Dimension, $\mathcal{Z}$-stability, and classification, of nuclear C^*-algebras

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Consider the following regularity properties for a C*-algebra A.

(A) A is topologically finite-dimensional.

(B) A absorbs a suitable strongly self-absorbing C*-algebra tensorially.

(Γ) A has sufficiently regular homological invariants.

(Γ') The homological invariants of A are algebraically finite-dimensional.

What do these properties mean?
How are they related?
When do they ensure classification?

We will use decomposition rank (Kirchberg–W) in the stably finite case, and nuclear dimension (W–Zacharias) in the general case.

Problem

If $\dim_{\mathrm{nuc}} A < \infty$ and A is (simple and) stably finite, then what is $\mathrm{dr}\, A$?

The Jiang–Su algebra $\mathcal{Z}$ is a finite analogue of $\mathcal{O}_\infty$; it may be described as follows:

- $\mathcal{Z}$ is the uniquely determined initial object in the category of strongly self-absorbing C*-algebras (W, using ideas of Dadarlat–Rørdam; 2009).
- $\mathcal{Z}$ can be written as a stationary inductive limit

  $$\lim_{\rightarrow}(Z_{2^\infty,3^\infty},\alpha),$$

  where

  $$Z_{2^\infty,3^\infty} = \{f \in \mathcal{C}([0,1], M_{2^\infty}\otimes M_{3^\infty}) \mid f(0) \in M_{2^\infty}\otimes\mathbf{1},\ f(1) \in \mathbf{1}\otimes M_{3^\infty}\}$$

  and α is a trace-collapsing endomorphism of $Z_{2^\infty,3^\infty}$ (Rørdam–W; 2008).

$\mathcal{Z}$ was originally constructed as an inductive limit of algebras of the form

$$Z_{p,q} = \{f \in \mathcal{C}([0,1], M_p \otimes M_q) \mid f(0) \in M_p \otimes \mathbf{1},\ f(1) \in \mathbf{1} \otimes M_q\}$$

with p, q relatively prime; the connecting maps are also not so easy to describe.

Problem

Present $\mathcal{Z}$ as a universal C*-algebra with (countably many) generators and relations.

I do have a solution*

*but at this point it's not a very nice one.

Recall that there is a bijection

$$\mathrm{CPC}_{\mathrm{ord0}}(M_q, B) \longleftrightarrow \mathrm{Hom}(\mathcal{C}_0(0,1] \otimes M_q, B).$$

We use this correspondence to define the "universal C*-algebra generated by an order zero map on M_q" by

$$\begin{aligned}&\mathrm{C}^*(\phi^{(q)} \mid \phi^{(q)} \text{ is c.p.c. order zero with domain } M_q)\\ &:= \mathrm{C}^*(e_{ij}^{(q)} \mid e_{ij}^{(q)} \text{ (with } i,j = 1,\ldots,q) \text{ satisfy } \mathcal{R}_{\mathrm{ord0}}^{(q)}).\end{aligned}$$

Here, $\mathcal{R}_{\mathrm{ord0}}^{(q)}$ are the same relations as for $\mathrm{id}_{(0,1]} \otimes e_{ij} \in \mathcal{C}_0((0,1]) \otimes M_q$, when writing the latter as a universal C*-algebra, so

$$\mathrm{C}^*(\phi^{(q)} \mid \phi^{(q)} \text{ is c.p.c. order zero with domain } M_q) \cong \mathcal{C}_0(0,1] \otimes M_q.$$

Next, define

$$Z^{(q)} := \mathrm{C}^*(\varphi^{(q)}, \psi^{(q)} \mid \mathcal{R}^{(q)}),$$

where $\mathcal{R}^{(q)}$ denotes the following set of relations:

- $\varphi^{(q)}$ and $\psi^{(q)}$ are c.p.c. order zero maps with domains M_q and M_2, respectively
- $\psi^{(q)}(e_{11}) = \mathbf{1} - \varphi^{(q)}(\mathbf{1}_{M_q})$
- $\varphi^{(q)}(e_{11})\psi^{(q)}(e_{22}) = \psi^{(q)}(e_{22})\varphi^{(q)}(e_{11}) = \psi^{(q)}(e_{22})$.

It follows from Rørdam–W that in fact

$$Z^{(q)} \cong Z_{q,q+1}.$$

Define $q(k) \in \mathbb{N}$ by setting $q(1) := 2$ and $q(k+1) := q(k)^3$.

