On the classification of non-simple real rank zero graph C^*-algebras

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Classification of amenable C^*-algebras
BIRS
23 September 2010

A ***(directed) graph*** $E = (E^0, E^1, r, s)$ consists of a countable set E^0 of vertices, a countable set E^1 of edges, and maps $r, s : E^1 \to E^0$ which is called the ***range*** and ***source*** maps.

A ***Cuntz-Krieger E-family*** is a set of mutually orthogonal projections $\{p_v : v \in E^0\}$ and a set of partial isometries $\{s_e : e \in E^1\}$ with orthogonal ranges satisfying the ***Cuntz-Krieger relations:***

(1) $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$;

(2) $s_e s_e^* \leq p_{s(e)}$ for every $e \in E^1$; and

(3) for every $v \in E^0$ with $0 < |s^{-1}(\{v\})| < \infty$

$$p_v = \sum_{\{e \in E^1 : s(e) = v\}} s_e s_e^*$$

The ***graph*** C^****-algebra*** $C^*(E)$ is defined to be the universal C^*-algebra generated by a Cuntz-Krieger E-family.

Theorem (Raeburn et al)

Let E be a graph satisfying Condition (K), i.e., no vertex of E is the base point of exactly one simple cycle.

(1) *If $\mathfrak{I}$ is an ideal of $C^*(E)$, then there exists a graph E_1 such that $C^*(E_1) \otimes \mathbb{K} \cong \mathfrak{I} \otimes \mathbb{K}$*

(2) *If $\mathfrak{I}$ is an ideal of $C^*(E)$, then there exists a graph E_2 such that $C^*(E_2) \otimes \mathbb{K} \cong C^*(E)/\mathfrak{I} \otimes \mathbb{K}$.*

(3) *If $C^*(E)$ is simple, then $C^*(E)$ either an AF algebra or a purely infinite, nuclear, simple C^*-algebra satisfying the UCT.*

(4) *$C^*(E)$ has real rank zero.*

Goal. Find an algebraic invariant F such that

$$C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K} \iff \mathsf{F}(C^*(E_1)) \cong \mathsf{F}(C^*(E_2))$$

where E_1 and E_2 are graphs satisfying Condition (K).

Conjecture

If $C^*(E_1)$ and $C^*(E_2)$ are graph C^*-algebras satisfying Condition (K). Then

$$C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K}$$

if and only if

$$\mathrm{FK}_X^+(\mathfrak{A}) \cong \mathrm{FK}_Y^+(\mathfrak{B})$$

Goal. Find an algebraic invariant F such that

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where $C^*(E_i)$ has finitely many ideals.

Conjecture

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if and only if

$$\mathrm{FK}^+_X(\mathfrak{A}) \cong \mathrm{FK}^+_Y(\mathfrak{B})$$

Filtrated, ordered K-theory

Let $\mathfrak{A}$ be a C^*-algebra with finitely many ideals. Let $X = \mathrm{Prim}(\mathfrak{A})$. Suppose we have ideals $\mathfrak{I} \trianglelefteq \mathfrak{D} \trianglelefteq \mathfrak{B}$ of $\mathfrak{A}$. Then

$$0 \to \mathfrak{D}/\mathfrak{I} \to \mathfrak{B}/\mathfrak{I} \to \mathfrak{B}/\mathfrak{D} \to 0$$

is a short exact sequence of C^*-algebras. Hence, we get

$$\begin{array}{ccccc}
K_0(\mathfrak{D}/\mathfrak{I}) & \xrightarrow{\iota_*} & K_0(\mathfrak{B}/\mathfrak{I}) & \xrightarrow{\pi_*} & K_0(\mathfrak{B}/\mathfrak{D}) \\
\partial \uparrow & & & & \downarrow \partial \\
K_1(\mathfrak{B}/\mathfrak{D}) & \xleftarrow[\pi_*]{} & K_1(\mathfrak{B}/\mathfrak{I}) & \xleftarrow[\iota_*]{} & K_1(\mathfrak{D}/\mathfrak{I})
\end{array}$$

$\mathrm{FK}_X^+(\mathfrak{A})$ of $\mathfrak{A}$ is the collection of all K-groups, equipped with order on K_0 and the natural transformation $\{\iota_*, \pi_*, \partial\}$.

