Linear Algebra Test
November 30, 2019

(1) For $k \geq 2$ let

$$A_k = \frac{1}{k}\begin{pmatrix} 1 & 1 & \dots 1 \\ 1 & 1 & \dots 1 \\ \dots & & \\ 1 & 1 & \dots 1 \end{pmatrix}$$

(a $k \times k$ matrix, rank 1 projection).
Is there $m \in \mathbb{N}$ such that for every k there exists $k \times k$ matrix B_k that satisfies

1. $\|A_k - B_k\| < \frac{1}{3}$,[1]
2. in every row B_k has at most m nonzero entries, and
3. in every column B_k has at most m nonzero entries.

[1] This is the operator norm.

Operator algebras then and now (a biased selection)

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Rigidity for uniform Roe algebras

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Large Scale Geometry and Coarse Equivalence

If (X, d) is a metric space, the *coarse structure* $\mathcal{E}_d$ on X is the set of all $E \subseteq X^2$ such that

$$\sup_{(x,x') \in E} d(x, x') < \infty$$

Definition

An (abstract) *coarse structure* on a set X is $\mathcal{E} \subseteq \mathcal{P}(X^2)$ such that

1. The diagonal Δ_X is in $\mathcal{E}$.
2. $E \in \mathcal{E}$ and $F \in \mathcal{E}$ implies $E \cup F \in \mathcal{E}$ and $E \circ F \in \mathcal{E}$.
3. $E \in \mathcal{E}$ and $F \subseteq E$ implies $F \in \mathcal{E}$.

The sets in $\mathcal{E}$ are said to be *controlled* (or *entourages*).

A coarse space $(X, \mathcal{E})$ is *connected* if $A^2 \in \mathcal{E}$ for all $A \Subset X$.

Lemma

A coarse space $(X, \mathcal{E})$ is metrizable iff it is countably generated and connected.

Basic Definitions

Given $(X, \mathcal{E})$ and $(Y, \mathcal{F})$, some $f\colon X \to Y$ is

$$\begin{aligned} \text{coarse if} \quad & (f \times f)[\mathcal{E}] \subseteq \mathcal{F} \\ \text{expanding if} \quad & (f \times f)^{-1}[\mathcal{F}] \subseteq \mathcal{E}, \\ \text{coarse embedding if} \quad & \text{both coarse and expanding.} \end{aligned}$$

We say X and Y are *coarsely equivalent*, $X \sim Y$, if there are a coarse $f\colon X \to Y$ and a coarse $g\colon Y \to X$ such that both $\{(x, g(f(x)) \mid x \in X\}$ and $\{(y, f(g(y)) \mid y \in Y\}$ are controlled.

Example

$\mathbb{R} \sim \mathbb{Z}$. $\mathbb{C} \sim \mathbb{Z}^2$.

Definition

A coarse space $(X, \mathcal{E})$ is *uniformly locally finite (u.l.f.)*, or has *bounded geometry*, if $(\forall E \in \mathcal{E})$:

$$\sup_{x \in X} |\{y \mid (x, y) \in E \text{ or } (y, x) \in E\}| < \infty$$

Example

Some u.l.f. metric spaces:

1. k-regular graphs, with path distance, for $k \in \mathbb{N}$.
2. Finitely generated groups (Cayley graph).
3. If G is a group and $S \subseteq G$ is a generating set, then the sets

$$E_{P,n} := \prod_{j \leq n} (P \cup P^{-1})$$

   for $P \Subset S$ and $n \in \mathbb{N}$ generate a coarse structure on G.
4. $\mathcal{E}_{\max} := \{E \subseteq \mathbb{N}^2 : \sup_m |\{n | (m, n) \in E \text{ or } (n, m) \in E\}| < \infty\}$ is the maximal u.l.f. coarse structure on $\mathbb{N}$.

The Uniform Roe Algebra

Definition

For $T \in \mathcal{B}(\ell_2(X))$ let $\mathrm{Supp}(T) := \{(x, x') : (T\delta_x | \delta_{x'}) \neq 0\}$.

