WWU MÜNSTER

Exotic group C^*-algebras of simple Lie groups with real rank one

Tim de Laat

(Joint work with Timo Siebenand)

Richard Kadison and his mathematical legacy – A memorial conference

November 30, 2019

living.knowledge

MM Mathematics Münster Cluster of Excellence

Universal and reduced group C^*-algebra

G – locally compact group

- $C_r^*(G)$ – reduced group C^*-algebra
- $C^*(G)$ – universal group C^*-algebra

Universal and reduced group C^*-algebra

G – locally compact group

- $C_r^*(G)$ – reduced group C^*-algebra
- $C^*(G)$ – universal group C^*-algebra

Theorem

$C^*(G) = C_r^*(G)$ if and only if G is amenable.

Exotic group C^*-algebras

Definition

An exotic group C^*-algebra of G is a C^*-completion A of $C_c(G)$ such that the identity map on $C_c(G)$ extends to proper quotient maps

$$C^*(G) \twoheadrightarrow A \twoheadrightarrow C_r^*(G).$$

Exotic group C^*-algebras

Definition

An exotic group C^*-algebra of G is a C^*-completion A of $C_c(G)$ such that the identity map on $C_c(G)$ extends to proper quotient maps

$$C^*(G) \twoheadrightarrow A \twoheadrightarrow C_r^*(G).$$

Question: Does every non-amenable G have exotic group C^*-algebras?

Exotic group C^*-algebras

Definition

An exotic group C^*-algebra of G is a C^*-completion A of $C_c(G)$ such that the identity map on $C_c(G)$ extends to proper quotient maps

$$C^*(G) \twoheadrightarrow A \twoheadrightarrow C_r^*(G).$$

Question: Does every non-amenable G have exotic group C^*-algebras?

This question is still open!

Why are they interesting?

- Exotic group C^*-algebras are C^*-algebras associated with groups which reflect structural properties of the underlying group.

Why are they interesting?

- Exotic group C^*-algebras are C^*-algebras associated with groups which reflect structural properties of the underlying group.
- Baum–Connes conjecture with coefficients: K-theory of reduced crossed product in terms of the topological K-theory.

Why are they interesting?

- Exotic group C^*-algebras are C^*-algebras associated with groups which reflect structural properties of the underlying group.
- Baum–Connes conjecture with coefficients: K-theory of reduced crossed product in terms of the topological K-theory. Counterexamples due to Higson–Lafforgue–Skandalis (2002).

Why are they interesting?

- Exotic group C^*-algebras are C^*-algebras associated with groups which reflect structural properties of the underlying group.
- Baum–Connes conjecture with coefficients: K-theory of reduced crossed product in terms of the topological K-theory. Counterexamples due to Higson–Lafforgue–Skandalis (2002).

  Baum–Guentner–Willet (2013): Reformulation of the conjecture. The reformulated conjecture expresses the K-theory of a certain nice special crossed product in terms of the topological K-theory.

Why are they interesting?

- Exotic group C^*-algebras are C^*-algebras associated with groups which reflect structural properties of the underlying group.
- Baum–Connes conjecture with coefficients: K-theory of reduced crossed product in terms of the topological K-theory. Counterexamples due to Higson–Lafforgue–Skandalis (2002).

  Baum–Guentner–Willet (2013): Reformulation of the conjecture. The reformulated conjecture expresses the K-theory of a certain nice special crossed product in terms of the topological K-theory.

  Can be constructed from exotic group C^*-algebras.

What is known? [Discrete groups]

First systematic approach: Ideal completions [Brown–Guentner (2013)].

What is known? [Discrete groups]

First systematic approach: Ideal completions [Brown–Guentner (2013)].

Γ countable group, D alg. two-sided ideal of $\ell^\infty(\Gamma)$. The ideal completion $C^*_D(\Gamma)$ is the completion of the group ring with respect to the norm defined through all un. reps with many matrix coefficients in D. (Matrix coefficient: $\pi_{\xi,\zeta} : g \mapsto \langle \pi(g)\xi, \zeta \rangle$.)

What is known? [Discrete groups]

First systematic approach: Ideal completions [Brown–Guentner (2013)].

