The ubiquitous hyperfinite II_1 factor

To Dick Kadison, in memoriam

Sorin Popa

Murray-von Neumann work on R (1936-43)

- The *hyperfinite* II_1 *factor* R, endowed with its trace state τ, is defined as $(R,\tau) = \overline{\otimes}_n(\mathbb{M}_2(\mathbb{C}), tr)_n$.

Proved that any II_1 factor (M,τ) that's AFD (*approx. finite dim.*) $\| \|_2$-separable is isomorphic to R (where $\|x\|_2 = \tau(x^*x)^{1/2}$, $x \in M$) [MvN43].

Showed that R embeds in any II_1 factor [MvN43], and comment: "the possibility exists that any factor in the case II_1 is isomorphic to a sub-ring of any other such factor".

Gave examples of proper subfactors $R_0 \subset R$ that are irreducible (or *ergodic*), i.e., $R_0' \cap R = \mathbb{C}1$, thus failing the bicommutant property $(R_0' \cap R)' \cap R = R \neq R_0$ [MvN36].

Asked the question of whether all non-type I factors M contain subfactors $M_0 \subset M$ with $(M_0' \cap M)' \cap M \neq M_0$ [MvN36].

Early developments (1950-1970)

• Fuglede-Kadison 1951: if R_0 is a maximal hyperfinite subfactor of a non-hyperfinite II_1 factor M, then $R_0' \cap M$ has non-trivial center, thus $(R_0' \cap M)' \cap M$ is not a factor, so it cannot be equal to R_0.
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• It became very important to decide whether any subfactor of R is isomorphic to R.

Connes Fundamental Theorem (1976)

• Connes Thm: Any separable *amenable* II_1 factor is AFD and is thus isomorphic to R. As a consequence, one has:

• If $N \subset R$ is a II_1 factor, then $N \simeq R$ (because such N is amenable).

• If Γ countable amenable ICC then $L(\Gamma) \simeq R$ (because $L(\Gamma)$ amen. iff Γ amenable). Thus, if $\Gamma = S_\infty$, or $\Gamma = \mathbb{Z} \wr \mathbb{Z}^n$, $n \geq 1$, then $L(\Gamma) \simeq R$.

• If Γ is a countable amenable group and $\Gamma \curvearrowright X$ is free ergodic p.m.p., then $L(\Gamma \curvearrowright X) \simeq R$ (because $L(\Gamma \curvearrowright X)$ amenable iff Γ amenable).

On the importance of "special" embeddings of R

- During 1950 - 1970 it has been recognized by Kadison, Dixmier, Glimm, Sakai, Johnson-Kadison-Ringrose, that being able to "push" elements x into the commutant of a vN algebra M by averaging over $\mathcal{U}(M)$ may be useful to Stone-Weierstrass type problems and vanishing Hochschild cohomology problems in vN algebras. But J. Schwartz results showed that this can be done iff M is amenable.

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• Fortunately, for certain questions it is sufficient to be able to "push" only the x's of some larger $\mathcal{M} \supset M$ into the relative commutant of a "large subalgebra" of M. Hence the importance of finding large copies of R inside M, more generally embeddings $R \hookrightarrow M$ satisfying various specific constraints. Many other reasons for seeking "special" embeddings $R \hookrightarrow M$ appeared over the years, notably in deformation-rigidity theory.

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- I will first present 2 results, and then a conjecture, about R-embeddings.

Ergodic R-embeddings into arbitrary factors

Theorem ([P1981], [P2019])

Any non-type I factor acting on a separable Hilbert space, $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$, contains an ergodic copy of R, i.e., $\exists$ hyperfinite subfactor $R \subset \mathcal{M}$ with $R' \cap \mathcal{M} = \mathbb{C}1$.