Example

1. $T \in \ell_\infty(X) \Leftrightarrow \mathrm{Supp}(T) \subseteq \Delta_X$.
2. If S is the shift on the basis of $\ell_2\mathbb{Z}$, $S\delta_m = \delta_{m+1}$, then $\mathrm{Supp}(S) = \{(m, m+1) : m \in \mathbb{Z}\}$.

Definition

If $(X, \mathcal{E})$ is a coarse space, let

$\mathrm{C}^*_\mathrm{u}[X, \mathcal{E}] = \{T \in \mathcal{B}(\ell_2(X)) : \mathrm{Supp}(T) \in \mathcal{E}\}$ (the *algebraic* uniform Roe algebra)

$\mathrm{C}^*_\mathrm{u}(X, \mathcal{E}) = \overline{\mathrm{C}^*_\mathrm{u}[X, \mathcal{E}]}^{\|\cdot\|}$, (the *uniform Roe algebra*).

Properties of $\mathsf{C}^*_{\mathsf{u}}(X) = \mathsf{C}^*_{\mathsf{u}}(X, \mathcal{E})$

1. It is a C^*-algebra.
2. All compact operators belong to it: $\mathcal{K}(\ell_2(X)) \subseteq \mathsf{C}^*_{\mathsf{u}}(X)$.
3. $\ell_\infty(X) \subseteq \mathsf{C}^*_{\mathsf{u}}[X] \subseteq \mathsf{C}^*_{\mathsf{u}}(X)$ is a maximal abelian subalgebra (masa). It is a Cartan masa; will come back to this later.
4. If X is *uniformly locally finite* and metrizable, then $\mathsf{C}^*_{\mathsf{u}}(X)$ is generated by $\ell_\infty(X)$ together with a countable subset of its normalizer.

(Our) main problem

Question

What is the relation between the following assertions for u.l.f. coarse spaces?

1. $X \sim Y$.
2. $\mathrm{C}^*_\mathrm{u}(X) \cong \mathrm{C}^*_\mathrm{u}(Y)$.
3. $C^*_u(X)/\mathcal{K}(\ell_2(X)) \cong C^*_u(Y)/\mathcal{K}(\ell_2(Y))$ *(Not in this talk.)*

We will also consider the analogous question for embeddings. (The original motivation for (2) $\Rightarrow^?$ (1) comes from the Baum–Connes conjecture.)

Lemma

If X and Y are u.l.f. then $X \sim Y$ implies $\mathrm{C}^*_\mathrm{u}(X) \otimes \mathcal{K} \cong \mathrm{C}^*_\mathrm{u}(Y) \otimes \mathcal{K}$.

$(2) \Rightarrow (1)$: From $C^*_u(X) \cong C^*_u(Y)$ to $X \sim Y$

Lemma (Špakula–Willett, 2013)

If $C^*_u(X) \cong C^*_u(Y)$, then the isomorphism is implemented by a unitary $u\colon \ell_2(X) \to \ell_2(Y)$, via $\Phi(T) = uTu^*$.

Theorem (Špakula–Willett, 2013)

For u.l.f. metric spaces X and Y the following are equivalent

1. $X \sim Y$ *bijectively.*
2. $C^*_u[X] \cong C^*_u[Y]$.
3. $(\exists \Phi)\colon\ C^*_u(X) \cong C^*_u(Y)$, *and* $\Phi[\ell_\infty(X)] = \ell_\infty(Y)$.

Cartan masas

Definition

A masa D in $\mathrm{C}^*_{\mathrm{u}}(X)$ is *Cartan* if

1. There exists a conditional expectation from $\mathrm{C}^*_{\mathrm{u}}(X)$ onto D,
2. The normalizer of D generates $\mathrm{C}^*_{\mathrm{u}}(X)$.

It is *co-separable* if $\mathrm{C}^*_{\mathrm{u}}(X) = \mathrm{C}^*(D, Z)$ for some countable subset Z of the normalizer of D.

