The Murray-von Neumann algebra and the unitary group of a II_1-factor

Andreas Thom
TU Dresden, Germany

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København

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Theorem (Kadison, 1952)

The group $\mathrm{PU}(M, \tau)$ is topologically simple.

Outline

1. Bounded normal generation of $\mathrm{PU}(M, \tau)$
2. The Lie algebra of $\mathrm{U}(M, \tau)$
3. The Heisenberg-von Neumann-Kadison puzzle

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Theorem (Liebeck-Shalev)

There exists a contant c, such that for any non-abelian finite simple group G and non-trivial $g \in G$ we have:

$$G = (g^G)^k \quad \text{if} \quad k \geq \frac{c \log |G|}{\log |g^G|}.$$

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This is optimal up to a multiplicative constant.

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Theorem

There exists a constant c, such that for any $n \geq 2$ and non-trivial $u \in PU(n)$, we have

$$PU(n) = (u^{PU(n)})^k, \quad \textit{if} \quad k \geq \frac{c|\log \ell(u)|}{\ell(u)}.$$

Consequences I – joint work with Philip Dowerk

Theorem

Let M be a II_1-factor von Neumann algebra. For any non-trivial $u \in PU(M)$, we have

$$PU(M) = (u^{PU(M)})^k, \quad \text{if} \quad k \geq \frac{c|\log \ell(u)|}{\ell(u)}.$$

Consequences II – joint work with Philip Dowerk

Recall, a polish group is called SIN if it has a basis of conjugation invariant neighborhoods of 1.

Theorem

Let M be a finite factorial von Neumann algebra.

1. *Any homomorphism from $PU(M)$ into a polish SIN group is automatically continuous.*
2. *$PU(M)$ carries a unique polish group topology.*

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Question

Is the first claim true for II_1-factors without the assumption that the target group is SIN?

Lie theory for infinite dimensional groups

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- Define $\mathcal{A}(M, \tau)$ the the completion of (M, τ) with respect to the metric

$$d(s, t) := \tau([s - t]),$$

where $[x]$ denotes the source projection of the operator $x \in M$.

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The world can be so easy...

We set

$$\mathrm{Lie}(M,\tau) := \{x \in \mathcal{A}(M,\tau) \mid x^* = -x\}.$$

Theorem (Ando-Matsuzawa)

There is a bijective correspondence between SOT*-continuous 1-parameter semigroups in* $\mathrm{U}(M,\tau)$ *and* $\mathrm{Lie}(M,\tau)$*.*

Moreover, $\mathrm{Lie}(M,\tau)$ *is a topological Lie algebra and analogues of familiar formulas from Lie theory hold.*

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Is every element in $\mathcal{A}(M, \tau)$ *equal to a commutator?*

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Which operators in a II_1*-factor are commutators?*

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Sketch of proof: Note that log-integrable operators have a well-defined Brown spectral measure μ_x. It is characterized by the property:

$$\log \Delta(x - \lambda 1) = \int_{\mathbb{C}} \log |t - \lambda| d\mu_x(t),$$

where Δ denotes the Fuglede-Kadison determinant.

Continuation of the proof

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Question

Does a generalization of Brown's spectral measure with suitable properties exist for all operators in $\mathcal{A}(M, \tau)$?

Thank you for your attention.