A characterization of semiprojectivity for commutative C^*-algebras.

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We study the following question:

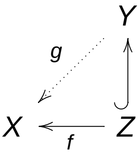
Question 1.1

When is a (separable), commutative C^*-algebra

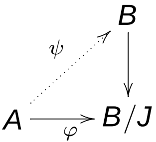
- semiprojective?
- weakly semiprojective?
- projective?
- weakly projective?

Reminder on definitions 1

A space X is an **(approximative) absolute retract**, abbreviated by **(A)AR**, if:
$\forall\, Z \subset Y, f : Z \to X$ (and $\varepsilon > 0$)
$\exists\, g : Z \to X$ such that the following diagram commutes (up to ε):

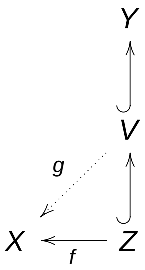


A C^*-algebra A is **(weakly) projective**, abbreviated by **(w-)P**, if:
$\forall$ quotients $B \to B/J$,
$\varphi : A \to B/J$ (and $\varepsilon > 0$ and finite subset $F \subset A$)
$\exists\, \psi : A \to B$ such that the following diagram commutes (up to ε on F):

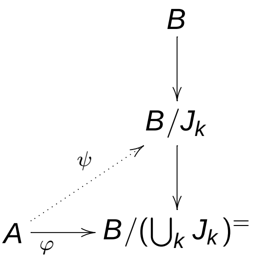


Reminder on definitions 2

X is an **(approximative) absolute neighborhood retract**, abbreviated by **(A)ANR**, if:
$\forall\, Z \subset Y, f : Z \to X$ (and $\varepsilon > 0$)
$\exists$ neighborhood V of Z and $g : V \to X$ such that the following diagram commutes (up to ε):



A is **(weakly) semiprojective**, abbreviated by **(w-)SP**, if:
$\forall\, B$ with increasing sequence of ideals $J_1 \lhd J_2 \lhd \ldots$,
$\varphi : A \to B/(\bigcup_k J_k)^=$ (and $\varepsilon > 0$ and finite subset $F \subset A$)
$\exists\, k$ and $\psi : A \to B/J_k$ such that the following diagram commutes (up to ε on F):



Partial answers

- Loring 1989: X finite graph $\Rightarrow C(X)$ is SP
- Chigogidze, Dranishnikov 2010:
  $C(X)$ is P $\Leftrightarrow X$ is AR & $\dim(X) \leq 1$
- $C(D^2)$ is not w-SP
  moreover: if $D^2 \hookrightarrow X$, then $C(X)$ not w-SP

Our main result is:

Theorem 1.2

Let X be a compact, metric space. TFAE:

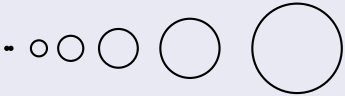
1. $C(X)$ *is SP*
2. X *is ANR*, $\dim(X) \leq 1$

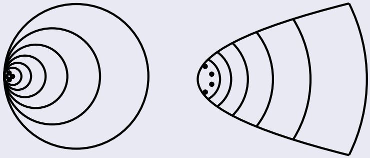
- this was conjectured by Blackadar
- it is a generalization of Lorings result
- it is the analogue of the result of Chigogidze, Dranishnikov

Proving Necessity 1

Lemma 2.1

Let X be a Peano continuum (e.g. an ANR), $\dim(X) \geq 2$*. Then X contains one of the following three spaces:*





If a space X contains one of these spaces, then $C(X)$ is not SP.

Proving Necessity 2

sketch of proof.

- dimension = local dimension, i.e. there exists a point $x_0 \in X$ s.t. $\dim(D) \geq 2$ for every neighborhood D of x_0
- a Peano space with $\dim \geq 2$ admits an emdedding of S^1
- Thus: at one point can embed smaller and smaller circles
- use unsolvable lifting problem shows $C(X)$ is not SP:

$$
\begin{array}{ccc}
 & & (\bigoplus_{\mathbb{N}} \mathcal{T})^+ \\
 & & \downarrow \\
 & & (\bigoplus_{\mathbb{N}} \mathcal{T})^+ / (\mathbb{K} \oplus \ldots_k \oplus \mathbb{K}) \\
 & \nearrow & \downarrow \\
C(X) & \longrightarrow & (\bigoplus_{\mathbb{N}} C(S^1))^+
\end{array}
$$

