Ergodicity and type of nonsingular Bernoulli actions

Richard Kadison and his mathematical legacy – A memorial conference

University of Copenhagen

29 - 30 November 2019

KU LEUVEN

Stefaan Vaes

Bernoulli actions

Bernoulli actions of a countable group G

For any standard probability space (X_0, μ_0), consider

$$G \curvearrowright (X_0, \mu_0)^G = \prod_{g \in G} (X_0, \mu_0) \text{ given by } (g \cdot x)_h = x_{g^{-1}h}.$$

- $(G = \mathbb{Z})$ Kolmogorov-Sinai : entropy of μ_0 is a conjugacy invariant.
- $(G = \mathbb{Z})$ Ornstein : entropy is a complete invariant.
- Bowen : beyond amenable groups, sofic groups.
- Popa : orbit equivalence rigidity, von Neumann algebra rigidity.

$\rightsquigarrow$ What about $G \curvearrowright \prod_{g \in G} (X_0, \mu_g)$ given by $(g \cdot x)_h = x_{g^{-1}h}$?

Main motivation: produce interesting families of type III group actions.

Group actions of type III

- The classical Bernoulli action $G \curvearrowright (X, \mu) = (X_0, \mu_0)^G$
  - is ergodic,
  - preserves the probability measure μ.
- An action $G \curvearrowright (X, \mu)$ is called **non-singular** if $\mu(g \cdot \mathcal{U}) = 0$ whenever $\mu(\mathcal{U}) = 0$ and $g \in G$.
- Write $\mathcal{U} \sim \mathcal{V}$ if there exists a measurable bijection $\Delta : \mathcal{U} \to \mathcal{V}$ with $\Delta(x) \in G \cdot x$ for a.e. $x \in \mathcal{U}$.
- A nonsingular ergodic $G \curvearrowright (X, \mu)$ is of **type III** if $\mathcal{U} \sim \mathcal{V}$ for all non-negligible $\mathcal{U}, \mathcal{V} \subset X$.
  - There is no G-invariant measure in the measure class of μ.
  - The **Radon-Nikodym derivative** $d(g \cdot \mu)/d\mu$ must be sufficiently wild.

Group actions of type III_1

Let $G \curvearrowright (X, \mu)$ be a nonsingular group action.

- Write $\omega(g, x) = \dfrac{d(g^{-1} \cdot \mu)}{d\mu}(x)$, the Radon-Nikodym 1-cocycle.
- The action $G \curvearrowright X \times \mathbb{R}$ given by $g \cdot (x, s) = (g \cdot x, s + \log(\omega(g, x)))$ preserves the (infinite) measure $\mu \times e^{-s} ds$.
- This is called the **Maharam extension.** It is the ergodic analogue of the **Connes-Takesaki continuous core** for von Neumann algebras.

$\rightsquigarrow$ An ergodic nonsingular action $G \curvearrowright (X, \mu)$ is of **type III_1** if its Maharam extension remains ergodic.

$\rightsquigarrow$ Associated ergodic flow $\mathbb{R} \curvearrowright L^\infty(X \times \mathbb{R})^G$.

$\rightsquigarrow$ $G \curvearrowright (X, \mu)$ is of type III iff this flow is not just $\mathbb{R} \curvearrowright \mathbb{R}$.

$\rightsquigarrow$ $G \curvearrowright (X, \mu)$ is of type III_λ iff this flow is $\mathbb{R} \curvearrowright \mathbb{R}/\mathbb{Z} \log \lambda$.

Bernoulli actions of type III

Consider $G \curvearrowright (X,\mu) = \prod_{g \in G} (X_0, \mu_g)$ given by $(g \cdot x)_h = x_{g^{-1}h}$.

1. All μ_g are equal : type II_1, ergodic, probability measure preserving.
2. **Interesting gray zone :** when is $G \curvearrowright (X,\mu)$ of type III, or type III_1 ?
3. The μ_g are quite different : type I, the action is **dissipative**, meaning that $X = \bigsqcup_{g \in G} g \cdot \mathcal{U}$ up to measure zero.
4. The μ_g are very different : the action is singular.

Kakutani's criterion

- The action $G \curvearrowright \prod_{g \in G} (X_0, \mu_g)$ is nonsingular if and only if

  for every $g \in G$, we have $\sum_{h \in G} d(\mu_{gh}, \mu_h)^2 < \infty$.

- Take $X_0 = \{0, 1\}$ with $0 < \mu_g(0) < 1$.

  Assume that $\delta \leq \mu_g(0) \leq 1 - \delta$ for all $g \in G$.

  Then, the action is nonsingular if and only if

  $\sum_{h \in G} |\mu_{gh}(0) - \mu_h(0)|^2 < \infty$ for all $g \in G$.

