to \mathrm{Aut}(C/J_n)$ be the action of G on C/J_n.

Since $C(G)$ is semiprojective without the action, for some n we can lift $\varphi(u)$ to a unitary $v_0 \in C/J_n$ such that $v_0^d = 1$. Increasing n, we may assume that $\|(\gamma_n)_\lambda(v_0) - \lambda^{-1}v_0\|$ is small for $\lambda \in G$.

Now "average" over G: set

$$a = \frac{1}{d}\sum_{\lambda \in G} \lambda \cdot (\gamma_n)_\lambda(v).$$

This element is approximately unitary and satisfies $(\gamma_n)_\lambda(a) = \lambda^{-1}a$ for all $\lambda \in G$. Moreover, its image in C/J is $\varphi(u)$.

G acting on $C(G)$ by translation (continued)

$a \in C/J_n$ is approximately unitary and satisfies $(\gamma_n)_\lambda(a) = \lambda^{-1}a$ for all $\lambda \in G$. Moreover, its image in C/J is $\varphi(u)$.

Set $w = a(a^*a)^{-1/2}$. Its image in C/J is also $\varphi(u)$.

For $\lambda \in G$ we then have $(\gamma_n)_\lambda(a^*) = \lambda a^*$, so $(\gamma_n)_\lambda(a^*a) = a^*a$, and $(\gamma_n)_\lambda\left((a^*a)^{-1/2}\right) = (a^*a)^{-1/2}$, whence $(\gamma_n)_\lambda(w) = \lambda^{-1}w$.

Since

$$\mathrm{sp}(\varphi(u)) \subset G = \left\{1,\ e^{2\pi i/d},\ e^{4\pi i/d},\ \ldots,\ e^{2(d-1)\pi i/d}\right\},$$

we can further increase n and assume that $\mathrm{sp}(w)$ is contained in a small neighborhood U of G.

G acting on $C(G)$ by translation (continued)

$G = \{1, e^{2\pi i/d}, e^{4\pi i/d}, \ldots, e^{2(d-1)\pi i/d}\}$.

We have a unitary $w \in C/J_n$ satisfying:

- The image of w in C/J is $\varphi(u)$.
- $\gamma_\lambda^{(n)}(w) = \lambda^{-1} w$ for all $\lambda \in G$.
- $\mathrm{sp}(w)$ is contained in a small neighborhood U of G.

We can take U to consist of arc the arcs I_k from $e^{2\pi ik/d-\delta}$ to $e^{2\pi ik/d+\delta}$ for $k = 0, 1, \ldots, d$, with $\delta > 0$ so small that they are disjoint. Then there is a continuous function $f \colon S^1 \to S^1$ which is invariant under rotation by $e^{2\pi i/d}$ and such that $f(I_k) = \{e^{2\pi ik/d}\}$ for $k = 0, 1, \ldots, d$. Set $v = f(v_0)$. Then:

- The image of v in C/J is $\varphi(u)$.
- $\gamma_\lambda^{(n)}(v) = \lambda^{-1} v$ for all $\lambda \in G$.
- $\mathrm{sp}(v)$ is contained in G.

There is now a unique equivariant unital homomorphism $\psi \colon C(G) \to C/J_n$ such that $\psi(u) = v$, and this homomorphism lifts φ. So equivariant semiprojectivity of $C(G)$ is proved.

G acting on $C(G)$ by translation (continued)

To get from finite cyclic groups G to arbitrary finite groups, we use an induction argument on the number of generators. The cyclic case is used in the induction step. The notation is somewhat messy. (For example, if H is a subgroup of G and H and g_0 generate G, then one need not be able to find $g \in G$ such that H and g generate G and such that $g_l \in H$ implies $g^l = 1$.)

In any case, without an improvement to the UCT, we need G to be cyclic of prime order in the intended main theorem.

Theorem

Let G be a finite group. Set $d = \mathrm{card}(G)$, and label the generators of $\mathcal{O}_d$ as s_g for $g \in G$. Then the quasifree action $\beta_g(s_h) = s_{gh}$ is equivariantly semiprojective.

Sketch of proof.

Let C, J_n, and J be as before (so G acts on everything and $J = \overline{\bigcup_{n=0}^{\infty} J_n}$), and let $\varphi \colon \mathcal{O}_d \to C/J$ be unital and equivariant.

The elements $s_g s_g^*$ generate a unital copy of $C(G)$ in $\mathcal{O}_d$, on which G acts by translation. Choose n such that one can lift $\varphi|_{C(G)}$ equivariantly to $\psi_0 \colon C(G) \to C/J_n$. Increasing n, we may assume that $\psi_0(s_1 s_1^*)$ is Murray-von Neumann equivalent to 1. That is, there exists $t \in C/J_n$ such that $t^* t = 1$ and $t t^* = \psi_0(s_1 s_1^*)$. Increasing n further and modifying t, we may assume its image in C/J is $\varphi(s_1)$. Set $t_g = (\gamma_n)_g(t)$ for $g \in G$.

Equivariance of ψ_0 implies that $t_g t_g^* = \psi_0(s_g s_g^*)$ for all $g \in G$. Thus $\sum_{g \in G} t_g t_g^* = 1$. We can define an equivariant unital homomorphism $\psi \colon \mathcal{O}_d \to C/J_n$ by $\psi(s_g) = t_g$ for $g \in G$, and ψ lifts φ. □

Equivariant semiprojectivity of further quasifree actions

G is finite with $\mathrm{card}(G) = d$. We showed that the quasifree action of G on $\mathcal{O}_d$ coming from the regular representation is equivariantly semiprojective.

Since $C(G)^m$ is equivariantly semiprojective, the same proof works for the direct sum of m copies of the regular representation.

Since $\mathbb{C}$ is equivariantly semiprojective, so is $C(G)^m \oplus \mathbb{C}$, and we can show that the analogous quasifree action of G on E_{md} is equivariantly semiprojective.

One can now adapt Blackadar's method to get equivariant semiprojectivity of the quasifree action $\iota \colon G \to \mathrm{Aut}(\mathcal{O}_\infty)$ coming from the direct sum of infinitely many copies of the regular representation is equivariantly semiprojective.