Simple : $\mathrm{FK}_X^+(\mathfrak{A}) = (K_0(\mathfrak{A}), K_0(\mathfrak{A})_+, K_1(\mathfrak{A}))$

One-ideal: $\mathrm{FK}_X^+(\mathfrak{A})$ is the six-term exact sequence in K-theory

$$\begin{array}{ccccc} K_0(\mathfrak{I}) & \xrightarrow{\iota_*} & K_0(\mathfrak{A}) & \xrightarrow{\pi_*} & K_0(\mathfrak{A}/\mathfrak{I}) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(\mathfrak{A}/\mathfrak{I}) & \xleftarrow[\pi_*]{} & K_1(\mathfrak{A}) & \xleftarrow[\iota_*]{} & K_1(\mathfrak{I}) \end{array}$$

induced by

$$0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0$$

If $\mathfrak{B}$ is a C^*-algebra with $Y = \mathrm{Prim}(\mathfrak{B})$,

$$\mathrm{FK}^+_X(\mathfrak{A}) \cong \mathrm{FK}^+_Y(\mathfrak{B})$$

if there exists a lattice isomorphism $\beta : \mathrm{Lat}(\mathfrak{A}) \to \mathrm{Lat}(\mathfrak{B})$ and for all $\mathfrak{I} \trianglelefteq \mathfrak{D}$ ideals of $\mathfrak{A}$, there exists a group isomorphism

$$\alpha_*^{\mathfrak{D},\mathfrak{I}} : K_*(\mathfrak{D}/\mathfrak{I}) \to K_*(\beta(\mathfrak{D})/\beta(\mathfrak{I}))$$

preserving all natural transformations and order.

Theorem (Restorff)

Let $\mathfrak{A}$ and $\mathfrak{B}$ be Cuntz-Krieger algebras satisfying Condition (II). Then

$$\mathfrak{A} \otimes \mathbb{K} \cong \mathfrak{B} \otimes \mathbb{K}$$

if and only if

$$\mathrm{FK}_X^+(\mathfrak{A}) \cong \mathrm{FK}_Y^+(\mathfrak{B})$$

Theorem (Eilers-Sørensen)

Let E_1 and E_2 be finite graphs and let $\overline{E}_i$ be the amplification of E_i. Then

$$C^*(\overline{E}_1) \otimes \mathbb{K} \cong C^*(\overline{E}_2) \otimes \mathbb{K}$$

if and only if

$$\mathrm{FK}_X^+(C^*(\overline{E}_1)) \cong \mathrm{FK}_Y^+(C^*(\overline{E}_2))$$

$$E_1 : \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_2 : \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overline{E}_1 : \begin{bmatrix} 0 & \infty & 0 & 0 & 0 \\ \infty & 0 & \infty & 0 & 0 \\ 0 & 0 & 0 & \infty & 0 \\ 0 & 0 & 0 & 0 & \infty \\ 0 & 0 & 0 & 0 & \infty \end{bmatrix} \qquad \overline{E}_2 : \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ 0 & 0 & 0 & \infty & \infty \\ 0 & 0 & 0 & 0 & \infty \\ 0 & 0 & 0 & 0 & \infty \end{bmatrix}$$

Simple sub-quotients of $C^*(\overline{E}_i)$: $\mathbb{K}$, $\mathcal{O}_\infty$, $\mathcal{O}_{\infty^2}, \ldots, \mathcal{O}_{\infty^k}$

$$C^*(\overline{E}_1) \otimes \mathbb{K} \cong C^*(\overline{E}_2) \otimes \mathbb{K}$$

Primitive ideal space is finite and linear

If the primitive ideal space of $C^*(E)$ is $X : 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$, then

$$0 \lhd \mathfrak{I}_1 \lhd \mathfrak{I}_2 \lhd \cdots \lhd \mathfrak{I}_n = C^*(E)$$

Theorem (Eilers-Restorff-R)

Let $C^*(E_1)$ and $C^*(E_2)$ be graph C^*-algebras with primitive ideal space X. If the finite and infinite simple sub-quotients are separated. Then

$$C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K} \iff \mathrm{FK}^+_X(C^*(E_1)) \cong \mathrm{FK}^+_X(C^*(E_2))$$

Theorem (Eilers-Restorff-R)

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- Ideas from Mikael Rørdam's classification result of extensions of Kirchberg algebras by Kirchberg algebras
- UCT of Meyer-Nest
- Kirchberg's result of lifting $\mathrm{KK}(X)$-equivalence to a C^*-algebra isomorphism

Example

Let p be a prime number and consider the class of graph C^*-algebras given by adjacency matrices

$$\begin{bmatrix} 0 & 0 & 0 \\ z & p+1 & 0 \\ y & x & p+1 \end{bmatrix}$$

for $y, z > 0$.

$$0 \lhd \mathfrak{I}_1 \lhd \mathfrak{I}_2 \lhd C^*(E)$$

with $\mathfrak{I}_1 = \mathbb{K}$, $\mathfrak{I}_2/\mathfrak{I}_1 = \mathcal{O}_{p+1} \otimes \mathbb{K}$, and $\mathfrak{A}/\mathfrak{I}_2 = \mathcal{O}_{p+1}$.

