Normalizers of group algebras and mixing

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1 Introduction

Recall that if $1 \in B \subset M$ is a pair of von Neumann algebras, the **normalizer** of B in M is the group

$$\mathcal{N}_M(B) = \{u \in U(M) : uBu^* = B\}.$$

Problem. Let G be a (discrete) group and let H be a subgroup of G. What can be said about the normalizer $\mathcal{N}_{L(G)}(L(H))$ of the von Neumann subalgebra $L(H) \subset L(G)$?

For instance, if $G \curvearrowright (Q, \tau)$, when is it true that

$$\mathcal{N}_{Q \rtimes G}(Q \rtimes H)'' = Q \rtimes \mathcal{N}_G(H)?$$

J. Fang, M. Gao & R. R. Smith, Internat. J. Math., 2011: If $H < G$ is an abelian subgroup such that $L(H)$ is a MASA in $L(G)$, then

$$\mathcal{N}_{L(G)}(L(H))'' = L(\mathcal{N}_G(H)).$$

In particular, $L(H)$ is a singular MASA if $\mathcal{N}_G(H) = H$. It follows from our last example that the equality can fail if $L(H)$ is not a MASA in $L(G)$.

Goal.
We look for conditions on the triple $H < \mathcal{N}_G(H) < G$ in order that the above equality holds.

Motivation.
I. Chifan (2006) proved that the triple

$$A \subset \mathcal{N}_M(A)'' \subset M$$

has a property named *relative weak asymptotic homomorphism property* (see Definition below), and he used it to prove that, if $(M_i)_{i \geq 1}$ is a sequence of finite von Neumann algebras and if $(A_i)_{i\geq 1}$ is a sequence such that $A_i \subset M_i$ is a MASA for every i, then

$$\overline{\bigotimes}_{i\geq 1}\mathcal{N}_{M_i}(A_i)'' = (\mathcal{N}_{\overline{\otimes}_i M_i}(\overline{\bigotimes}_i A_i))''.$$

2 Weak mixing and one-sided quasi-normalizers

Definitions. Let $1 \in B \subset N \subset M$ be a triple of finite von Neumann algebras.
(1) We say that B is **weakly mixing in** M **relative to** N if $\exists (u_i)_{i\in I} \subset U(B)$ s.t.

$$\lim_{i\in I} \|\mathbb{E}_B(xu_iy) - \mathbb{E}_B(\mathbb{E}_N(x)u_i\mathbb{E}_N(y))\|_2 = 0$$
$$(= \lim_{i\in I} \|\mathbb{E}_B(xu_iyu_i*) - \mathbb{E}_B(\mathbb{E}_N(x)u_i\mathbb{E}_N(y)u_i^*)\|_2)$$

for all $x, y \in M$. One also says that $B \subset N \subset M$ has the **relative weak asymptotic homomorphism property** (Sinclair & Smith (Geom. Funct. Anal., 2002), Fang, Gao & Smith (2011)).
Compare with the case where $G \curvearrowright (Q,\tau)$: the action is weakly mixing iff there exists $(g_i)_{i\in I}$ s.t.

$$|\tau(a\alpha_{g_i}(b) - \tau(a)\tau(b)| \to 0 \quad \forall a, b \in Q.$$

(Here we consider the action of $U(B)$ on M by conjugation.)
(2) The **one-sided quasi-normalizer** of B in M is the set of elements $x \in M$ for which $\exists\{x_1, \ldots, x_n\} \subset M$ such that

$$Bx \subset \sum_{i=1}^{n} x_i B.$$

We denote the set of these elements by $q\mathcal{N}_M^{(1)}(B)$.

Theorem 1. (J. Fang, M. Gao & R. R. Smith, 2011) *Let* $1 \in B \subset N \subset M$ *be as above. Then the following conditions are equivalent:*

(1) B *is weakly mixing in* M *relative to* N*;*

(2) $q\mathcal{N}_M^{(1)}(B) \subset N$.

In particular, for arbitrary $B \subset M$*,* B *is weakly mixing in* M *relative to* $W^*(q\mathcal{N}_M^{(1)}(B))$.

The case of group algebras.
Let $H < G$ be a pair of groups. The one-sided quasi-normalizer has a natural analogue: Denote by $q\mathcal{N}_G^{(1)}(H)$ the set of elements $g \in G$ for which $\exists F \subset G$ finite s.t. $Hg \subset FH$.