□

Proving Necessity 3

- parts of the above proof are based on ideas of Chigogidze, Dranishnikov

Remark 2.2

$D^2 \not\hookrightarrow X$ does not imply $\dim(X) \leq 1$

- Bing '51: there exist spaces of arbitrary high dimension that do not contain a disc (not even an arc)
- for CW-complexes $D^2 \not\hookrightarrow X$ does imply $\dim(X) \leq 1$
- Bing, Borsuk '64: there exists a three-dimensional AR which contains no disc

Proving Sufficiency 1

- results about structure of one-dimensional ANR are based on work of Nadler, Meilstrup and others
- results about lifting of generators and relations based on work of Loring, Chigogidze, Dranishnikov and others

Theorem 3.1

Let X be a Peano continuum with $\dim(X) \leq 1$. TFAE:

(1) *X is ANR*

(2) *$\pi_1(X)$ is finitely generated*

(3) *$\exists\, Y \subset X$ finite graph s.t. $\pi_1(Y) \xrightarrow{\cong} \pi_1(X)$*

Proving Sufficiency 2

Theorem 3.2

Let X be an ANR with $\dim(X) \leq 1$*. Then there exist finite graphs graphs* $Y_1 \subset Y_2 \subset \ldots X$ *s.t.*

1. $(\bigcup_k Y_k)^{=} = X$
2. Y_{k+1} *is obtained form* Y_k *by attaching an arc at one point, i.e.* $\overline{Y_{k+1} \setminus Y_k}$ *is an arc with an end point* p_k *such that* $\overline{Y_{k+1} \setminus Y_k} \cap Y_k = \{p_k\}$
3. *there exist natural strong deformation retractions* $r_k : X \to Y_k$

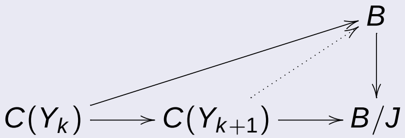
Remark 3.3

Y_1 contains all homotopy information, i.e., $\pi_k(Y_1) \xrightarrow{\cong} \pi_k(X)$ for all k. There exists an minimal such subgraph that is even essentially unique (the "homotopy core").

Proving Sufficiency 3

Lemma 3.4

In the above setting, we can solve the following lifting problem for every B and J $\lhd$ *B (with right triangle commuting, and left triangle commuting up to* $\varepsilon > 0$ *on finite set of generators):*



Proving Sufficiency 4

sketch of Sufficiency.

$$\begin{array}{ccccccc} & & & & & & B/J_k \\ & & & & & \nearrow\nearrow & \downarrow \\ C(Y_1) & \longrightarrow & C(Y_2) & \cdots\cdots\to & C(X) & \longrightarrow & B/(\bigcup_k J_k)^{=} \end{array}$$

- lift $C(Y_1)$ (using Lorings result for finite graphs)
- apply lemma inductively with $\varepsilon = 1/2^k$
- define lift as limit (of Cauchy sequence)

□

Applications

- can solve non-unital case: use the general result that A is SP if and only if its minimal unitalization $\widetilde{A}$ is SP
- can answer essentially all questions about semiprojectivity for commutative C^*-algebras; use it as testcase for conjectures about semiprojectivity (all the conjectures that we checked do hold in the commutative case)

Applications for weak (semi-)projectivity

For a continuum (compact, connected metric space) X consider:

a) for each $\varepsilon > 0$ there exists a map $f : X \to Y \subset X$ such that Y is an AR (an ANR), and $\text{dist}(f(x), x) \leq \varepsilon$ for all $x \in X$

b) X is an AAR (an AANR)

In general: a)$\Rightarrow$ b)

Theorem 4.1

If $\dim(X) \leq 1$, *then a)* $\Leftrightarrow$ *b), and these are even equivalent to:*

c) *for each* $\varepsilon > 0$ *there exists a map* $f : X \to Y \subset X$ *such that* Y *is a finite tree (a finite graph), and* $\text{dist}(f(x), x) \leq \varepsilon$ *for all* $x \in X$

Corollary 4.2

If X *is AA(N)R &* $\dim(X) \leq 1$, *then* $C(X)$ *is w-(S)P.*

Summary

Summarizing our results and the result of Chigogidze, Dranishnikov we get:

Corollary 4.3

Let X be a compact space with $\dim(X) \leq 1$*. Then:*

$C(X)$ is SP	$\Leftrightarrow$	*X is ANR*
$C(X)$ is w-SP	$\Leftrightarrow$	*X is AANR*
$C(X)$ is P	$\Leftrightarrow$	*X is AR*
$C(X)$ is w-P	$\Leftrightarrow$	*X is AAR*

Moreover: If $C(X)$ is SP, then $\dim(X) \leq 1$. However, we do not know whether $C(X)$ w-SP implies $\dim(X) \leq 1$.