$\rightsquigarrow$ Then $c : G \to \ell^2(G) : c_g(h) = \mu_h(0) - \mu_{g^{-1}h}(0)$ is a **1-cocycle** for the left regular representation,

meaning that $c_{gh} = c_g + \lambda_g c_h$.

An easy no-go theorem

Theorem (V-Wahl, 2017)

If $H^1(G, \ell^2(G)) = \{0\}$, there are **no nonsingular Bernoulli actions of type III.** More precisely,

every nonsingular Bernoulli action of G is the sum of a classical, probability measure preserving Bernoulli action and a dissipative Bernoulli action.

- The groups with $H^1(G, \ell^2(G)) = \{0\}$ are precisely the nonamenable groups with $\beta_1^{(2)}(G) = 0$.
- Large classes of nonamenable groups have $\beta_1^{(2)}(G) = 0$:
  - property (T) groups,
  - groups that admit an infinite, amenable, normal subgroup,
  - direct products of infinite groups.

What if $H^1(G, \ell^2(G)) \neq \{0\}$?

This is very delicate ! Even for the case $G = \mathbb{Z}$.

- (Krengel, 1970)

  The group $G = \mathbb{Z}$ admits a nonsingular Bernoulli action without invariant probability measure.

- (Hamachi, 1981)

  The group $G = \mathbb{Z}$ admits a nonsingular Bernoulli action of type **III**.

- (Kosloff, 2009)

  The group $G = \mathbb{Z}$ admits a nonsingular Bernoulli action of type $\mathbf{III_1}$.

⤳ In all cases: no explicit constructions.

Dissipative versus conservative

Recall: $G \curvearrowright (X, \mu)$ is dissipative iff $X = \bigsqcup_{g \in G} g \cdot \mathcal{U}$ up to measure zero.

$G \curvearrowright (X, \mu)$ is conservative iff we return to every $\mathcal{U} \subset X$ with $\mu(\mathcal{U}) > 0$.

Theorem (V-Wahl, 2017)

Let $G \curvearrowright \prod_{g \in G}(\{0,1\}, \mu_g)$ be nonsingular. Let $c_g(h) = \mu_h(0) - \mu_{g^{-1}h}(0)$.

- If $\sum_{g \in G} \exp\left(-\frac{1}{2}\|c_g\|_2^2\right) < \infty$, the action is dissipative.
- If $\mu_g(0) \in [\delta, 1-\delta]$ for all $g \in G$

  and if $\sum_{g \in G} \exp\left(-3\delta^{-2}\|c_g\|_2^2\right) = +\infty$, the action is conservative.

$\rightsquigarrow$ The growth of $g \mapsto \|c_g\|_2$ should be sufficiently slow.

A naive example

Take $\mathbb{Z} \curvearrowright \prod_{n\in\mathbb{Z}}(\{0,1\},\mu_n)$ where

- $\mu_n(0) = p$ if $n < 0$,
- $\mu_n(0) = q$ if $n \geq 0$.

One might expect: if $p \neq q$, then the action is of type III_λ.

But (Krengel 1970 and Hamachi 1981): if $p \neq q$, the action is dissipative.

Indeed: $\|c_n\|_2^2 \sim |n|$ and $\sum_{n\in\mathbb{Z}} \exp(-\varepsilon\,|n|) < +\infty$ for every $\varepsilon > 0$.

Ergodicity of nonsingular Bernoulli actions

Let $G \curvearrowright (X, \mu) = \prod_{g \in G} (\{0,1\}, \mu_g)$ be any nonsingular Bernoulli action.

Assume that $\mu_g(0) \in [\delta, 1-\delta]$ for all $g \in G$.

- (Kosloff, 2018) When $G = \mathbb{Z}$ and $G \curvearrowright (X, \mu)$ is conservative, then $G \curvearrowright (X, \mu)$ is ergodic.
- (Danilenko, 2018) When G is amenable and $G \curvearrowright (X, \mu)$ is conservative, then $G \curvearrowright (X, \mu)$ is ergodic.

Tool: let $\mathcal{R}$ be the tail equivalence relation on (X, μ) given by $x \sim y$ iff $x_g \neq y_g$ for at most finitely many $g \in G$.

- They prove that any G-invariant function is $\mathcal{R}$-invariant.
- Key role: Hurewicz ratio ergodic theorem (K) / a new pointwise ergodic theorem (D).

Ergodicity of nonsingular Bernoulli actions

Let $G \curvearrowright (X, \mu) = \prod_{g \in G}(\{0,1\}, \mu_g)$ be any nonsingular Bernoulli action.