Question. (R. Smith) Assume that $H < K < G$ is a triple of groups. Is it true that $L(H)$ is weakly mixing in $L(G)$ relative to $L(K)$ if and only if $q\mathcal{N}_G^{(1)}(H) \subset K$?

Fang, Gao & Smith: True if $K = H$, as a corollary of their Theorem 1.

Theorem 2. (J, 2010) Let $H < K < G$ be as above. TFAE:
(1) $L(H)$ *is weakly mixing in* $L(G)$ *relative to* $L(K)$*;*
(2) $\exists (h_i)_{i\in I} \subset H$ *s.t.*

$$\lim_{i\in I} \|\mathbb{E}_{L(H)}(x\lambda_{h_i}y) - \mathbb{E}_{L(H)}(\mathbb{E}_{L(K)}(x)\lambda_{h_i}\mathbb{E}_{L(K)}(y))\|_2 = 0$$

for all $x, y \in L(G)$;
(3) $q\mathcal{N}_G^{(1)}(H) \subset K$;
(4) $H < K < G$ *satisfies* ***condition (SS)****: for every finite* $F \subset G \setminus K$, $\exists h \in H$ *s.t.* $FhF \cap H = \emptyset$*;*
(5) *the subspace of H-fixed vectors* $\ell^2(G/H)^H$ *in the quasi-regular representation* mod H *is contained in* $\ell^2(K/H)$.

It turns out that if the triple $H < K < G$ satisfies condition (SS) and if $G \curvearrowright (Q, \tau)$, then $Q \rtimes H$ is weakly mixing in $Q \rtimes G$ relative to $Q \rtimes K$, and we get:

Theorem 3. (J, 2011) *Let $H < G$ be a pair of groups, and put $K = \mathcal{N}_G(H)$. If $H < K < G$ satisfies condition (SS), then*

$$\mathcal{N}_{L(G)}(L(H))'' = L(K).$$

Moreover, if $G \curvearrowright (Q, \tau)$, then

$$Q \rtimes K = \mathcal{N}_{Q\rtimes G}(Q \rtimes H)''.$$

Comments.
(i) Condition (SS) was first introduced by Robertson, Sinclair and Smith in 2003 for pairs of groups $H < G$ where H is abelian. They proved that if it is the case, then $L(H)$ is a (strongly) singular MASA of $L(G)$.
(ii) A relative weakly mixing condition was introduced by S. Popa in 2005: If $1 \in B \subset M$ and if $\Gamma \curvearrowright_\alpha M$ so that $\alpha_g(B) = B$ for every g, then α is called

weakly mixing relative to B if, for every finite subset $F \subset \ker(\mathbb{E}_B)$ and for every $\varepsilon > 0$, one can find $g \in \Gamma$ such that

$$\|\mathbb{E}_B(x\alpha_g(y))\|_2 < \varepsilon \quad \forall x, y \in F.$$

3 Examples

The special case $H = K$.

If $H < G$ satisfies condition (SS), the equality $\mathcal{N}_G(H) = H$ is automatic. Thus, if $G \curvearrowright (Q, \tau)$,

$$\mathcal{N}_{Q \rtimes G}(Q \rtimes H)'' = Q \rtimes H.$$

Geometric examples (Robertson, Sinclair & Smith (2003)): Assume that $G \curvearrowright (X, d)$ and $\exists Y \subset X$ s.t. Y is H-invariant and satisfies two conditions:

(C1) there is a compact set $C \subset Y$ such that $Y = HC$;
(C2) if $Y \subset_\delta \bigcup_{g \in F} gY$ for some $\delta > 0$ and $F \subset G$ finite, then $F \cap H \neq \emptyset$.
Then $H < G$ satisfies condition (SS).
It is the case if $G = \Gamma$ is a cocompact lattice of a suitable s.s. Lie group $\mathcal{G}$, hence

(1) X is a symmetric space,

(2) $\Gamma = \pi(M)$ for M a compact manifold

(3) and $H = \mathbb{Z}^r = \pi(T^r)$ where r is the rank of X.

