From Kadison-Singer to Ramanujan (after Marcus-Spielman-Srivastava)

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1 MSS on KS

Theorem 1.1. *(solution to Kadison-Singer, Marcus-Spielman-Srivastava 2013) Every pure state of $\ell^\infty(\mathbf{N})$ uniquely extends as a state of $B(\ell^2(\mathbf{N}))$.*

To prove this (actually, Weaver's translation in linear algebra), MSS had to prove two results on random matrices:

Let $A_1, ..., A_d$ be independent random variables with values in rank 1 positive semi-definite matrices. Set $A = \sum_{i=1}^d A_i$. Let $p_A(z) = \det(z\mathbf{1}_m - A)$ be the characteristic polynomial of A.

Theorem 1.2. *(MSS)*

1. *Assume $\mathbb{E}A = \mathbf{1}_m$ and $\mathbb{E}\|A_i\| \leq \varepsilon$ for all $i = 1, ..., d$. Then $\mathbb{E}p_A$ is real-rooted with biggest root at most $(1 + \sqrt{\varepsilon})^2$.*

2. *Assume that the A_i's take finitely many values. Then for some realization of the A_i's, $\|A\|$ (=biggest root of p_A) is less or equal to the biggest root of $\mathbb{E}p_A$.*

It turn out that the second part also solved another famous question , going back to 1986: the existence of infinite families of d-regular Ramanujan graphs, for every $d \geq 3$.

2 Ramanujan graphs

Let $X = (V, E)$ be a finite, connected d-regular connected graph, on n vertices. Let A be its adjacency matrix:

$$A_{xy} = \begin{cases} 1 & \text{if } x \text{ adjacent to } y \\ 0 & \text{otherwise} \end{cases} \quad (x, y \in V)$$

By linear algebra, the spectrum $Sp(A)$ consists of n eigenvalues (counting multiplicities):

$$\lambda_0 = d > \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{n-1} (\geq -d).$$

Proposition 2.1. *X is bipartite if and only if $\lambda_{n-1} = -d$. In this case the spectrum of A is symmetric with respect to 0.*

The *spectral gap* of X is $d - \lambda_1(X)$.

Let $(X_m)_{m>0}$ be a family of d-regular, finite, connected graphs with $|X_m| \to \infty$ for $m \to \infty$.

Definition 2.2. *$(X_m)_{m>0}$ is an expander family if the spectral gap of the X_m's is bounded below by a positive constant ε:*

$$d - \lambda_1(X_m) \geq \varepsilon$$

for every $m > 0$.

Asymptotically the spectral gap is at most $d - 2\sqrt{d-1}$:

Theorem 2.3. *(Alon-Boppana)* $\liminf_{m\to\infty} \lambda_1(X_m) \geq 2\sqrt{d-1}$.

Definition 2.4. *X is Ramanujan if, for every eigenvalue λ of A, with $\lambda \neq \pm d$, we have $|\lambda| \leq 2\sqrt{d-1}$.*

Example 1. *The complete graph K_n is $(n-1)$-regular Ramanujan; the complete bipartite graph $K_{n,n}$ is n-regular Ramanujan.*

Infinite families of d-regular Ramanujan graphs, if they exist, provide expander families with the largest possible spectral gap. But do they exist?

Theorem 2.5. *(Lubotzky-Phillips-Sarnak 1986, Margulis 1986, Morgenstern 1994) When* $d-1$ *is a prime power, there exists explicit infinite families of d-regular Ramanujan graphs, both bipartite and non-bipartite.*

The proof uses deep number theory (proof by Deligne of the Ramanujan conjecture).

Question 1. *For arbitrary* $d \geq 3$, *does there exist infinite families of d-regular Ramanujan graphs?*

3 2-lifts

Definition 3.1. *A signing of X is a map $s : E \to \{\pm 1\}$. For a signing s, the signed adjacency matrix $A^{(s)}$ is:*

$$A^{(s)}_{xy} = \begin{cases} s(x,y) & \text{if } x \text{ adjacent to } y \\ 0 & \text{otherwise} \end{cases} \quad (x, y \in V).$$

To every signing s, we associate the *2-lift* $\tilde{X}^{(s)}$, a graph on $V_1 \coprod V_2$, with $V_1 = V_2 = V$ (see flipchart).

Lemma 3.2. *(Bilu-Linial 2006): For a signing s of X:*

$$Sp(\tilde{X}^{(s)}) = Sp(A) \cup Sp(A^{(s)}).$$

Conjecture 1. *(Bilu-Linial) For any d-regular X, there exist a signing s such that $Sp(A^s) \subset [-2\sqrt{d-1}, 2\sqrt{d-1}]$.*

Theorem 3.3. *(MSS) The Bilu-Linial conjecture holds for bipartite graphs.*

The conjecture is still open for non-bipartite graphs!

Corollary 3.4. *(MSS) Every d-regular bipartite Ramanujan graph admits a 2-lift which is also d-regular bipartite Ramanujan.* □

Taking $K_{d,d}$ as seed and iterating, we get:

Corollary 3.5. *(MSS) For every $d \geq 3$, there exists infinite families of d-regular bipartite Ramanujan graphs.*

Remark 3.6.

