Weak containment and amenability

joint work with

Alcides Buss and Rufus Willett

Richard Kadison and his mathematical legacy - A memorial conference.

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Motivation

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- G is amenable.
- $1_G \prec \lambda_G$ (:$\Leftrightarrow$ there exists a net of compactly supported positive definite functions $\varphi_i : G \to \mathbb{C}$ which approximate 1_G uniformly on compact subsets of G).
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Weak containment problem (Anantharaman-Delaroche): Suppose $\alpha : G \to \mathrm{Aut}(A)$ is a strongly continuous action of a l.c. group G on a C^*-algebra A. Is it true that

$$A \rtimes_{\max} G = A \rtimes_r G \quad \overset{?}{\Longleftrightarrow} \quad \alpha : G \to \mathrm{Aut}(A) \text{ amenable.}$$

What is the correct definition of an amenable action?

Some history

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If $M = L^\infty(X, \mu)$, this is equivalent to Zimmer's definition.
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Ozawa 2000, Brodzki-Cave-Li 2017. A l.c. group G is exact if and only if: $\exists$ topologically amenable $G \curvearrowright X$ with X compact.

Functions of positive type (Anantharaman-Delaroche)

Let $\alpha : G \to \mathrm{Aut}(A)$ be a strongly continuous action. A continuous function $\theta : G \to A$ is said to be *of positive type*, if for all $g_1, \ldots, g_l \in G$ we have

$$\big(\alpha_{g_i}(\theta(g_i^{-1}g_j))\big)_{i,j} \in M_l(A)^+.$$

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(2) Let $L^2(G,A) = \overline{C_c(G,A)}^{\langle \cdot,\cdot \rangle_A}$ w.r.t $\langle \xi, \eta\rangle_A = \int_G \xi(g)^* \eta(g)\, dg$ and let

$$\lambda^\alpha : G \to \mathrm{Aut}(L^2(G,A)); \lambda_g^\alpha(\xi)(h) = \alpha_g(\xi(g^{-1}h)).$$

Then every compactly supported p.t. function is of the form

$$g \mapsto \langle \xi, \lambda_g^\alpha(\xi)\rangle_A, \quad \text{for some } \xi \in L^2(G,A).$$

The enveloping G-von Neumann algebra

Recall: If G is discrete, $G \curvearrowright A$ is amenable iff $G \curvearrowright A^{**}$ is amenable. Problem: If G is loc. comp. then $G \curvearrowright A^{**}$ can fail to be (ultra) weakly continuous! We need to replace it!

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Defintion. Let $\iota \rtimes U : A \rtimes_{\max} G \to (A \rtimes_{\max} G)^{**}$ be the inclusion. We define $A''_\alpha := \iota(A)'' \subseteq (A \rtimes_{\max} G)^{**}$ and $\alpha'' := \mathrm{Ad}\, U$.

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Universal property: For every nondeg covariant represent. $(\pi, u) : (A, G) \to \mathcal{B}(H_\pi)$ there exists a unique normal $\alpha'' - \operatorname{Ad} u$ equivariant $*$-homomorphism $\pi^{''} : A_\alpha^{''} \twoheadrightarrow \pi(A)^{''}$ extending π.

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Notice: If G is discrete, then $\iota^{**} : A^{**} \to (A \rtimes_{\max} G)^{**}$ is faithful, hence $A''_\alpha = A^{**}$. But this does not hold in general!

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Example: Let $\tau : G \curvearrowright C_0(X)$. Then $C_0(G) \rtimes G \cong \mathcal{K}(L^2(G))$.

Hence $$(C_0(G) \rtimes G)^{**} = \mathcal{K}(L^2(G))^{**} = \mathcal{B}(L^2(G))$$

Thus $C_0(G)''_\tau \cong L^\infty(G) \ncong C_0(G)^{**}$ for G non-discrete.

Amenable actions of locally compact groups

Notation: If $\beta : G \to \mathrm{Aut}(B)$ is a possibly non-continuous action, we write $B_c := \{b \in B : g \mapsto \beta_g(b) \text{ norm continuous}\}$.

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Definition (A.-D., Brown-Ozawa, Buss-E-Willett) $\alpha : G \curvearrowright A$ is

(1) strongly amenable if $\exists$ a net $\phi_i : G \to ZM(A)_c$ of cont. compactly supported p.t. functions s.t. $\|\phi_i(e)\| = 1$ and $\phi_i(g) \to 1_A$ strictly & unif. on compact subsets of G.