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Proof consists in constructing recursively an increasing sequence of dyadic fin.dim. factors Q_n inside $\mathcal{M}$ such that their diagonals $D_n \subset Q_n$ become "more and more" a MASA in $\mathcal{M}$, while at the same time "more and more" of a dense countable set of unit vectors in $\mathcal{H}$ implement asymptotically the trace τ on Q_n. But then, $Q := \overline{\cup_n Q_n} \subset \mathcal{M}$ will be so that on the one hand $Q \subset \mathcal{B}(\mathcal{H})$ is a rep. of the hyperfinite II_1 factor R, while at the same time $D := \overline{\cup_n D_n}$ is a MASA in $\mathcal{M}$. Thus, $Q' \cap \mathcal{M} \subset Q' \cap D = \mathbb{C}$.

Coarse decomposition of II_1 factors

Coarse subalgebras and coarse pairs

A proper inclusion $B \subset M$ is **coarse** if the vN algebra generated by left-right multiplication by elements in B on $L^2(M \ominus B)$ is $B \overline{\otimes} B^{op}$. The vN subalgebras $B, Q \subset M$ form a **coarse pair** if the vN algebra generated by left multiplication by B and right multiplication by Q on L^2M is $B \overline{\otimes} Q^{op}$.

Examples

- If $M = L(\Gamma)$ and $H \subset \Gamma$ is an infinite subgroup, then $B = L(H) \subset L(\Gamma) = M$ is coarse iff $\forall g \in \Gamma \setminus H$ one has $gHg^{-1} \cap H = \{e\}$. Also, if $H_0 \subset \Gamma$ is another group, then $L(H)$, $L(H_0)$ is a coarse pair iff $gHg^{-1} \cap H_0 = \{e\}$, $\forall g \in \Gamma$. For instance, if $\Gamma = \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ then $L(\Gamma) = R$ and $H = \mathbb{Z}$ gives rise to a coarse MASA inclusion, $L(\mathbb{Z}) = A \subset R$.

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• If Γ is an infinite group, N_0 is non-trivial tracial vN and $\Gamma \curvearrowright N = N_0^{\overline{\otimes}\Gamma}$ is the Bernoulli Γ-action with base N_0, then $L(\Gamma) \subset M = N \rtimes \Gamma$ is coarse. Also, $L(\Gamma), N_0 \subset M$ is a coarse pair.

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• If B, B_0 are tracial vN algebras with B diffuse and B_0 non-trivial, then $M = B * B_0$ is a II_1 factor and $B = B * 1 \subset M$ is coarse, while B_0, B is a coarse pair.

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Theorem (P 2018-19)

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In particular, there exists a pair of hyp. factors $R_0, R_1 \subset M$ so that each one is coarse and $R_0 \vee R_1^{op} \simeq R_0 \overline{\otimes} R_1^{op}$ (R_0, R_1 mutually coarse).

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Proof. Let $\{\xi_n\}_n \subset L^2 M$ be $\| \ \|_2$-dense in $(L^2 M)_1$. One needs to build iteratively an increasing sequence of dyadic factors $Q_n \subset M$ such that for each n the "new part" is more and more coarse with respect to $S_n = \{E_{Q_{n-1}}(\xi_j) \mid 1 \leq j \leq n\}$ (this amounts to $Q'_{n-1} \cap Q_n$ being almost 2-independent to S_n). If this is done carefully/rapidly enough, then the resulting $R = \overline{\cup_n Q_n}$ will be so that the restriction of $R \vee R^{op}$ on $L^2(M \ominus R)$ gives a rep. of $R \overline{\otimes} R^{op}$.

Tightness and the coarseness trap

• Proofs of Theorems show that one can construct hyperfinite II_1 factors $R_0, R_1 \subset M$ recursively, as inductive limit of dyadic finite dimensional factors $R_{0,n} \nearrow R_0$, $R_{1,n} \nearrow R_1$, so that at each step n more and more of the vectors in a countable dense subset $\{\xi_n\}_n \subset L^2M$ implement asymptotically a specific type of state on $R_{0,n} \vee R_{1,n}^{op} \simeq R_{0,n} \otimes R_{1,n}^{op}$, namely the trace $\tau \otimes \tau$.

