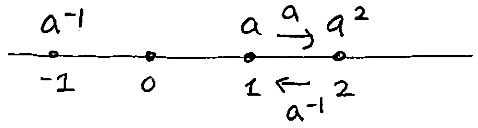
Kaimanovich 1

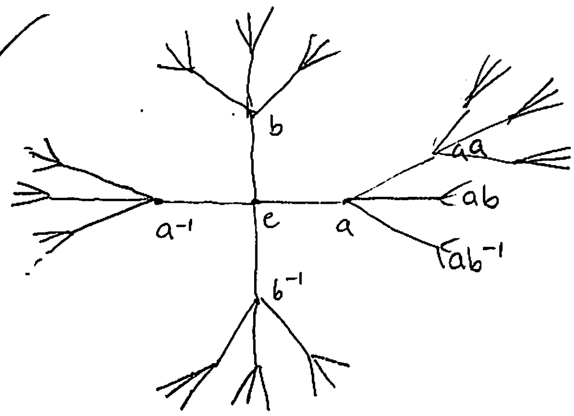
Subject: infinite countable groups

1) $\mathbb{Z} = \langle a \rangle = \{a^n\}_{n\in\mathbb{Z}}$



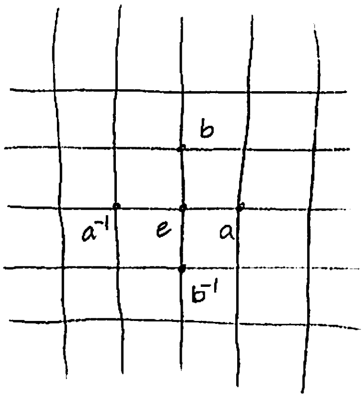
Cayley graph.

2) $F_2 = \langle a, b \rangle$ $\quad aba^{-1}, \; b^{-1}a^2, \ldots$ irreducible words.



Homogeneous tree of degree 4.

$\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$



amenability

(generalization of finiteness)

G is finite $\Leftrightarrow$ $\exists\, m \in \mathcal{P}(G)$ s.t. $g \cdot m = m$ $\forall g \in G$

(probability measures on G.
$m = \{ m(g) \mid m(g) \geq 0, \sum_{g\in G} m(g) = 1 \}$)

Here, $(g\cdot m)(x) = m(g^{-1}x)$, $x \in G$.

(If G: finite $\quad m = \sum_{g\in G} \frac{1}{|G|}\delta_g$ works.

If G infinite $\quad g\cdot m = m$, $\forall g$

$\Rightarrow m \equiv 0$ contradiction)

$\mathcal{P}(G) \subseteq \ell^1(G)_{+,1} \subseteq \ell^1(G)$

$(\ell^1)^* = \ell^\infty$ so $\ell^1 \subseteq (\ell^\infty)^*_+$

$\varphi \in \mathcal{M}(G) = (\ell^\infty)^*_{+,=1}$

space of all means on G

mean : finitely additive measure on G.

$\varphi(A) = \langle \varphi, 1_A \rangle \quad A \subseteq G.$

<u>Von Neumann</u> G is amenable $\Leftrightarrow \exists m \in \mathcal{M}(G)$ s.t. $\forall g \in G\ [g.m = m]$.

equivalent convenient def : $\exists (m_n)_{n=1}^{\infty} \subseteq \mathcal{P}(G)$

(Day-Reiter, 50s.) s.t. $\forall g \in G \quad \| g \cdot m_n - m_n \|_1 \xrightarrow{n \to \infty} 0$

$\| v \|_1 = \sum_{g \in G} | v(g) |$ (Total variation)

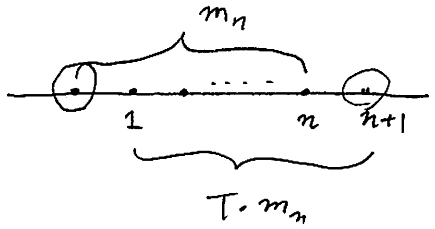
<u>Fixed point property</u>

G is amenable $\Leftrightarrow$ any continuous action of G on a compact space has an invariant measure.

Krylov - Bogolyubov (1935)
$G = \mathbb{Z}, (\Rightarrow)$.

<u>Cesaro</u> $m_n = \frac{1}{n}(\delta_1 + \cdots + \delta_n)$ $\quad \| m_n - T \cdot m_n \|_1 = \frac{1}{n} \| \delta_1 - \delta_{n+1} \|_1$

$= \frac{2}{n} \to 0$



$\mathbb{Z} \curvearrowright K$: cpt $\quad \lambda \in \mathcal{P}(K)$

$m_n * \lambda$

$\| m_n * \lambda - T m_n * \lambda \|$

$\leq \| m_n - T m_n \| \to 0$

Action $G \times K \to K$

$\rightsquigarrow$

$\mathcal{P}(G) \times \mathcal{P}(K) \to \mathcal{P}(K)$

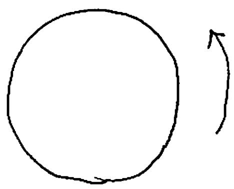
$(m, \lambda) \mapsto m * \lambda$

$\mathcal{P}(K)$, w^*– compact.