(Almost) malnormal subgroups.
For $g, h \in G$, set $E(g, h) = \{\gamma \in H : g\gamma h \in H\} = g^{-1}Hh^{-1} \cap H$.
In particular, $E(g) := E(g^{-1}, g) = gHg^{-1} \cap H$ is a subgroup of G.
If $g, h \in G$, for arbitrary $\gamma_0 \in E(g, h)$, one has $E(g, h) \subset E(g^{-1})\gamma_0$.

Proposition. *TFAE:*
(1) $E(g, h)$ *is finite for all* $g, h \in G \setminus H$;
(2) $E(g)$ *is finite for every* $g \in G \setminus H$;
(3) *(Condition (ST), J & Y. Stalder, 2008) for every nonempty finite* $F \subset G \setminus H$, $\exists E \subset H$ *finite s.t.*

$$FhF \cap H = \emptyset \quad \forall h \in H \setminus E.$$

If (1)-(3) hold, H is **almost malnormal** in G. It is **malnormal** if $gHg^{-1} \cap H = \{1\}$ for all $g \notin H$.

Examples.
(1) Let $F = \langle x_0, x_1, \dots | x_i^{-1} x_n x_i = x_{n+1}, 0 \leq i < n \rangle$ be Thompson's group F and let $H = \langle x_0 \rangle$. Then H is malnormal in G (J, 2005).
(2) Suitable HNN-extentions, e.g. let $n \neq m$ be positive integers; then let $G = BS(m,n) = \langle a, b | ab^m a^{-1} = b^n \rangle$ be the associated **Baumslag-Solitar group**. Then $\langle a \rangle$ is malnormal in G.
(3) Let H and H' be non trivial groups. Then H is malnormal in $H * H'$.
See recent arXiv article by de la Harpe & Weber for more examples.

Almost malnormality has a characterisation in terms of $L(H) \subset L(G)$: H is almost malnormal in G iff for every net $(u_i)_{i \in I} \subset U(L(G))$ s.t. $u_i \to 0$ weakly, then

$$\lim_{i \in I} \|\mathbb{E}_{L(H)}(x u_i y)\|_2 = 0 \quad \forall x, y \in \ker(\mathbb{E}_{L(H)}).$$

The latter property is a strongly mixing condition on the pair $L(H) \subset L(G)$.

Semidirect products and generalized wreath products.

Let K be a (countable) group acting on some (countable) group A, and assume that K contains some subgroup H. Set $G = A \rtimes K$.
E.g. G can be a generalized wreath product group: Assume that $K \curvearrowright X$ where X is a countable set, and let Γ be a nontrivial (countable) group. Then K acts in a natural way on $A = \Gamma^{(X)}$ by left translation and $G = \Gamma^{(X)} \rtimes K =: \Gamma \wr_X K$ is the corresponding wreath product.

Let H, K, A and $G = A \rtimes K$ be as above.
Assumptions in Theorem 3 depend on the action $K \curvearrowright A$:
(a) Assume that $H \triangleleft K$. If e is the only element $a \in A$ s.t. $h \cdot a = a \ \forall h \in H$, then $\mathcal{N}_G(H) = K$.
(b) $H < K < G$ satisfies condition (SS) iff $\forall E \subset A \setminus \{e\}$ finite, $\exists h \in H$ s.t. $E \cap h \cdot E = \emptyset$ (iff, for every $a \in A \setminus \{e\}$, its H-orbit $H \cdot a$ is infinite (Neumann)).

Assume that $K \curvearrowright X$ as above, and take $A = \Gamma^{(X)}$. Then condition (b) above is satisfied iff the action of $K \curvearrowright X$ has infinite orbits.

Kechris & Tsankov, 2007: Let (Y, ν) be a standard probability space. Then $K \curvearrowright X$ has infinite orbits iff the generalized Bernoulli shift action of K on (Y^X, ν^X) is weakly mixing.

Counterexamples for normalizers.
Take H, K and A s.t.

(i) $\forall a \in A \setminus \{e\}$, $\exists h \in H$ s.t. $h \cdot a \neq a$ (i.e. $\mathcal{N}_G(H) = K$);

(ii) $\exists a_0 \in A \setminus \{e\}$ s.t. $H \cdot a_0$ is finite (i.e. $H < K < G$ does not satisfy condition (SS)).

Then $L(K) \subsetneqq \mathcal{N}_{L(G)}(L(H))''$. More precisely, one can find a unitary element $u \in L(H)' \cap L(G)$ s.t. $u \notin L(K)$.