Suppose $(\varphi^{(q(k))}, \psi^{(q(k))})$ and $(\varphi^{(q(k+1))}, \psi^{(q(k+1))})$ are pairs of c.p.c. order zero maps satisfying $\mathcal{R}^{(q(k))}$ and $\mathcal{R}^{(q(k+1))}$, respectively.
Define additional relations $\mathcal{S}^{(q(k))}$ by

$$\begin{aligned} \varphi^{(q(k))} &= f(\varphi^{(q(k+1))}) \circ \varrho^{(q(k))}, \\ (\psi^{(q(k))})^{\frac{1}{2}}(e_{12}) &= (\mathbf{1} - f(\varphi^{(q(k+1))})(\mathbf{1}_{q^{(k+1)}}) \\ &\quad + g(\varphi^{(q(k+1))})(\mathbf{1}_{q^{(k+1)}} - \varrho^{(q(k))}(\mathbf{1}_{q(k)})))^{\frac{1}{2}} d(\psi^{(q(k+1))})(e_{12}) \\ &\quad + h(\varphi^{(q(k+1))})(\mathbf{1}_{q^{(k+1)}} - \varrho^{(q(k))}(\mathbf{1}_{q(k)}))^{\frac{1}{2}} f(\varphi^{(q(k+1))})(v), \end{aligned}$$

where $d, f, g, h \in \mathcal{C}([0,1])$ are certain piecewise linear functions, $v \in M_{q(k+1)}$ is a certain partial isometry, and

$$\varrho : M_{q(k)} \to M_{q(k+1)} \cong M_{q(k)} \otimes M_{q(k)} \otimes M_{q(k)}$$

is the c.p.c. order zero map given by

$$(\mathrm{id}_{M_{q(k)}} \otimes \mathbf{1}_{q(k)-1} \otimes \mathbf{1}_{q(k)}) \oplus \bigoplus_{i=1}^{q(k)} \frac{i}{q(k)} \cdot (\mathrm{id}_{M_{q(k)}} \otimes e_{q(k),q(k)} \otimes e_{ii}).$$

We then define

$$\mathcal{Z}_{\mathrm{u}} := \mathrm{C}^*(\varphi^{(q(k))}, \psi^{(q(k))} \mid \mathcal{R}^{(q(k))}, \mathcal{S}^{(q(k))}; k = 1, 2, \ldots).$$

This is a universal C^*-algebra given by countably many generators and relations; the latter are complicated, but explicit.

¿Theorem? (W, Jacelon; 2010)

$\mathcal{Z}_{\mathrm{u}} \cong \mathcal{Z}$.

Next steps:

- Write the relations in a more intuitive manner.
- Prove directly that $\mathcal{Z}_{\mathrm{u}}$ is strongly self-absorbing.
- Handle ‘the’ monotracial stably projectionless example (as studied by Kishimoto–Kumjian, Razak, Dean, Jacelon, Robert) in an analogous manner (to obtain a stably finite version of $\mathcal{O}_2$).

Recall that the Cuntz semigroup

$$W(A) = M_\infty(A)_+/\sim$$

carries an order $\leq$ modeled after Murray–von Neumann subequivalence.

Definition (Rørdam)

$W(A)$ is almost unperforated if, for all $x, y \in W(A)$ and $n, m \in \mathbb{N}$,

$$(nx \leq my \text{ and } n > m) \Longrightarrow x \leq y.$$

Definition

$W(A)$ is almost divisible, if for any $x \in W(A)$ and $n \in \mathbb{N}$, there is $y \in W(A)$ such that

$$ny \leq x \leq (n+1)y.$$

Definition

Let A be separable, simple, unital, $m \in \mathbb{N}$. We say A has

- m-comparison, if for any nonzero positive contractions $a, b_0, \ldots, b_m \in M_\infty(A)$ we have

$$a \precsim b_0 \oplus \ldots \oplus b_m$$

  whenever

$$d_\tau(a) < d_\tau(b_i)$$

  for every $\tau \in QT(A)$ and $i = 0, \ldots, m$.

- strong tracial m-comparison, if for any nonzero positive contractions $a, b \in M_\infty(A)$ we have

$$a \precsim b$$

  whenever

$$d_\tau(a) < \frac{1}{m+1}\tau(b)$$

  for every $\tau \in QT(A)$.

In a similar manner, define

- m-almost divisibility

  and

- tracial m-almost divisibility.

Questions

- How are these notions related?

  In particular, when does m-comparison imply $\tilde{m}$-almost divisibility?

  (Promising results by Dadarlat–Toms.)
- Can these notions help to find range results for the Cuntz semigroup?