Theorem (Björklund-Kosloff-V, 2019)

- If G is abelian and $G \curvearrowright (X, \mu)$ is conservative, then $G \curvearrowright (X, \mu)$ is ergodic.

  So, no assumption that $\mu_g(0) \in [\delta, 1-\delta]$.

- If G is arbitrary and $G \curvearrowright (X, \mu)$ is strongly conservative, then $G \curvearrowright (X, \mu)$ is ergodic.

  So, no amenability assumption.

Assume that $\mu_g(0) \in [\delta, 1-\delta]$. Write $c_g(h) = \mu_h(0) - \mu_{g^{-1}h}(0)$.

If $\sum_{g \in G} \exp(-8\delta^{-1} \|c_g\|_2^2) = +\infty$, then $G \curvearrowright (X, \mu)$ is strongly conservative and thus ergodic.

Type of nonsingular Bernoulli actions

Let $G \curvearrowright (X, \mu) = \prod_{g \in G}(\{0,1\}, \mu_g)$ be a conservative Bernoulli action.

- Basically no systematic results on the type of $G \curvearrowright (X, \mu)$.
- (Björklund-Kosloff, 2018) If G is amenable and $\lim_{g \to \infty} \mu_g(0)$ exists, then $G \curvearrowright (X, \mu)$ is either II_1 or III_1.

Theorem (Björklund-Kosloff-V, 2019)

Let G be abelian and not locally finite.

- If $\lim_{g \to \infty} \mu_g(0)$ does not exist: type III_1.
- If $\lim_{g \to \infty} \mu_g(0) = \lambda$ and $0 < \lambda < 1$, then type II_1 or type III_1, depending on $\sum_{g \in G}(\mu_g(0) - \lambda)^2$ being finite or not.
- If $\lim_{g \to \infty} \mu_g(0) = \lambda$ and $\lambda \in \{0,1\}$, then type III.

$\rightsquigarrow$ Answering Krengel: a Bernoulli action of $\mathbb{Z}$ is never of type II_∞.

Type of nonsingular Bernoulli actions

Let $G \curvearrowright (X, \mu) = \prod_{g \in G}(\{0, 1\}, \mu_g)$ be nonsingular and $\mu_g(0) \in [\delta, 1 - \delta]$.

Write $c_g(h) = \mu_h(0) - \mu_{g^{-1}h}(0)$.

Theorem (Björklund-Kosloff-V, 2019)

Assume that G has only one end.

Assume that $\sum_{g \in G} \exp(-8\delta^{-1} \|c_g\|_2^2) = +\infty$.

Then, $G \curvearrowright (X, \mu)$ is of type III_1, unless

for some $0 < \lambda < 1$, we have $\sum_{g \in G}(\mu_g(0) - \lambda)^2 < +\infty$. Then type II_1.

Corollary (answering conjecture of V-Wahl): a group G admits a type III_1 Bernoulli action iff $H^1(G, \ell^2(G)) \neq \{0\}$.

Recall: the growth condition on the cocycle implies that $G \curvearrowright (X, \mu)$ is strongly conservative.

Ends of groups

Recall. A finitely generated group G has **more than one end** if its Cayley graph has more than one end: there exists a finite subset $\mathcal{F} \subset G$ with disconnected complement.

Proposition. A finitely generated group G has more than one end iff there exists a subset $W \subset G$ such that

- W is almost invariant: $|gW \triangle W| < \infty$ for all $g \in G$,
- both W and $G \setminus W$ are infinite.

⇝ Use this as definition of "having more than one end" for arbitrary countable groups.

Ends of groups

Stallings' Theorem

A countable group G has more than one end if and only if G is in one of the following families.

- Nontrivial amalgamated free products and HNN extensions over finite subgroups.
- Virtually cyclic groups.
- Locally finite groups.

$\rightsquigarrow$ Due to Stallings for finitely generated groups.

$\rightsquigarrow$ Due to Dicks & Dunwoody for arbitrary groups.

Ends of groups and nonsingular Bernoulli actions

Let $W \subset G$ be almost invariant. Define

- $\mu_g(0) = p$ if $g \in W$,
- $\mu_g(0) = q$ if $g \notin W$.

Then: $G \curvearrowright (X, \mu) = \prod_{g \in G}(\{0,1\}, \mu_g)$ is a nonsingular Bernoulli action.

But (remember $G = \mathbb{Z}$ and $W = \mathbb{N}$) : the action could be dissipative.

Theorem (Björklund-Kosloff-V, 2019)

- Infinite, locally finite groups admit Bernoulli actions of each possible type: II_1, II_∞, III_0, III_λ and III_1.
- Every nonamenable group with more than one end admits nonsingular Bernoulli actions of type III_λ for each λ close enough to 1.