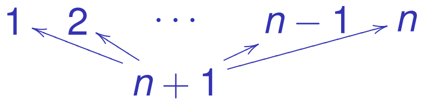
Let E_n be the graph with adjacency matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ n & 3 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$

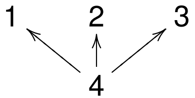
Then

$$C^*(E_n) \otimes \mathbb{K} \cong C^*(E_{n+4}) \otimes \mathbb{K}$$

Primitive ideal space is V :



Suppose $\mathfrak{A} = C^*(E)$ has primitive ideal space



Then we have a smallest ideal $\mathfrak{A}(4)$ of $\mathfrak{A}$ and simple C^*-algebras $\mathfrak{A}(1), \mathfrak{A}(2), \mathfrak{A}(3)$ such that

$$0 \to \mathfrak{A}(4) \to \mathfrak{A} \to \mathfrak{A}(1) \oplus \mathfrak{A}(2) \oplus \mathfrak{A}(3) \to 0$$

is an exact sequence.

Theorem (Eilers-Restorff-R)

Let $C^*(E_1)$ and $C^*(E_2)$ be graph C^*-algebras with primitive ideal space V. Then

$$C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K} \iff \mathrm{FK}_V^+(C^*(E_1)) \cong \mathrm{FK}_V^+(C^*(E_2))$$

for the following cases:

(1) *AF-(combination of PI and AF)*

(2) *PI-(combination of PI and AF with at most 3 PI)*

(2) requires

- UCT of Bentmann for accordion spaces for the case with 2 PI
- (Arklint-Restorff-R) lifting isomorphism of filtrated K-theory to $\mathrm{KK}(X)$-equivalence for the case with 3 PI

Consider graphs given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ x & p+1 & 0 & 0 \\ y & 0 & p+1 & 0 \\ z & 0 & 0 & p+1 \end{bmatrix}$$

$x, y, z > 0$. We get an extension

$$0 \to \mathbb{K} \to C^*(E) \to \mathcal{O}_{p+1} \oplus \mathcal{O}_{p+1} \oplus \mathcal{O}_{p+1} \to 0$$

With graphs E_1 and E_2 given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ x & p+1 & 0 & 0 \\ y & 0 & p+1 & 0 \\ z & 0 & 0 & p+1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ x' & p+1 & 0 & 0 \\ y' & 0 & p+1 & 0 \\ z' & 0 & 0 & p+1 \end{bmatrix}$$

Then

$$C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K}$$

$$\iff$$

$$|\{r \in \{x, y, z\} \, : \, p \mid r\}| = |\{r \in \{x', y', z'\} \, : \, p \mid r\}|$$

Ideas of the proof

Set $\mathfrak{A}_i = C^*(E_i) \otimes \mathbb{K}$. Let $\alpha \ : \ \mathrm{FK}_X^+(\mathfrak{A}_1) \cong \mathrm{FK}_X^+(\mathfrak{A}_2)$.

Primitive ideal space linear with 2 points

Theorem (Rørdam, Eilers-Restorff-R)

There exist invertible elements

$$\beta_0 \in \mathrm{KK}(\mathfrak{I}_1, \mathfrak{I}_2) \quad \textit{and} \quad \beta_2 \in \mathrm{KK}(\mathfrak{A}_1/\mathfrak{I}_1, \mathfrak{A}_2/\mathfrak{I}_2)$$

such that

$$[\mathfrak{e}_1] \times \beta_0 = \beta_2 \times [\mathfrak{e}_2] \quad \textit{in} \quad \mathrm{KK}^1(\mathfrak{A}_1/\mathfrak{I}_1, \mathfrak{I}_2)$$

Ideas of the proof

Set $\mathfrak{A}_i = C^*(E_i) \otimes \mathbb{K}$. Let $\alpha : \mathrm{FK}_X^+(\mathfrak{A}_1) \cong \mathrm{FK}_X^+(\mathfrak{A}_2)$.