$(m_n * \lambda)$ has a w^*–lim pt. $\rightsquigarrow$ G–invariant.

Free groups are non-amenable.

S^1 $\qquad \alpha \in \mathbb{R} \setminus \mathbb{Q}$



$T_1 : x \mapsto x + \alpha \mod 1$

uniquely ergodic.

(unique inv – measure = Lebesgue on S^1)

Take another homeo that doesn't preserve Lebesgue T_2

$\mathbb{F}_2 \curvearrowright S^1 \quad a \mapsto T_1 \quad b \mapsto T_2$

doesn't have an invariant measure!

Boundary actions

∂F : boundary.

$$\begin{array}{ccc} G \times G & \to & G \\ \downarrow & & \downarrow \\ G \times \partial G & \to & \partial G \end{array}$$

has to extend

(right) infinite irreducible words.

no invariant measures on ∂F

$a^{\pm}$ $\qquad a^{\infty} = aa\cdots$

$a^n \; b^{\pm}$ $\qquad a^{-\infty} = a^{-1}a^{-1}\cdots$

$a^n \gamma \to a^{\infty}$ for $\gamma \neq a^{-\infty}$

1) Space of ends (Freudenthal 1940's)

countable graph X (locally finite, connected, etc)

$K \subseteq X$. finite subset of vertices $\qquad \mathcal{E}(X \setminus K)$ = the set of connected components. (finite)

$K \subseteq K' \quad \mathcal{E}(X \setminus K') \to \mathcal{E}(X \setminus K)$

$K_1 \subseteq K_2 \subseteq \cdots\cdots \subseteq K_n \subseteq \cdots\cdots \quad \bigcup K_n = X.$

$\mathcal{E}_{K_1} \leftarrow \mathcal{E}_{K_2} \leftarrow \cdots \leftarrow \mathcal{E}_{K_n} \leftarrow \cdots$

$\varprojlim \mathcal{E}_{K_n} = \mathcal{E}(X)$ the space of ends.

does not depend on the choice of generating set.

<u>Stallings.</u> $1, 2, \infty$:

$\mathbb{Z}$ free groups and like

Busemann compactification (boundary)

(X, d) metric space. $\quad K : X \times X \longrightarrow \mathbb{R}$

$X \ni x \longmapsto K(x, \cdot) \in \mathrm{func}(X, \mathbb{R})$ space of maps $X \longrightarrow \mathbb{R}$

$X \hookrightarrow \mathrm{func}(X, \mathbb{R})$ and compactify X

$d : X \times X \longrightarrow \mathbb{R}$

$X \ni x \longmapsto d(x, \cdot)/\mathrm{const}$ (functions are identified up to additive constants.)

$o \in X \qquad X \ni x \longmapsto \varphi_x$

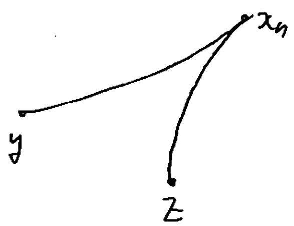
$\varphi_x(y) = d(x, y) - d(x, o)$

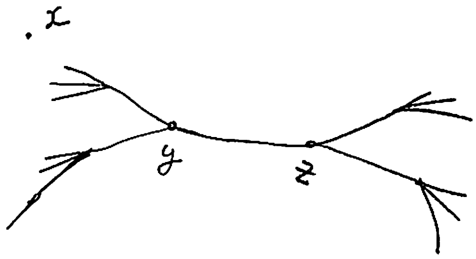
(x_n) convergent $\iff \varphi_{x_n}$ converge pointwise.

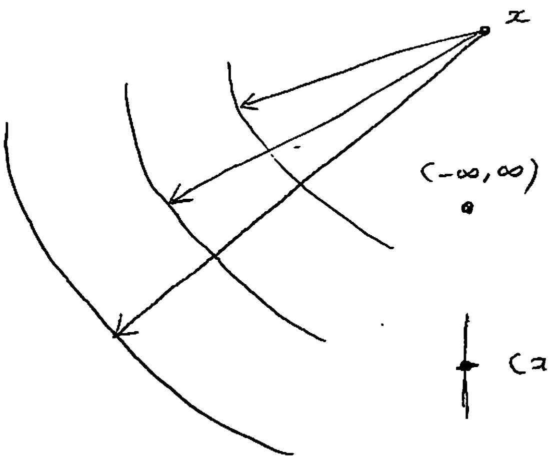
$\iff d(x_n, y) - d(x_n, o)$ converge $\forall y \in X$.

$x \in X \quad \beta_x(y, z) = d(x, y) - d(x, z)$

x_n converges $\iff \beta_{x_n}$ converges







spheres $\to$ horospheres
(limit spheres)