Theorem (Robert; 2010)

If $\dim_{\mathrm{nuc}} A \leq m$, then A has m-comparison.

Proposition (W; 2010)

If A is separable, simple, unital, with $\dim_{\mathrm{nuc}} A \leq m$, then A has tracial $\tilde{m}$-almost divisibility and strong tracial $\bar{m}$-comparison for some $\tilde{m}, \bar{m} \in \mathbb{N}$.

Theorem (W; 2010)

Let A be simple, separable, unital, with locally finite nuclear dimension.
If A has strong tracial m-comparison and tracial $\tilde{m}$-almost divisibility for some $m, \tilde{m} \in \mathbb{N}$, then A is $\mathcal{Z}$-stable.

Corollary

Let A be simple, separable, unital, with locally finite nuclear dimension. Then,

$$A \cong A \otimes \mathcal{Z} \Longleftrightarrow W(A) \cong W(A \otimes \mathcal{Z}).$$

Corollary (Using results of Gong, Elliott–Gong–Li, Lin, W)

The class of simple, separable, unital AH algebras with slow dimension growth satisfies the Elliott conjecture.

Corollary

Let A be simple, separable, unital, with finite nuclear dimension.
Then, A is $\mathcal{Z}$-stable.

(This generalizes the earlier result on finite decomposition rank.)

Let us return to

Theorem

Let A be simple, separable, unital, with locally finite nuclear dimension. If A has strong tracial m-comparison and tracial $\tilde{m}$-almost divisibility for some $m, \tilde{m} \in \mathbb{N}$, then A is $\mathcal{Z}$-stable.

1. We need a unital $*$-homomorphism $\mathcal{Z} \to A_\infty \cap A'$; then $A \cong A \otimes \mathcal{Z}$ by an intertwining argument and since $\mathcal{Z}$ is strongly self-absorbing.
2. With
$Z_{p,p+1} = \{f \in \mathcal{C}_0([0,1], M_p \otimes M_{p+1} \mid f(0) \in M_p \otimes \mathbf{1},\ f(1) \in \mathbf{1} \otimes M_{p+1}\}$,
one can write $\mathcal{Z} = \lim Z_{p_l,p_l+1}$, hence an approximately central sequence of unital $*$-homomorphisms
$$Z_{p,p+1} \to A$$
for any $p \in \mathbb{N}$ will do.
3. By Rørdam–W, we need to find a c.p.c. order zero map
$$\Phi : M_p \to A$$
and $v \in A$ such that
$$vv^* = \mathbf{1}_A - \Phi(\mathbf{1}_{M_p}) \text{ and } v^*v \leq \Phi(e_{11})$$
and such that $\Phi(M_p)$ and v are approximately central.

The following is a key result for constructing both Φ and v.

Lemma

For $m, \tilde{m} \in \mathbb{N}$, there is $\alpha_{m,\tilde{m}} > 0$ such that the following holds:
Let A be separable, simple, unital, with tracial $\tilde{m}$-almost divisibility.
Let $\mathbf{1}_A \in B \subset A$ be a C*-subalgebra with $\dim_{\mathrm{nuc}} B \leq m$, and let $k, l \in \mathbb{N}$.
If

$$\varphi : M_l \to A_\infty \cap B'$$

is c.p.c. order zero, then there is a c.p.c. order zero map

$$\psi : M_k \to A_\infty \cap B' \cap \varphi(M_l)'$$

such that

$$\tau(\psi(\mathbf{1}_k)\varphi(\mathbf{1}_l)b) \geq \alpha_{m,\tilde{m}} \cdot \tau(\varphi(\mathbf{1}_l)b)$$

for all $b \in B_+$ and $\tau \in T_\infty(A)$.

This in turn uses careful analysis of the m-decomposable approximations for B, m-comparison of B, and tracial $\tilde{m}$-almost divisibility, together with the following:

Proposition

Let A be separable, simple, unital, with tracial $\tilde{m}$-almost divisibility, and let $d \in A_\infty$ be a positive contraction.
Then, there are orthogonal positive contractions

$$d_0, d_1 \in A_\infty \cap \{d\}'$$

satisfying

$$\tau(d_i f(d)) \geq \frac{1}{4(\tilde{m}+1)} \cdot \tau(f(d)),\ i = 0, 1,$$

for all $\tau \in T_\infty(A)$ and all $f \in \mathcal{C}_0((0,1])_+$.

Remark

The proof does not at any stage involve a dichotomy (stably finite vs. purely infinite).