For linear primitive ideal space with n points

Theorem (The UCT of Meyer-Nest)

There exists an invertible element $\beta \in \mathrm{KK}(X; \mathfrak{A}_1, \mathfrak{A}_2)$ *such that*

$$\mathrm{KK}(X; \beta) = \alpha$$

Theorem (Eilers-Restorff-R)

There exist invertible elements

$$\beta_0 \in \mathrm{KK}(U; \mathfrak{I}_1, \mathfrak{I}_2) \quad \textit{and} \quad \beta_2 \in \mathrm{KK}(Z; \mathfrak{A}_1/\mathfrak{I}_1, \mathfrak{A}_2/\mathfrak{I}_2)$$

such that

$$[\mathfrak{e}_1] \times \mathrm{KK}(\beta_0) = \mathrm{KK}(\beta_2) \times [\mathfrak{e}_2] \quad \textit{in} \quad \mathrm{KK}^1(\mathfrak{A}_1/\mathfrak{I}_1, \mathfrak{I}_2)$$

Theorem (Kirchberg)

Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable, nuclear, simple, stable C^*-algebras such that $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathcal{O}_\infty$-absorbing and $\mathrm{Prim}(\mathfrak{A}) \cong \mathrm{Prim}(\mathfrak{B}) \cong Y$. If β is an invertible element in $\mathrm{KK}(Y; \mathfrak{A}, \mathfrak{B})$, then there exists an isomorphism φ from $\mathfrak{A}$ to $\mathfrak{B}$ such that $\mathrm{KK}(\varphi) = \beta$.

Theorem (Elliott)

Let $\mathfrak{A}$ and $\mathfrak{B}$ be AF-algebras. If $\alpha \in \mathrm{KK}(\mathfrak{A} \otimes \mathbb{K}, \mathfrak{B} \otimes \mathbb{K})$ such that

$$K_0(\alpha) : (K_0(\mathfrak{A} \otimes \mathbb{K}), K_0(\mathfrak{A} \otimes \mathbb{K})_+) \cong (K_0(\mathfrak{B} \otimes \mathbb{K}), K_0(\mathfrak{B} \otimes \mathbb{K})_+)$$

then there exists an isomorphism φ from $\mathfrak{A} \otimes \mathbb{K}$ to $\mathfrak{B} \otimes \mathbb{K}$ such that $\mathrm{KK}(\varphi) = \alpha$.

Lift $\mathrm{KK}(\beta_0)$ and $\mathrm{KK}(\beta_2)$ to isomorphisms φ_0 and φ_2

$$
\begin{array}{lccccccccc}
\mathfrak{e}_1 & 0 & \longrightarrow & \mathfrak{I}_1 & \longrightarrow & \mathfrak{A}_1 & \longrightarrow & \mathfrak{A}_1/\mathfrak{I}_1 & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \phi_0} & & \downarrow & & \| & & \\
\widetilde{\mathfrak{e}}_1 & 0 & \longrightarrow & \mathfrak{I}_2 & \longrightarrow & \widetilde{\mathfrak{A}}_1 & \longrightarrow & \mathfrak{A}_1/\mathfrak{I}_1 & \longrightarrow & 0
\end{array}
$$

$$
\begin{array}{lccccccccc}
\widetilde{\mathfrak{e}}_2 & 0 & \longrightarrow & \mathfrak{I}_2 & \longrightarrow & \widetilde{\mathfrak{A}}_2 & \longrightarrow & \mathfrak{A}_1/\mathfrak{I}_1 & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow{\scriptstyle \varphi_2} & & \\
\mathfrak{e}_2 & 0 & \longrightarrow & \mathfrak{I}_2 & \longrightarrow & \mathfrak{A}_2 & \longrightarrow & \mathfrak{A}_2/\mathfrak{I}_2 & \longrightarrow & 0
\end{array}
$$

$$[\,\widetilde{\mathfrak{e}}_1] = [\,\widetilde{\mathfrak{e}}_2] \quad \text{in} \quad \mathrm{KK}^1(\mathfrak{A}_1/\mathfrak{I}_1, \mathfrak{I}_2)$$

Theorem (Eilers-Tomforde, Eilers-Restorff-R)

Let $C^*(E)$ be a graph C^*-algebra with linear ideal lattice. If finite and infinite simple sub-quotients are separated and $C^*(E)$ is not an AF algebra, then

$$0 \to \mathfrak{I} \otimes \mathbb{K} \to C^*(E) \otimes \mathbb{K} \to C^*(E)/\mathfrak{I} \otimes \mathbb{K} \to 0$$

is a full extension.

Results of Kucerovsky-Ng on full extensions implies that

$$\widetilde{\mathfrak{A}}_1 \cong \widetilde{\mathfrak{A}}_2 \quad \Longrightarrow \quad \mathfrak{A}_1 \cong \mathfrak{A}_2$$