Some open questions on asymptotics of groups and algebras.

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Harald Bohr Lecture

1. The Burnside Problem.

W. Burnside (1902)

$G = \langle a_1, \ldots, a_m \rangle$, $\exists n \geq 1$:

$\forall g \in G \quad g^n = 1 \overset{?}{\Longrightarrow} |G| < \infty$

This is about one group

$$B(m,n) = F_m / F_m^n,$$

F_m = the free group of rank m.

<u>YES</u>: $n=2$ (obvious), $n=3$ (Burnside), $n=4$ (Sanov), $n=6$ (M. Hall).

Linear Groups (Burnside, Schur)

<u>NO</u>: odd $n \geq 4381$ (Novikov-Adian), even, $n=2^k$, $k \geq 36$ (S. Ivanov, I. Lysenok).

<u>Problem 1.</u> $\exists$? an infinite torsion finitely presented group

<u>The Restricted Version</u>

is about finite groups.

There are finitely many *finite* m-generated groups of exponent $n \Leftrightarrow B_0(m,n) = B(m,n) / \cap \{ H < B(m,n) \,/\, |B(m,n) : H| < \infty \}$ is finite.

P. Hall - G. Higman (56) : reduction to $n = p^k$

A. Kostrikin (59) : $n = p$

E. Z. (89) : $n = p^k$

For some n:

$$\left. m^{m^{\cdot^{\cdot^{\cdot^{m}}}}} \right\} n \leq |B_0(m,n)| \leq \left. m^{m^{\cdot^{\cdot^{\cdot^{m}}}}} \right\} n^{n^n}$$

(M. Vaughan-Lee - E.Z., M. Newman).

<u>Conjecture 2</u>. The class of nilpotency of $B_0(m, p^n)$ is $\leq f_{p,n}(m)$, where $f_{p,n}$ is a polynomial of degree n.

True for $p^n = 3, 4, 5$ (G. Higman).

A group G is <u>residually finite</u> if $\exists\ \varphi_i : G \to G_i$, $|G_i| < \infty$,

$\bigcap\limits_i \ker \varphi_i = (1) \iff \bigcap \{H \mid |G:H| < \infty\} = (1)$

The Restricted Burnside Problem = The Burnside Problem for residually finite groups.

Infinite Groups

Hopelessly Infinite Groups (Geometric Group Theory)

Residually Finite Groups (Number Theory, Combinatorics)

Problem 3 (Ol'shanskii, Rips)

$\exists ?\ N(m,n) : G = \langle a_1, \ldots, a_m \rangle$,

$|G| < \infty \quad \forall a = a_{i_1}^{\pm 1} \cdots a_{i_k}^{\pm 1}$,

$\text{length}(a) = k \leq N(m,n) \quad a^n = 1$

$\Rightarrow G^n = (1)$.

Motivation

G is LEF (locally embeddable into finite groups) if

$\forall$ finite subset $S \subset G$, viewed as a partial group,

$S \hookrightarrow$ a finite group.

R. Grigorchuk: limits of groups

LEF = a (Grigorchuk) limit of finite groups

LEF $\Leftrightarrow G \hookrightarrow$ ultraproduct of finite groups

Residually Finite $\subsetneqq$ LEF

Problem 3 = The Burnside Problem for LEF groups.

A group is <u>hyperbolic</u> if in the Cayley graph geodesic triangles are thin 

<u>Problem 4.</u> $\exists$? a non residually finite hyperbolic group

Positive solution of Problem 3 $\Rightarrow$ "Yes"

<u>Adian - Lysenok</u> : for a large prime p $\quad G = \langle x_1, \dots, x_m \mid w_1^p = 1, \dots, w_k^p = 1 \rangle$ is hyperbolic.

Now, suppose that $N(m,p)$ exists and $\{w_1, \dots, w_k\}$ = all words of length $\leq N(m,p)$. Then G is not residually finite.

C. Martinez (94):

(1) Yes, if G is solvable.
Hence, the Burnside Problem has positive solution for groups that are locally embeddable in finite solvable groups;

(2) reduction to simple groups.

Problem 5. $\exists ?\, N$: $A_n, n \geq 5$, is not generated by 2 elements such that $\forall$ words of length $\leq N$ are of order 5.