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• It should be possible to carry out such an "iterative construction with constraints" of increasing $R_{0,n}, R_{1,n}$ so that the vectors in $\{\xi_n\}_n \subset L^2M$ implement asymptotically states that "stay away" from $\tau \otimes \tau$, a fact that's equivalent to $R_0 \vee R_1^{op}$ properly infinite, even equal $\mathcal{B}(L^2M)$ ("tightness").

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• But by [Ge-P 1998], if $M = L(\mathbb{F}_n)$, then any choice of increasing dyadic factors $R_{0,n}, R_{1,n}$ produces $R_0, R_1 \subset M$ with $R_0 \vee R_1^{op}$ having a coarse part ("coarseness trap") !

Escaping the coarseness trap

I believe the condition for escaping the coarseness trap is the following:

The SSG property

A II_1 factor M is *stably single generated* (*SSG*) if M^t is single generated for any $t > 0$ (equivalently, $\exists t_n \searrow 0$ s.t. M^{t_n} is single generated $\forall n$).

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The tightness conjectures

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- If M is SSG then there exist $R_0, R_1 \subset M$ s.t. $R_0 \vee R_1^{op} \subset \mathcal{B}(L^2M)$ is properly infinite (M is *weakly tight*).

Relevance to the free group factor problem

Fact

If (weak) tightness conjecture holds true, then $L(\mathbb{F}_\infty)$ cannot be generated by finitely many elements and $L(\mathbb{F}_n)$, $2 \leq n \leq \infty$, are mutually isomorphic.

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Proof. Indeed, since $\mathbb{Q}_+ \subset \mathcal{F}(L(\mathbb{F}_\infty))$ ([Voiculescu 1989]), if $M = L(\mathbb{F}_\infty)$ finitely generated, then it is SSG, which by the conjecture would imply existence of hyperfinite $R_0, R_1 \subset M$ with $R_0 \vee R_1^{op}$ properly infinite, thus having a cyclic vector $\xi \in L^2M$, with $[R_0 \xi R_1] = L^2M$, contradicting [Ge-P 1998]. Thus, $L(\mathbb{F}_\infty)$ is infinitely generated, so $\not\simeq L(\mathbb{F}_2)$, so by [Radulescu 1994] all $L(\mathbb{F}_n)$, $2 \le n \le \infty$, are non-isomorphic.

A dynamical approach to proving "SSG $\Rightarrow$ tight"

Tightness requires constructing increasing sequences of dyadic factors $R_{0,n}$, $R_{1,n} \subset M$ so that by averaging over unitaries in $R_{0,n}, R_{1,n}^{op}$ at each n, one obtains that "larger and larger" finite subsets of a countable dense subset $\{T_k\}_k$ of $\mathcal{L}_0^1 := \{T \in \mathcal{B}(L^2M) \mid \|T\|_{1,Tr} := Tr(|T|) \leq 1, Tr(T) = 0\}$ get "more and more annihilated".

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Then $u_{i_0} u_{i_1}^*$ and $v_{j_0} v_{j_1}^*$ would give two (dyadic) "abelian directions" $D_{0,n}, D_{1,n} \subset M$ around which one can construct dyadic $R_{0,n}, R_{1,n} \subset M$, allowing to move on with the iterative construction due to stability of SSG (*Step 3*).

A recent solution to Step 1 (MV property)

Fact: he above $\| \ \|_{1,Tr}$ MV-property of $\mathcal{U}(M) \times \mathcal{U}(M^{op}) \curvearrowright \mathcal{L}_0^1$ is equivalent to: $\forall T \in \mathcal{B}(L^2M)$, the wo-closure of the convex hull of $uv^{op}Tu^*v^{op*}$ intersects $\mathbb{C}1$ (hint: use Hahn-Banach)

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Note that this result does not require M to be SSG, thus putting all the weight of a possible proof of the tightness conjecture on the shoulders of Step 2, the solution of which MUST therefore use SSG!