- *This is an existence result!*
- *The assumption "bipartite" is used only as follows: the MSS techniques allow them to control the top eigenvalue. For a bipartite graph, you also control the lowest eigenvalue.*
- *In 2015, using their theory of free finite convolutions, MSS could prove that for every n and d, there exists a d-regular bipartite Ramanujan graph on n vertices.*
- *The name "Ramanujan" might not be the best one - after all!*

4 Main steps in the proof

Fix a signing s. For $e = (u, v) \in E$ define a positive rank 1 operator $A_e^{(s)} \in M_n(\mathbf{C})$; for $f \in \mathbf{C}^V$:

$$A_e^{(s)}(f) = \begin{cases} \langle f|\delta_u - \delta_v\rangle(\delta_u - \delta_v) & if \quad s(u,v) = -1 \\ \langle f|\delta_u + \delta_v\rangle(\delta_u + \delta_v) & if \quad s(u,v) = 1 \end{cases}$$

Then:

$$d.\mathbf{1}_n + A^{(s)} = \sum_{e \in E} A_e^{(s)}$$

Endow the set of $2^{|E|}$ signings with the uniform probability, and view the $s \mapsto A_e^{(s)}$ (for $e \in E$) as a collection of independent random variables. By the 2nd part of the MSS result: for some realization of $A^{(s)}$:

$$\text{max-root}(p_{d.\mathbf{1}_n + A^{(s)}}) \leq \text{max-root}(\mathbf{E}_s p_{d.\mathbf{1}_n + A^{(s)}})$$

So let $\lambda_{max}^{(s)}$ be the largest eigenvalue of $A^{(s)}$. The LHS of the previous inequality is $d + \lambda_{max}^{(s)}$.

Definition 4.1. *An r-matching of X is a collection of r disjoint edges. We denote by p_r the number of r-matchings, and by $\mu_X(z) = \sum_{r\geq 0}(-1)^r p_r z^{n-2r}$ the matching polynomial of X.*

Theorem 4.2. *(Godsil-Gutman 1978)* $\mathbf{E}_s p_{A^{(s)}} = \mu_X$.

To proceed: for $u \in V$, the *path-tree* $T(X,u)$ is a finite subtree of the universal cover $\tilde{X}$ obtained by lifting all injective paths from u in X.

Theorem 4.3. *(Heilbronn-Lieb 1972) The matching polynomial μ_X divides the characteristic polynomial p_T of the adjacency matrix of $T(X,u)$.*

Consequence: from Perron-Frobenius and the above:

$$\text{max-root}(\mu_X) \leq \text{max-root}(p_T) \leq 2\sqrt{d-1}$$

So the max-root of $\mathbf{E}_s p_{d.\mathbf{1}_n + A^{(s)}}(z) = \mathbf{E}_s p_{A^{(s)}}(z-d)$ is at most $d + 2\sqrt{d-1}$ and we are done.

Remark 4.4. : *For a finite tree T, the matching polynomial μ_T coincides with the characteristic polynomial p_T of the adjacency matrix.*

Indeed, denoting by $Sym(S)^{nf}$ the set of fixed-point free permutations of S:

$$p_T(z) = \det(z.\mathbf{1}_n - A) = \sum_{\sigma \in Sym(n)} \epsilon(\sigma) \prod_{i=1}^{n} (z.\mathbf{1}_n - A)_{i,\sigma(i)}$$

$$= \sum_{k=0}^{n} z^{n-k}(-1)^k \sum_{|S|=k} \sum_{\pi \in Sym(S)^{nf}} \epsilon(\pi) \prod_{i \in S} A_{i,\pi(i)}.$$

The product is 0 or 1, and is 1 if and only i is adjacent to $\pi(i)$ for every $i \in S$: this means every cycle in π is a cycle in S. As T is a tree, the only possibilities for π are disjoint transpositions associated with perfect matchings of S.

5 (If time left) The MSS proof of GG

Recall: we want to prove $\mathbf{E}_s p_{A^{(s)}} = \mu_X$
As in the above remark:

$$p_{A^{(s)}}(z) = \sum_{k=0}^{n} z^{n-k}(-1)^k \sum_{|S|=k} \sum_{\pi \in Sym(S)^{nf}} \epsilon(\pi) \prod_{i \in S} A^{(s)}_{i,\pi(i)}$$

$$= \sum_{k=0}^{n} z^{n-k}(-1)^k \sum_{|S|=k} \sum_{\pi \in Sym(S)^{nf}} \epsilon(\pi) \prod_{i \in S} s(i,\pi(i)).$$

Since $\mathbf{E}_s s(i,j) = 0$, taking $\mathbf{E}_s$ and using independence, we see that $\mathbf{E}_s(\prod_{i\in S} s(i,\pi(i))) = 0$ as soon as π has a cycle of length ≥ 3. So π contributes if and only if it is a product of disjoint transpositions, if and only if it corresponds to a perfect matching of S. Hence $\mathbf{E}_s[\sum_{\pi\in Sym(S)^{nf}} \epsilon(\pi) \prod_{i\in S} s(i,\pi(i))]$ is $(-1)^r \times |\{\text{Perfect matchings of } S\}|$ if $|S| = 2r$, and is 0 if $|S| = 2r+1$. □