2. Primitive Elements.

F_m the free group on $x_1, \dots, x_m$.

Def. $w \in F_m$ is primitive if

$\exists\, \varphi \in \operatorname{Aut} F_m : w = \varphi(x_1)$.

Problem 6 (torsion only for primitive elements)

$\exists ?\ f(m,n)$: G a finite group, $G = \langle a_1, \dots, a_m \rangle$, $m \geqslant 3$, and $w(a_1, \dots, a_m)^n = 1$ for all primitive w $\Rightarrow |G| < f(m,n)$.

There is a big difference between $m = 2$ and $m \geqslant 3$. For $m = 2$ the answer is NO.

Not known even for linear groups.

$G = \langle a_1, \ldots, a_m \rangle < GL(k, \mathbb{C})$, $m \geq 3$.

$W = \{$all primitive elements in $F_m\}$.

Problem 7.

(1) $\forall w \in W \quad w(a_1, \ldots, a_m)^n = 1 \overset{?}{\Longrightarrow} |G| < \infty$;

(2) $\exists\, f(t) \in \mathbb{C}[t] : \forall w \in W$
$f(w(a_1, \ldots, a_m)) = 0 \overset{?}{\Longrightarrow} G$ is virtually nilpotent;

(3) $\forall w \in W \quad (w(a_1, \ldots, a_m) - 1)^k = 0 \overset{?}{\Longrightarrow}$
G is unipotent.

The part (2) would imply: $\forall$ representation $\varphi : \mathrm{Aut}\, F_m \to GL(k, \mathbb{C})$, $m \geq 3$, $\varphi(F_m)$ is virtually nilpotent.

Formanek - Procesi : nonlinearity of

Aut F_m.

Platonov - Potapchik : enough to consider $f(t) = (t^r - 1)^s$.

Doron Puder (recently):

- (with Parzanchevski) measure theoretic characterization of primitive elements $G^m \to G$, $(a_1, \dots, a_m) \to w(a_1, \dots, a_m)$, $|G| < \infty$;
- measure of $W = \{$primitive elements$\}$
- almost all are "obviously" primitive elements.

Pro-P Groups.

p a prime number; G is **residually-p** if $\exists\, \varphi_i : G \to G_i$, G_i are finite p-groups, $\bigcap_i \operatorname{Ker} \varphi_i = (1)$.

Then $\{\operatorname{Ker} \varphi_i\}$ is a basis of neighborhoods of 1. If the topology is complete then

G = an inverse limit of finite p-groups = a pro-p group

If not complete, then $G \hookrightarrow G_{\hat{p}}$, the pro-$p$ completion.

$\mathbb{Z}_p$ = p-adic integers.

EX.1. $GL'(n,\mathbb{Z}_p) = \{ A = (a_{ij})_{n\times n}, a_{ij} \in \mathbb{Z}_p,$
$A \equiv I_n \mod p \}$

M. Lazard (65): $\forall$ p-adic analytic group has an open subgroup, which is embeddable in $GL'(n,\mathbb{Z}_p)$.

EX. 2 R a commutative local Noetherian complete ring ($\mathbb{Z}_p$, $\mathbb{Z}_p[[x_1,\dots,x_m]]$, $GF(p^k)[[x_1,\dots,x_m]]$)

$J \triangleleft R$ max ideal, $R/J \cong GF(p^k)$.

Then $GL'(n,R) = I_n + M_n(J)$ is a pro-p group (congruence subgroup).

$(F_m)_{\hat{p}}$ the free pro-p group.

Let $1 \neq w(x_1, \dots, x_m) \in (F_m)_{\hat{p}}$

Def. $w = 1$ is an identity on a pro-p group G if $\forall a_1, \dots, a_m \in G$ $w(a_1, \dots, a_m) = 1$.

The following two problems are equivalent:

Problem 8. $\exists ?\ 1 \neq w_n \in (F_m)_{\hat{p}}$: $w_n = 1$ is an identity on all $GL'(n, R)$

Problem 9. $(F_m)_{\hat{p}} \xhookrightarrow{?} GL'(n, R)$ for some R

For comparison, let's switch to algebras.

$$S_n(x_1, \ldots, x_n) = \sum_{\sigma \in P_n} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$$

Amitsur-Levitzky (51) : for any commutative ring R

$S_{2n}(x_1, \ldots, x_{2n}) = 0$ is an identity on $M_n(R)$.