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(2) amenable if $\exists$ a net $\phi_i : G \to Z(A_\alpha^{''})_c$ of cont. compactly supported p.t. functions s.t. $\|\phi_i(e)\| = 1$ and $\phi_i(g) \to 1_A$ weakly & unif. on compact subsets of G.

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Remarks: We always have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and if G is discrete, then (2) $\Leftrightarrow$ (3). If G is discrete and $A = C_0(X)$, then (1)$\Leftrightarrow$ (2)$\Leftrightarrow$ (3). (A.-D. 2003). Note that (2)$\not\Rightarrow$(1) (Suzuki 2018).

Amenable actions of locally compact groups

Recall that an l.c. group G is exact iff $(A, \alpha) \mapsto A \rtimes_r G$ preserves short exact sequences.

Theorem (Brodzski-Cave-Li '17) A l. c. group G is exact if and only if $\tau : G \curvearrowright C_{lub}(G)$ is strongly amenable.

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α is amenable $\Longleftrightarrow$ α is von-Neumann amenable.

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Proof of $\Longleftarrow$: Let $L^\infty(G, A''_\alpha) \xrightarrow{P} A''_\alpha$ be a G-projection. It restricts to a projection $L^\infty(G, Z(A''_\alpha)) \xrightarrow{P} Z(A''_\alpha)$. Then consider

$$\Phi : C_{lub}(G) \to L^\infty(G, Z(A''_\alpha)) \xrightarrow{P} Z(A''_\alpha); \quad f \mapsto P(f \otimes 1).$$

By BCL $\exists$ a net $\{\theta_i : G \to C_{lub}(G)\}$ of comp. supp. positive type funct. with $\theta_i(e) = 1_G$ and $\theta_i(g) \to 1_G$ in norm & unif. on comp. in G. Then $(\Phi \circ \theta_i)$ is a net as in the definition of amenability for α. q.e.d.

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Amenable actions of locally compact groups

Proposition (A.-D. '87, BEW '19) Let $G \curvearrowright A$ be a continuous action of the locally compact group G.

(1) If $G \curvearrowright A$ is amenable, then $A \rtimes_{\max} G = A \rtimes_r G$.

(2) If $G \curvearrowright A$ is amenable, then so is $\alpha \otimes \beta : G \curvearrowright A \otimes_{\max} B$ for every $\beta : G \curvearrowright B$.

(3) If $G \curvearrowright A$ is amenable, then A nuclear $\Longrightarrow A \rtimes_r G$ nuclear. If G is discrete, the converse holds as well.

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Proof: (2) Univ. property of (A_α'', α'') implies: $\exists$ a G-hom. $\Phi : A_\alpha'' \to (A \otimes_{\max} B)_{\alpha\otimes\beta}''$ mapping $Z(A_\alpha'')$ into $Z((A \otimes_{\max} B)_{\alpha\otimes\beta}'')$. Thus, if $\{\theta_i : G \to Z(A_\alpha'')_c\}$ implements amenability of α, then $(\Phi \circ \theta_i)$ implements amenability for $\alpha \otimes \beta$.

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(3) Is an easy consequence of (1) and (2):

$$(A \rtimes_r G) \otimes_{\max} B = (A \rtimes_{\max} G) \otimes_{\max} B = (A \otimes_{\max} B) \rtimes_{\max} G$$

$$= (A \otimes_{\min} B) \rtimes_r G = (A \rtimes_r G) \otimes_{\min} B \quad \text{q.e.d.}$$

Matsumura's theorem

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Theorem (Matsumura 2014) Suppose G is discrete and exact and A is unital.

- $A = C(X)$, then $A \rtimes_{\max} G = A \rtimes_r G \iff G \curvearrowright A$ amenable
- If A is nuclear, then $(A \otimes A^{op}) \rtimes_{\max} G = (A \otimes A^{op}) \rtimes_r G \Leftrightarrow G \curvearrowright A$ is amenable.

