Amenability and approximations of $C^\star$-algebras

Wilhelm Winter
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Richard Kadison and his mathematical legacy
Copenhagen, November 2019

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Let us recall some sample results.

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In particular, they satisfy Kadison's similarity property:
Every (bounded) representation is similar to a $\star$-representation.

THEOREM [Christensen–Sinclair–Smith–White–W]

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Separable nuclear $C^\star$-algebras satisfy the Kadison–Kastler conjecture:
They are stable under small perturbations.

More precisely, if two nuclear $C^\star$-algebras act on the same Hilbert space, if one of them is separable and nuclear, and if their unit balls are within $\frac{1}{420000}$ of each other, then they are isomorphic.

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Think of the Jiang–Su algebra $\mathcal{Z}$ as the smallest possible C*-version of the hyperfinite II_1 factor.

THEOREM [many hands]

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$$\left\{ A \otimes \mathcal{Z} \mid A \text{ separable, simple, unital, nuclear, with UCT} \right\}$$

is classified by the Elliott invariant

$$\Big(\mathrm{K}_0(A),\ \mathrm{K}_0(A)_+,\ [1_A]_0,\ \mathrm{K}_1(A),\ \mathrm{T}(A),\ \mathrm{r}_A : \mathrm{T}(A) \to \mathrm{S}(\mathrm{K}_0(A)) \Big).$$

For all of these results, understanding completely positive approximations is key.
Let us look at these in more detail now.

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A C*-algebra A has *nuclear dimension* at most d, $\dim_{\text{nuc}} A \leq d$, if there is a system

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- for each λ, $F_\lambda = F_\lambda^{(0)} \oplus \ldots \oplus F_\lambda^{(d)}$ and for each $k \in \{0, \ldots, d\}$, $\varphi_\lambda|_{F_\lambda^{(k)}}$ is contractive and has *order zero*, i.e., preserves orthogonality.

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Note that, even for $C([0,1])$, such approximations are not entirely straightforward to write down.

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When does a system $\left(F_0 \xrightarrow{\varrho_{0,1}} F_1 \xrightarrow{\varrho_{1,2}} \ldots \right)$ come from a C*-algebra?

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How to read of information on A (K-theory, traces, ...) from $\left(F_0 \xrightarrow{\varrho_{0,1}} F_1 \xrightarrow{\varrho_{1,2}} \ldots \right)$?

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What structure does the inductive limit of $\left(F_0 \xrightarrow{\varrho_{0,1}} F_1 \xrightarrow{\varrho_{1,2}} \ldots \right)$ have?

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such that for every $M \leq m < n$ we have

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For every $K \in \mathbb{N}$, $x, y \in F_K$ and $\epsilon > 0$ there are $K \leq M \in \mathbb{N}$ and $\delta > 0$
such that for every $M \leq m < n$ we have

$$\|\varrho_{K,m}(x)\varrho_{K,m}(y)\|_{F_m} < \delta \Longrightarrow \|\varrho_{K,n}(x)\varrho_{K,n}(y)\|_{F_n} < \epsilon.$$

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- the induced map $\Psi : A \longrightarrow X := \overline{\lim_{\rightarrow}(F_k, \varrho_{k,k+1})} \subset \prod_{\mathbb{N}} F_k / \bigoplus_{\mathbb{N}} F_k$ is an orthogonality preserving complete order isomorphism.

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- Ψ is an injective *-homomorphism
- the maps $\varrho_{k,k+1}$ induce affine maps $\mathrm{T}^{\leq 1}(F_k) \longleftarrow \mathrm{T}^{\leq 1}(F_{k+1})$ between the simplices of positive trace functionals sucht that $\mathrm{T}^{\leq 1}(A) \approx \lim_{\leftarrow} \mathrm{T}^{\leq 1}(F_k)$.

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Since our constructions are compatible with matrix amplification, we can describe K-theory in this context.

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Similar identities hold for unitaries and partial isometries, and so one can *always* describe K-theory for nuclear C*-algebras in terms of their approximating systems. [Joint work in progress with Kristin Courtney.]

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In other words, given

$$F_0 \xrightarrow{\varrho_{0,1}} F_1 \xrightarrow{\varrho_{1,2}} \ldots \longrightarrow X,$$

when can X be equipped with a C*-algebra structure?

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- $X^2 = eX$.

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Then, X is a *-algebra with the unique product $\bullet : X \times X \longrightarrow X$ satisfying

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With this product and norm, X becomes a unital pre-C*-algebra.