C*-algebras of stable rank one and their Cuntz semigroups

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Richard Kadison and his mathematical legacy
A memorial conference
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Application: Non-stable K-theory (Rieffel 1983):

- $K_1(A) = \mathcal{U}(A)/\mathcal{U}_0(A)$. (No matrix amplifications needed.)
- projections p and q are equivalent iff $[p] = [q] \in K_0(A)$

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$\rightsquigarrow$ Question: Do all simple group C*-algebras have stable rank one?

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Theorem (CEI 2008)

There is a category **Cu** *of order-complete, partially ordered semigroups such that* $A \mapsto \mathrm{Cu}(A)$ *is a functor* $\mathbf{C}^* \to \mathbf{Cu}$.

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Theorem (APT 2017)

- **Cu** *admits internal-hom (closed monoidal category)*
- *starting point to develop UCT for bivariant Cuntz semigroups:* $\mathrm{Cu}(A, B) \to [\![\mathrm{Cu}(A), \mathrm{Cu}(B)]\!]$

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Ultraproduct $\prod_{\mathcal{U}} A_k$ *is simple iff either:*

1. *almost all* A_k *are simple, purely infinite; or:*
2. *there is n such that almost all* A_k *are isomorphic to* $M_n(\mathbb{C})$.

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Characterization of when limit (quasi)traces on $\prod_{\mathcal{U}} A_k$ *are dense in* $\mathrm{QT}(\prod_{\mathcal{U}} A_j)$.

(Generalizing Ozawa's 2013 'no silly traces' result. Uses that $\mathrm{Cu}(-)$ encodes simplex of (quasi)traces.)

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Every quasitrace $\tau\colon A \to \mathbb{C}$ induces a dimension function:

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Corollary

If A has stable rank one and strict comparison, then

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Corollary (T 2018)

Toms-Winter conjecture holds for:

- *C*-algebras with stable rank one and locally finite nuclear dimension (in particular, stable rank one ASH-algebras).*
- *minimal crossed products $C(X) \rtimes \mathbb{Z}$ (using Lutley 2017).*

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If A has stable rank one, then $\mathrm{Cu}(A)$ has Riesz interpolation:
If $x_j \leq z_k$ for $j, k = 1, 2$, then there is y with $x_1, x_2 \leq y \leq z_1, z_2$.

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- We can apply methods from semilattice theory to study C*-algebras of stable rank one.

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dimension function = finitely additive, noncommutative measure

- $\mathrm{DF}(C(X))$ = finitely additive probability measures on X

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dimension function = finitely additive, noncommutative measure

- $\mathrm{DF}(C(X))$ = finitely additive probability measures on X

Blackadar-Handelman conjecture 1982

$\mathrm{DF}(A)$ is a Choquet simplex.

Applications (1)

Definition

A dimension function on A is a functional $d\colon \mathrm{Cu}(A) \to [0,\infty]$*. It assigns a dimension to each Hilbert A-module such that:*

1. $d(E \oplus F) = d(E) + d(F)$,
2. $d(E) \leq d(F)$ *if* $E \hookrightarrow F$,
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$\mathrm{DF}(A)$ is a Choquet simplex.

Theorem (APRT 2018)

Blackadar-Handelman conjecture holds for stable rank one.

Applications (2)

Let A be unital, $k \in \mathbb{N}$.
Obstructions to irreducible representation of dimension $< k$:

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Theorem (APRT 2018, Global Glimm Halving)

Let A be unital with stable rank one, and $k \in \mathbb{N}$. TFAE:

1. *A has no irreducible representation of dimension $< k$.*
2. *there exists $M_k(C_0((0,1])) \to A$ with full image.*
3. *there exists a Hilbert A-module E such that $E^{\oplus k} \subseteq A \subseteq E^{\oplus n}$, for some n.*

Applications (3)

The *rank* of a Hilbert A-module E is

$$\widehat{E}\colon\ \mathrm{QT}(A) \to [0,\infty], \quad \widehat{E}(\tau) = d_\tau(E).$$

The rank problem

Describe $\{\widehat{E} : E \text{ Hilbert } A\text{-module}\}$.

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Theorem (APRT 2018)

Let A be unital, stable rank one, no finite-dimensional representations. Then for every $f \in \mathrm{LAff}(\mathrm{QT}(A))_{++}$ there is Hilbert A-module E with $\widehat{E} = f$.

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