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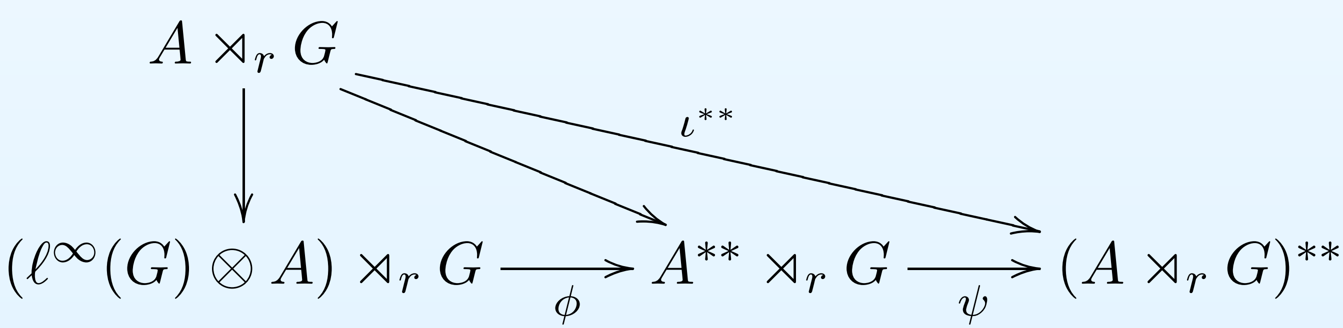
Idea of proof: Matsumura constructs a commutative diagram



This implies that $\iota^{**} : A \rtimes_r G \to (A \rtimes_r G)^{**}$ is nuclear!

Matsumura's theorem

Question: What is the role of exactness?

If $G \curvearrowright X$ is an action with G is discrete and exact and X compact, then Matsumura's result shows that

$$X \rtimes G \text{ amenable} \iff C^*(X \rtimes G) = C_r(X \rtimes G).$$

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Theorem (Willett '15) There exist (non-exact) étale groupoids $\mathcal{G}$ (with compact base $X = \mathcal{G}_0$) such that $C^*_{\max}(\mathcal{G}) = C^*_r(\mathcal{G})$ but $\mathcal{G}$ is not amenable!

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In what follows we want to show that the answer is positive!

The injective crossed product

Definition For a G-algebra A the *injective crossed product* is defined as

$$A \rtimes_{\mathrm{inj}} G := \overline{C_c(G, A)}^{\|\cdot\|_{\mathrm{inj}}}$$

$$\|f\|_{\mathrm{inj}} = \inf\{\|\phi \circ f\|_{B \rtimes_{\max} G} \mid \phi : A \hookrightarrow B \text{ a } G\text{-embedding.}\}.$$

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Theorem (Buss-E-Willett, 2018) $(A, \alpha) \mapsto A \rtimes_{\mathrm{inj}} G$ is the largest (exotic) crossed-product functor which is injective in the sense

$$\phi : A \hookrightarrow B \quad (G\text{-embedding}) \quad \Rightarrow \quad \phi \rtimes G : A \rtimes_{\mathrm{inj}} G \hookrightarrow B \rtimes_{\mathrm{inj}} G.$$

Moreover, if G is exact, then $A \rtimes_{\mathrm{inj}} G = A \rtimes_r G$.

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$$A \rtimes_{\max} G = A \rtimes_{\text{inj}} G \iff \left[A \hookrightarrow B \Rightarrow A \rtimes_{\max} G \hookrightarrow B \rtimes_{\max} G\right].$$

Injective covariant representations

Definition A covariant rep. $(\pi, u) : (A, G) \to \mathcal{B}(H)$ is injective if:

$$\forall \phi : A \hookrightarrow B \text{ (G-hom.) } \exists \left\{ \begin{array}{c} \text{ccp map } \sigma : B \to \mathcal{B}(H) \text{ s.t. } \sigma \circ \phi = \pi \\ \text{and } (\sigma, u) \text{ is covariant for } (B, G). \end{array} \right\}$$

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Theorem (BEW) Let A be a G-algebra. TFAE:

(a) $A \rtimes_{\max} G = A \rtimes_{\mathrm{inj}} G$.