Γ countable group, D alg. two-sided ideal of $\ell^\infty(\Gamma)$. The ideal completion $C^*_D(\Gamma)$ is the completion of the group ring with respect to the norm defined through all un. reps with many matrix coefficients in D.
(Matrix coefficient: $\pi_{\xi,\zeta} : g \mapsto \langle \pi(g)\xi, \zeta \rangle$.)

- $C^*_{\ell^p}(\mathbb{F}_d) \twoheadrightarrow C^*_{\ell^q}(\mathbb{F}_d)$ is not injective if $2 \leq q < p \leq \infty$ [Okayasu (2014)]. Same for discrete groups Γ with $\mathbb{F}_d < \Gamma$ [Wiersma (2016)].

What is known? [Discrete groups]

First systematic approach: Ideal completions [Brown–Guentner (2013)].

Γ countable group, D alg. two-sided ideal of $\ell^\infty(\Gamma)$. The ideal completion $C^*_D(\Gamma)$ is the completion of the group ring with respect to the norm defined through all un. reps with many matrix coefficients in D. (Matrix coefficient: $\pi_{\xi,\zeta} : g \mapsto \langle \pi(g)\xi, \zeta \rangle$.)

- $C^*_{\ell^p}(\mathbb{F}_d) \twoheadrightarrow C^*_{\ell^q}(\mathbb{F}_d)$ is not injective if $2 \leq q < p \leq \infty$ [Okayasu (2014)]. Same for discrete groups Γ with $\mathbb{F}_d < \Gamma$ [Wiersma (2016)].
- Other constructions and examples, many results due to Wiersma.

What is known? [Lie groups]

- $C^*_{\ell^p}(\mathrm{SL}(2,\mathbb{R})) \twoheadrightarrow C^*_{\ell^q}(\mathrm{SL}(2,\mathbb{R}))$ is not injective if $2 \leq q < p \leq \infty$ [Wiersma (2015)]. Analysis of the representations.

What is known? [Lie groups]

- $C^*_{\ell^p}(\mathrm{SL}(2,\mathbb{R})) \twoheadrightarrow C^*_{\ell^q}(\mathrm{SL}(2,\mathbb{R}))$ is not injective if $2 \leq q < p \leq \infty$ [Wiersma (2015)]. Analysis of the representations.
- Same for $\mathrm{SO}_0(n,1)$ and $\mathrm{SU}(n,1)$ [Samei–Wiersma (2018)]. Relies on the Kunze – Stein property and a strengthened version of the Haagerup property.

What is known? [Lie groups]

- $C^*_{\ell^p}(\mathrm{SL}(2,\mathbb{R})) \twoheadrightarrow C^*_{\ell^q}(\mathrm{SL}(2,\mathbb{R}))$ is not injective if $2 \leq q < p \leq \infty$ [Wiersma (2015)]. Analysis of the representations.
- Same for $\mathrm{SO}_0(n,1)$ and $\mathrm{SU}(n,1)$ [Samei–Wiersma (2018)]. Relies on the Kunze – Stein property and a strengthened version of the Haagerup property.

They are not so exotic.

What is known? [Lie groups]

- $C^*_{\ell^p}(\mathrm{SL}(2,\mathbb{R})) \twoheadrightarrow C^*_{\ell^q}(\mathrm{SL}(2,\mathbb{R}))$ is not injective if $2 \leq q < p \leq \infty$ [Wiersma (2015)]. Analysis of the representations.
- Same for $\mathrm{SO}_0(n,1)$ and $\mathrm{SU}(n,1)$ [Samei–Wiersma (2018)]. Relies on the Kunze – Stein property and a strengthened version of the Haagerup property.

They are not so exotic.

Question: Can we understand these results in terms of representations? What about Lie groups with property (T)?

L^p-integrability of matrix coefficients

Construct exotic group algebras of Lie groups through L^p-integrability properties of matrix coefficients of unitary representations.

L^p-integrability of matrix coefficients

Construct exotic group algebras of Lie groups through L^p-integrability properties of matrix coefficients of unitary representations.

G locally compact group, $p \in [1, \infty]$

Definition

A un. rep. $\pi \colon G \to \mathcal{B}(\mathcal{H})$ is an L^p-representation if there is a dense subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that $\pi_{\xi,\zeta} \in L^p(G)$ for all $\xi, \zeta \in \mathcal{H}_0$.

L^p-integrability of matrix coefficients

Construct exotic group algebras of Lie groups through L^p-integrability properties of matrix coefficients of unitary representations.