Zubkov (89) : $(F_m)_{\hat{p}} \not\hookrightarrow GL'(2,R), \ p > 2$

Theorem $(F_m)_{\hat{p}} \not\hookrightarrow GL'(n,R)$, $p >> n$

Hence $GL'(n,R)$'s satisfy nontrivial identities.

Dream : a theory of pro-p groups satisfying a nontrivial identity (parallel to the theory of PI-algebras).

Conjecture 10. Let G be a just infinite pro-p group satisfying a pro-p identity. Then it is analytic over $\mathbb{Z}_p$ or over $GF(p^k)[[t]]$.

Fontaine - Mazur Conjecture

S finite set of primes, $p \notin S$, $|S| = m > 4$; $K/\mathbb{Q}$ = max pro-p extension unramified outside of S. Then $\forall\ \rho : Gal(K/\mathbb{Q}) \to GL'(n, \mathbb{R})$ the image of ρ is finite.

<u>Thm</u> Gal$(K/\mathbb{Q})$ is not n-linear if $p \gg n$.

Algebras.

F a field, A_F

<u>Problem 11</u> (analog of Problem 1):

$\exists$? an infinite dimensional finitely presented nil ($\forall a \in A$ $a^{n(a)}=0$) algebra.

$$A = A_1 + A_2 + \cdots \quad \text{a graded algebra,}$$

$$A^{(n)} = A_n + A_{2n} + A_{3n} + \cdots$$

Veronese subalgebra.

Backelin: A finitely presented $\Rightarrow$ for a sufficiently large n $A^{(n)}$ is quadratic (presented by relations $\sum \alpha_{ij} x_i x_j = 0$)

Problem 11′ $\exists$? an infinite dimensional quadratic nil algebra

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F a field, $F\langle x_1, \dots, x_m \rangle$ the free associative algebra.

Def. An element $f(x_1, \dots, x_m)$ is primitive if $\exists\, \varphi \in \operatorname{Aut} F\langle x_1, \dots, x_m \rangle$: $f = \varphi(x_1)$.

Problem 12 (analog of Problem 7 (3))

$A = \langle a_1, \ldots, a_m \rangle$, $m \geq 3$, $\forall$ primitive $f(x_1, \ldots, x_m)$ we have $f(a_1, \ldots, a_m)^n = 0 \overset{?}{\Longrightarrow} A$ is nilpotent.

The same for Lie algebras.

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Growth of Algebras.

$A = \langle V \rangle$, $\dim_F V < \infty$,

$V^n = V + \cdots + \underbrace{V \cdots V}_{n}$, $V^1 \subset V^2 \subset \ldots$,

$\cup V^n = A$, $\dim_F V^1 \leq \dim_F V^2 \leq \ldots$

A is of polynomial growth if

$\exists$ a polynomial $p(t)$: $\dim_F V^n \leq p(n)$.

GROMOV THM $\Rightarrow$ a finitely generated torsion group of polynomial growth is finite.

A similar question for algebras: $\exists$? a finitely generated nonnilpotent nil algebra of polynomial growth

T. Lenagan - A. Smoktunowicz :

YES, if F is countable.

New examples of infinite dimensional nil algebras (before this work: only Golod - Shafarevich).

Problem 13. Can we represent Lenagan-Smoktunowicz algebras as limits (in some sense!) of Golod-Shafarevich algebras?

It may help with uncountable fields.

Problem 14. Study self-similar algebras (analogs of Grigorchuk, Gupta-Sidki etc. groups).

- Petrogradsky-Shestakov-Z. :

Lie algebras $\subseteq \mathrm{Der}\, F[t_1, t_2, \dots \mid t_i^p = 0]$, $\mathrm{char}\, F = p > 0$;

- L. Bartholdi : A associative commutative algebra, L Lie algebra

$L \to L \otimes A \rtimes \mathrm{Der}\, A$

- S. Sidki : $A \to M_2(A)$