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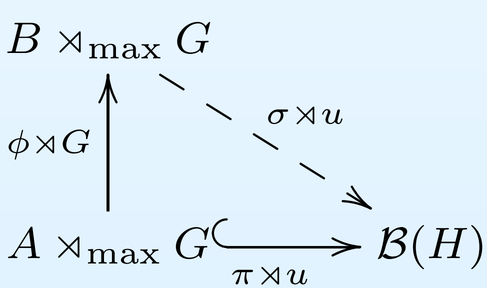

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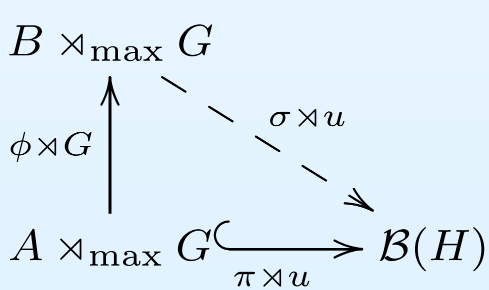
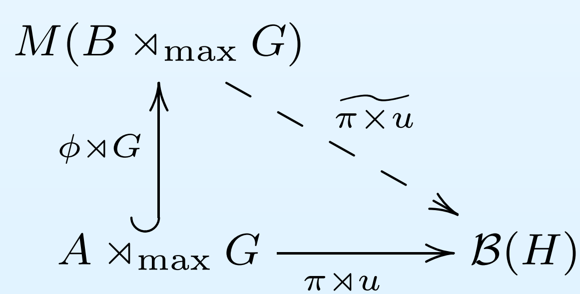


$(a) \Rightarrow (b)$



$\sigma := \widetilde{\pi \rtimes u}|_B$

Injective covariant representations

Lemma. Let $(\pi, u) : (A, G) \to \mathcal{B}(H)$ be injective with π nondeg..
Let C be any unital G-algebra.

Then there exists a ucp map

$$\phi : C \to \pi(A)' \subseteq \mathcal{B}(H) \quad \text{s.t. } (\phi, u) \text{ is covariant for } (C, G).$$

This applies in particular to $C = C_{lub}(G)$.

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$Proof$: Consider $\iota_A, \iota_C : A, C \hookrightarrow M(C \otimes A)_c$.

Injectivity of (π, u) implies:

$\exists$ ucp map $\sigma : M(C \otimes A)_c \to \mathcal{B}(H)$ s.t. $\sigma \circ \iota_A = \pi$ and (σ, u) covariant.

Put $\phi = \sigma \circ \iota_C$. Then (ϕ, u) is covariant and $\phi(C) \subseteq \pi(A)'$

(notice that $\iota_A(A)$ lies in the multiplicative domain of σ!)
q.e.d.

Commutant amenability

Definition $(\pi, u) : (A, G) \to \mathcal{B}(H)$ is **commutant amenable** if there exists a net of compactly supported continuous positive type functions $\theta_i : G \to (\pi(A)')_c$ (with resp. to $\beta = \mathrm{Ad}\, u$) such that

(i) $\theta_i(e) = 1$, and

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Remark If (π, u) is a nondeg. covariant rep. there exists $\pi'' : A''_\alpha \twoheadrightarrow \pi(A)''$. Then $\pi''\big(Z(A''_\alpha)\big) \subseteq Z(\pi(A)'') \subseteq \pi(A)'$.
Thus: $G \curvearrowright A$ amenable $\Rightarrow$ $G \curvearrowright A$ commutant amenable.

Commutant amenability

Theorem (BEW '19) If $\alpha : G \curvearrowright A$ is commutant amenable, then

$$A \rtimes_{\max} G = A \rtimes_r G.$$

Moreover, if G is exact, the converse holds as well!

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Proof "$\Rightarrow$": Use same ideas as of A.-D. for amenable actions.

"$\Leftarrow$" Assume that G is exact. If $A \rtimes_{\max} G = A \rtimes_r G$, then every nondeg. cov. rep. (π, u) is injective. Thus there exists a G-ucp map

$$\phi : C_{lub}(G) \to \pi(A)'.$$

Since G is exact there exists a net $\{\theta_i : G \to C_{lub}(G)\}$ of compactly sup. pos. type functions approximating 1_X. Then

$\{\theta_i = \phi \circ \eta_i : G \to \pi(A)'\}$ implements C-amenability of (π, U).

q.e.d.

Matsumura's theorem revisited

Theorem (BEW '19) Let $\alpha : G \curvearrowright A$ be an action with G exact. Then the following are equivalent:

(1) $G \curvearrowright A$ is amenable (von-Neumann amenable).

(2) $G \curvearrowright A \otimes_{\max} A^{op}$ is amenable.

(3) $G \curvearrowright A \otimes_{\max} A^{op}$ is commutant amenable.