G locally compact group, $p \in [1, \infty]$

Definition

A un. rep. $\pi \colon G \to \mathcal{B}(\mathcal{H})$ is an L^p-representation if there is a dense subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that $\pi_{\xi,\zeta} \in L^p(G)$ for all $\xi, \zeta \in \mathcal{H}_0$.

π is an L^{p+}-representation if π is $L^{p+\varepsilon}$ for all $\varepsilon > 0$.

($f \in C_b(G) \cap L^p(G)$ implies: f is contained in $L^q(G)$ for all $q \geq p$.)

Why L^{p+}-representations?

L^{p+}-representations $\sim$ weak containment.

Theorem [Cowling–Haagerup–Howe (1988)]

Let $(\mathcal{H}, \pi, \xi)$ be a cyclic unitary representation of a locally compact group G such that $\pi_{\xi,\xi} \in L^{2+\varepsilon}(G)$ for all $\varepsilon > 0$. Then π is weakly contained in the left-regular representation.

The algebras $C^*_{L^{p+}}(G)$

G locally compact group, $p \in [2, \infty]$

$C^*_{L^p}(G)$ and $C^*_{L^{p+}}(G)$ are the completions of $C_c(G)$ with respect to the C^*-norms

The algebras $C^*_{L^{p+}(G)}$

G locally compact group, $p \in [2, \infty]$

$C^*_{L^p}(G)$ and $C^*_{L^{p+}}(G)$ are the completions of $C_c(G)$ with respect to the C^*-norms

$$\| \cdot \|_{L^p} : C_c(G) \to [0, \infty),\ f \mapsto \sup\{\|\pi(f)\| \mid \pi \text{ is an } L^p\text{-rep.}\} \text{ and}$$
$$\| \cdot \|_{L^{p+}} : C_c(G) \to [0, \infty),\ f \mapsto \sup\{\|\pi(f)\| \mid \pi \text{ is an } L^{p+}\text{-rep.}\},$$

The Kunze–Stein property

G is called Kunze–Stein if $m\colon C_c(G) \times C_c(G) \to C_c(G),\ (f,g) \mapsto f * g$ extends to a bounded bil. map $L^q(G) \times L^2(G) \to L^2(G)$ for all $q \in [1, 2)$.

The Kunze–Stein property

G is called Kunze–Stein if $m\colon C_c(G) \times C_c(G) \to C_c(G),\ (f,g) \mapsto f * g$ extends to a bounded bil. map $L^q(G) \times L^2(G) \to L^2(G)$ for all $q \in [1,2)$.

Examples:

- $\mathrm{SL}(2,\mathbb{R})$ [Kunze–Stein (1960)]

The Kunze–Stein property

G is called Kunze–Stein if $m\colon C_c(G) \times C_c(G) \to C_c(G),\ (f,g) \mapsto f * g$ extends to a bounded bil. map $L^q(G) \times L^2(G) \to L^2(G)$ for all $q \in [1,2)$.

Examples:

- $\mathrm{SL}(2,\mathbb{R})$ [Kunze–Stein (1960)]
- Connected semisimple Lie groups with finite center [Cowling (1978)] and non-Archimedean analogues [Veca (2002)]

The Kunze–Stein property

G is called Kunze–Stein if $m\colon C_c(G) \times C_c(G) \to C_c(G),\ (f,g) \mapsto f * g$ extends to a bounded bil. map $L^q(G) \times L^2(G) \to L^2(G)$ for all $q \in [1,2)$.

Examples:

- $\mathrm{SL}(2,\mathbb{R})$ [Kunze–Stein (1960)]
- Connected semisimple Lie groups with finite center [Cowling (1978)] and non-Archimedean analogues [Veca (2002)]
- Groups of automorphisms of trees [Nebbia (1988)]

The Kunze–Stein property

G is called Kunze–Stein if $m\colon C_c(G) \times C_c(G) \to C_c(G),\ (f,g) \mapsto f * g$ extends to a bounded bil. map $L^q(G) \times L^2(G) \to L^2(G)$ for all $q \in [1,2)$.