Lemma (White–Willett, 2018)

If X is u.l.f. and D is a Cartan masa in $C^*_u(X)$ isomorphic to some $\ell_\infty(Y)$, then there exist a coarse structure $(Y, \mathcal{F})$ and an isomorphism $\Phi\colon C^*_u(X) \cong C^*_u(Y, \mathcal{F})$ such that $\Phi[D] = \ell_\infty(Y)$. The space $(Y, \mathcal{F})$ is metrizable if and only if D is co-separable.

The first part reduces the rigidity question "does $C^*_u(X) \cong C^*_u(Y)$ imply $X \sim Y$?" to the question of classification of Cartan masas in $C^*_u(X)$ that are isomorphic to some ℓ_∞-space.

Question (essentially White–Willett, 2018)

If $\mathrm{C}^*_\mathrm{u}(X) \cong \mathrm{C}^*_\mathrm{u}(Y)$, X and Y are u.l.f., and X is metrizable, is Y necessarily metrizable?

Example (Braga–F.–Vignati, 2019)

There exists a non-metrizable, connected, coarse structure $\mathcal{E}_\mathcal{U}$ on $\mathbb{N}$ included in the coarse structure on $\mathbb{N}$ induced by the standard metric.

Hence $\mathrm{C}^*_\mathrm{u}(\mathbb{N}, \mathcal{E}_\mathcal{U})$ is a subalgebra of $\mathrm{C}^*_\mathrm{u}(\mathbb{N})$, with $\ell_\infty(\mathbb{N})$ as a (non-co-separable) Cartan masa.

From $C^*_u(X) \cong C^*_u(Y)$ to $X \sim Y$

Suppose $\Phi\colon C^*_u(X) \cong C^*_u(Y)$ and fix u such that $\Phi(T) = uTu^*$, for $m \geq 1$ let

$$X_m = \{x : (\exists y \in Y)|(u\delta_x, \delta_y)| > 1/m\},$$
$$Y_m = \{y : (\exists x \in X)|(\delta_x, u\delta_y)| > 1/m\}.$$

If $X = X_m$ and $Y = Y_m$ for some m, then Φ is *rigidly implemented*.

Proposition (Špakula–Willett)

Suppose both X and Y are metric and $\Phi\colon C^*_u(X) \cong C^*_u(Y)$.
If Φ is rigidly implemented then X and Y are coarsely equivalent.
If in addition X has property A ($\Leftrightarrow$ if $C^*_u(X)$ is nuclear) then Φ is rigidly implemented.

Ghosts

Definition (Yu)

An operator $T \in \mathrm{C}^*_\mathrm{u}(X)$ is a *ghost* if $\lim_{x,x' \to \infty} |(T\delta_x | \delta_{x'})| = 0$.

Theorem (Roe–Willett)

A u.l.f. space X has property A if and only if all ghosts $\mathrm{C}^*_\mathrm{u}(X)$ are compact.

Advanced Linear Algebra Test
November 30, 2019

(1) For $k \geq 2$ let

$$A_k = \frac{1}{k} \begin{pmatrix} 1 & 1 & \dots 1 \\ 1 & 1 & \dots 1 \\ \dots & & \\ 1 & 1 & \dots 1 \end{pmatrix}$$

(a $k \times k$ matrix, projection of rank 1).
Prove that for all $\varepsilon > 0$ there exist m_ε and a $k \times k$ matrix $B_{\varepsilon,k}$ (for every k) such that

1. $\|A_k - B_{\varepsilon,k}\| < \varepsilon$,
2. in every row $B_{\varepsilon,k}$ has at most m_ε nonzero entries, and
3. in every column $B_{\varepsilon,k}$ has at most m_ε nonzero entries.

Hints 1: Extract a proof from the counterexample to the coarse Baum–Connes conjecture (Higson–Laforgue–Skandalis, 2002).

(Expander graphs, f.g. groups with property (T), spectral gap of a Laplacian.)