(4) $(A \otimes_{\max} A^{op}) \rtimes_{\max} G = (A \otimes_{\max} A^{op}) \rtimes_{\max} G$.

If, moreover, $A = C_0(X)$ is abelian, then the above conditions are equivalent to

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Proof. Already know (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) and (1) $\Rightarrow$ (5) $\Leftrightarrow$ (6).

Thus it suffices to show (3) $\Rightarrow$ (1) and (5) $\Rightarrow$ (1) if $A = C_0(X)$.

Haagerup's standard form

Theorem (Haagerup '75) Let $\alpha : G \curvearrowright A$ be an action. Then there exist normal, unital, and faithful reps.

$$\pi : A_\alpha'' \to \mathcal{B}(H), \quad \pi^{op} : (A^{op})_{\alpha^{op}}'' \to \mathcal{B}(H)$$

and a strongly cont. unitary rep $u : G \to U(H)$ such that

(i) (π, u) and (π^{op}, u) are covariant;

(ii) $\pi(A)' = \pi^{op}((A_\alpha^{op})'') \cong (A_{\alpha^{op}}^{op})''$, $\quad \pi^{op}(A^{op})' = \pi(A_\alpha'') \cong A_\alpha''$;

(iii) if A is commutative, then $\pi(A)' \cong A_\alpha'' = Z(A_\alpha'')$.

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Corollary (a) $(\pi \otimes \pi^{op}, u)$ is a cov. rep. of $G \curvearrowright A \otimes_{\max} A^{op}$ such that $\pi \otimes \pi^{op}(A \otimes_{\max} A^{op})' \cong Z(A_\alpha'')$. Thus, if $G \curvearrowright A \otimes_{\max} A^{op}$ is commutant amenable, then $G \curvearrowright A$ is amenable.

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Theorem (Haagerup '75) Let $\alpha : G \curvearrowright A$ be an action. Then there exist normal, unital, and faithful reps.

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and a strongly cont. unitary rep $u : G \to U(H)$ such that

(i) (π, u) and (π^{op}, u) are covariant;

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(b) If A is commutative then (iii) implies:

$G \curvearrowright A$ commutant amenable $\Longrightarrow G \curvearrowright A$ amenable.

Measure-wise amenability

Recall that an action $G \curvearrowright (X, \mu)$ is amenable if there exists a G-equivariant cond. exp. $P : L^\infty(G \times X) \to L^\infty(X)$.

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Definition (Renault '80) Let $G \curvearrowright X$ be an action of the second countable l.c.. group G on the second countable l.c. space X.

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Theorem (BEW '19) Let $G \curvearrowright X$ be as above with G exact. Then the following are equivalent:

(1) $G \curvearrowright C_0(X)$ is amenable (resp. commutant amenable).

(2) $G \curvearrowright X$ is measure-wise amenable.

(3) $C_0(X) \rtimes_{\max} G \cong C_0(X) \rtimes_r G$.

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For this let μ be a quasi-invariant measure on X. Let (M, u) be the cov. rep. of $(C_0(X), G)$ on $L^2(X, \mu)$ given by

- $M : C_0(X) \to \mathcal{B}(L^2(X, \mu))$ by multiplication operators
- $\big(u_g\xi\big)(x) = \left(\frac{d\mu(g^{-1}x)}{d\mu(x)}\right)^{1/2}\xi(g^{-1}x)$ (Koopman repr.)

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Following A.-D. these allow the construction of approximately G-invariant projections $P_n : L^\infty(G \times X) \to L^\infty(X)$ and a suitable compactness argument then gives the desired G-projection $P : L^\infty(G \times X) \to L^\infty(X)$. q.e.d.

Resumé

We extended the notion of amenable actions $G \curvearrowright A$ due to Anantharaman-Delaroche for discrete groups G to the case of actions of locally compact groups and we introduced the new notion of commutant amenable actions.

For $A = C_0(X)$ and G exact we show equivalence of

- $G \curvearrowright C_0(X)$ is amenable.
- $G \curvearrowright C_0(X)$ is commutant amenable.
- $G \curvearrowright X$ is measure-wise amenable.
- $C_0(X) \rtimes_{\max} G = C_0(X) \rtimes_r G$.

extending an earlier result of Matsumura for G discrete and X compact.

Thank you for your attention!