Examples:

- $\mathrm{SL}(2,\mathbb{R})$ [Kunze–Stein (1960)]
- Connected semisimple Lie groups with finite center [Cowling (1978)] and non-Archimedean analogues [Veca (2002)]
- Groups of automorphisms of trees [Nebbia (1988)]

If G is non-compact and amenable and m extends to a bounded bil. map $L^q \times L^2 \to L^2(G)$, then $q = 1$.

L^{p+}-representations of Kunze–Stein groups

For $p \in [2, \infty]$, set

$$\widehat{G}_{L^{p+}} := \{[\pi] \in \widehat{G} \mid \pi \text{ is an } L^{p+}\text{-representation}\}$$

Theorem [dL – Siebenand (2019)]

Let G be a Kunze-Stein group. Then $\widehat{G}_{L^{p+}}$ is Fell-closed in $\widehat{G}$.

L^{p+}-representations of Kunze–Stein groups

For $p \in [2, \infty]$, set

$$\widehat{G}_{L^{p+}} := \{[\pi] \in \widehat{G} \mid \pi \text{ is an } L^{p+}\text{-representation}\}$$

Theorem [dL – Siebenand (2019)]

Let G be a Kunze-Stein group. Then $\widehat{G}_{L^{p+}}$ is Fell-closed in $\widehat{G}$.

($\overline{S}$ consists of all $[\pi]$ in $\widehat{G}$ which are weakly contained in S.)

L^{p+}-representations of Kunze–Stein groups

For $p \in [2, \infty]$, set

$$\widehat{G}_{L^{p+}} := \{[\pi] \in \widehat{G} \mid \pi \text{ is an } L^{p+}\text{-representation}\}$$

Theorem [dL – Siebenand (2019)]

Let G be a Kunze-Stein group. Then $\widehat{G}_{L^{p+}}$ is Fell-closed in $\widehat{G}$.

($\overline{S}$ consists of all $[\pi]$ in $\widehat{G}$ which are weakly contained in S.)

Was known for $SO_0(n, 1)$ and $SU(n, 1)$ from work of Shalom (2000).

Simple Lie groups

Have a well-behaved and rich structure.

Simple Lie groups

Have a well-behaved and rich structure.

G connected simple Lie group $\rightsquigarrow G = K\overline{A^+}K$,
K maximal compact subgroup, A abelian

Simple Lie groups

Have a well-behaved and rich structure.

G connected simple Lie group $\rightsquigarrow G = K\overline{A^+}K$,
K maximal compact subgroup, A abelian

Real rank of $G = \dim(\mathrm{Lie}(A))$

Simple Lie groups

Have a well-behaved and rich structure.

G connected simple Lie group $\rightsquigarrow G = K\overline{A^+}K$,
K maximal compact subgroup, A abelian

Real rank of $G = \dim(\mathrm{Lie}(A))$

(G, K) is a Gelfand pair $\rightsquigarrow$ spherical functions (Harish-Chandra).
These are diagonal matrix coefficients $\pi_{\xi,\xi}$, with $\xi \in \mathcal{H}^K \backslash \{0\}$.

$(\mathcal{H}, \pi)$ is class one if $\mathcal{H}^K$ is one-dimensional

Simple Lie groups with real rank one

G – connected simple Lie group with real rank one.
Then G is locally isomorphic to one of the following Lie groups:

$$\mathrm{SO}(n,1) = \{g \in \mathrm{SL}(n+1,\mathbb{R}) \mid g^* I_{n,1} g = I_{n,1}\},$$
$$\mathrm{SU}(n,1) = \{g \in \mathrm{SL}(n+1,\mathbb{C}) \mid g^* I_{n,1} g = I_{n,1}\},$$
$$\mathrm{Sp}(n,1) = \{g \in \mathrm{GL}(n+1,\mathbb{H}) \mid g^* I_{n,1} g = I_{n,1}\},$$
$$\mathrm{F}_{4(-20)}.$$

First three: Isometry groups of the classical rank one symmetric spaces of the non-compact type. Class one representation theory is well understood.

Locally compact group G:

$$\Phi(G) := \inf\{p \in [1,\infty] \mid \forall\, \pi \in \widehat{G} \setminus \{\tau_0\}, \pi \text{ is an } L^{p+}\text{-representation}\},$$

where τ_0 is the trivial representation.

Locally compact group G:

$$\Phi(G) := \inf\{p \in [1,\infty] \mid \forall \pi \in \widehat{G}\backslash\{\tau_0\}, \pi \text{ is an } L^{p+}\text{-representation}\},$$

where τ_0 is the trivial representation.