Back to our main rigidity question

A Cartan masa A in $\mathrm{C}^*_\mathrm{u}(X)$ is *ghostly* if there are orthogonal non-compact ghost projections $Q_n \in A$ such that $\bigvee_n Q_n = 1$.

Theorem (Braga–F., 2018)

If X and Y are u.l.f. metric spaces, and $\Phi\colon \mathrm{C}^*_\mathrm{u}(X) \cong \mathrm{C}^*_\mathrm{u}(Y)$ is not rigidly implemented, then at least one of $\mathrm{C}^*_\mathrm{u}(X)$ and $\mathrm{C}^*_\mathrm{u}(Y)$ contains a ghostly Cartan masa.

Corollary

If all ghost projections in $\mathrm{C}^*_\mathrm{u}(X)$ and $\mathrm{C}^*_\mathrm{u}(Y)$ are compact and $\mathrm{C}^*_\mathrm{u}(X) \cong \mathrm{C}^*_\mathrm{u}(Y)$, then X and Y are bijectively coarsely equivalent.

An 'almost ghostly' masa

Example (Willett, Braga–F., 2018)

There exist X, a non-compact ghost projection Q in $\mathrm{C}^*_\mathrm{u}(X)$, a unitary $u \in \ell_\infty(X)$, and a finite rank perturbation P_n of $u^n Q u^n$, for $n \geq 0$, such that

1. Each P_n is a non-compact ghost projection in $\mathrm{C}^*_\mathrm{u}(X)$,
2. $\bigvee_n P_n = 1$.

Question

Can X be chosen so that the projections P_n, $n \in \mathbb{N}$ are contained in a masa of $\mathrm{C}^*_\mathrm{u}(X)$ that is closed in the weak operator topology?

What about the not necessarily metrizable spaces?

Theorem (Braga–F., 2018)

If X and Y are (not necessarily metrizable) coarse spaces, X has property A, $\mathrm{C}^*_{\mathrm{u}}(X) \cong \mathrm{C}^*_{\mathrm{u}}(Y)$, and the isomorphism is forcing absolute, then $X \sim Y$.

The definition of 'forcing absolute' is of a heavily metamathematical nature, but I don't need to state it now.

Theorem (Braga–F.–Vignati, 2018)

Suppose X and Y are coarse u.l.f. spaces such that X has property A. If $\mathrm{C}^*_{\mathrm{u}}(X) \cong \mathrm{C}^*_{\mathrm{u}}(Y)$, then $X \sim Y$.

Corollary

If X is a metrizable u.l.f. space with property A and D is a Cartan masa in $\mathrm{C}^*_{\mathrm{u}}(X)$ isomorphic to $\ell_\infty(\mathbb{N})$, then D is co-countable.

(This generalizes to higher cardinals.)

When does $C^*_u(X) \hookrightarrow C^*_u(Y)$ imply that X coarsely embeds into Y?

If Q is the Higson–Laforgue–Skandalis noncompact ghost projection in $C^*_u(X)$ then the corner $Q\,C^*_u(X)Q$ is isomorphic to $C^*_u(\{n^2 : n \in \mathbb{N}\})$.

Example (Braga–F.–Vignati, 2019)

There is a (non-metrizable) u.l.f. coarse space $X_{\mathcal{U}}$ with property A such that $C^*_u(X)$ embeds into $C^*_u(\mathbb{N})$ but X does not coarsely embed into $\mathbb{N}$.

Theorem (Braga–F.–Vignati, 2019)

If X and Y are u.l.f. coarse spaces such that X has property A and $C^*_u(Y)$ is isomorphic to a hereditary subalgebra of $C^*_u(X)$, then Y injectively coarsely embeds into X

We also have ℓ_p-versions of these results, but it is getting late. . .

Book Advertisement:

I. Farah, Combinatorial Set Theory and C^*-algebras
Springer Monographs in Mathematics
December 2019.

Draft available at
`http://www.math.yorku.ca/∼ifarah/ilijas-book.pdf`