For the classical real rank one Lie groups:

$$\Phi(G) = \begin{cases} \infty & \text{if } G = \mathrm{SO}_0(n,1), \\ \infty & \text{if } G = \mathrm{SU}(n,1), \\ 2n+1 & \text{if } G = \mathrm{Sp}(n,1). \end{cases}$$

First two cases: Harish-Chandra
$\mathrm{Sp}(n,1)$: Li (1995).

Theorem [dL – Siebenand (2019)]

Let G be a classical simple Lie group with real rank one and finite center. Then for $2 \leq q < p \leq \Phi(G)$, the canonical quotient map

$$C^*_{L^{p+}}(G) \twoheadrightarrow C^*_{L^{q+}}(G)$$

has non-trivial kernel. Furthermore, for every $p, q \in [\Phi(G), \infty)$, we have

$$C^*_{L^{p+}}(G) = C^*_{L^{q+}}(G).$$

Theorem [dL – Siebenand (2019)]

Let G be a classical simple Lie group with real rank one and finite center. Then for $2 \leq q < p \leq \Phi(G)$, the canonical quotient map

$$C^*_{L^{p+}}(G) \twoheadrightarrow C^*_{L^{q+}}(G)$$

has non-trivial kernel. Furthermore, for every $p, q \in [\Phi(G), \infty)$, we have

$$C^*_{L^{p+}}(G) = C^*_{L^{q+}}(G).$$

For $F_{4(-20)}$, we get partial results.

Theorem [dL – Siebenand (2019)]

Let G be a classical simple Lie group with real rank one and finite center. Then for $2 \leq q < p \leq \Phi(G)$, the canonical quotient map

$$C^*_{L^{p+}}(G) \twoheadrightarrow C^*_{L^{q+}}(G)$$

has non-trivial kernel. Furthermore, for every $p, q \in [\Phi(G), \infty)$, we have

$$C^*_{L^{p+}}(G) = C^*_{L^{q+}}(G).$$

For $F_{4(-20)}$, we get partial results.

For finite coverings, the result is the same.

About the proof

- By the earlier theorem, the first claim reduces to finding L^{p+}-representations which are not L^{q+} for $2 \leq q < p \leq \Phi(G)$.

About the proof

- By the earlier theorem, the first claim reduces to finding L^{p+}-representations which are not L^{q+} for $2 \leq q < p \leq \Phi(G)$.
- Consider the strip of class one complementary series representations.

About the proof

- By the earlier theorem, the first claim reduces to finding L^{p+}-representations which are not L^{q+} for $2 \leq q < p \leq \Phi(G)$.
- Consider the strip of class one complementary series representations.
- Asymptotics (L^p-integrability) of spherical functions follows from Harish-Chandra's rich work. One can realize all necessary L^p-integability in this strip.

About the proof

- By the earlier theorem, the first claim reduces to finding L^{p+}-representations which are not L^{q+} for $2 \leq q < p \leq \Phi(G)$.
- Consider the strip of class one complementary series representations.
- Asymptotics (L^p-integrability) of spherical functions follows from Harish-Chandra's rich work. One can realize all necessary L^p-integability in this strip.
- The second claim follows from a result of Cowling (1979). (Quantitative version of property (T).)

Concluding remarks

Our approach also works for groups of automorphisms of trees.
[Samei – Wiersma (2018), Heinig – dL – Siebenand (2019)]

Concluding remarks

Our approach also works for groups of automorphisms of trees.
[Samei – Wiersma (2018), Heinig – dL – Siebenand (2019)]

What about

- simple Lie groups with higher real rank: $\mathrm{SL}(n,\mathbb{R})$ $(n \geq 3)$,
- almost simple algebraic groups over non-Archimedean local fields?

Concluding remarks

Our approach also works for groups of automorphisms of trees.
[Samei – Wiersma (2018), Heinig – dL – Siebenand (2019)]

What about

- simple Lie groups with higher real rank: $\mathrm{SL}(n, \mathbb{R})$ $(n \geq 3)$,
- almost simple algebraic groups over non-Archimedean local fields?

Another question:

- Are the algebras $C^*_{L^{p+}}(G)$ the only exotic group C^*-algebras coming from ideals in the Fourier–Stieltjes algebra?