Reminder Kaimanovich 3 (V+VI)

G : countable group

$\mu \in P(G)$

$\Pi_e = \mu \quad \Pi_g = g\Pi_e = g\mu$.

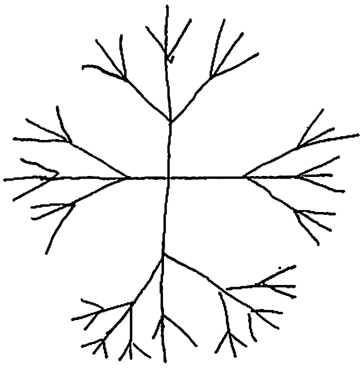
$x_0 h_1 h_2 \cdots h_n = x_n$.

$h_1 h_2 \cdots h_n$ when $x_0 = e$. $h_i \sim \mu$.

Take

$G = F_2 =: F \quad K$

$\mu = \frac{1}{4}(\underbrace{\delta_a + \delta_b + \delta_{a^{-1}} + \delta_{b^{-1}}}_{K})$



$RW(F_2, \mu) = SRW(\mathrm{Cayley}(F, K))$

$\text{bnd}: \{x_n\} \longrightarrow x_\infty \in \partial F$ boundary map

Suppose now that

μ is an arbitrary measure.

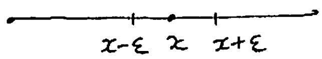
boundary convergence ?

Ans : Yes!

1) Krylov – Bogolubov thm (existence of invariant means)

2) Martingale convergence thm.

Lebesgue differentiation theorem

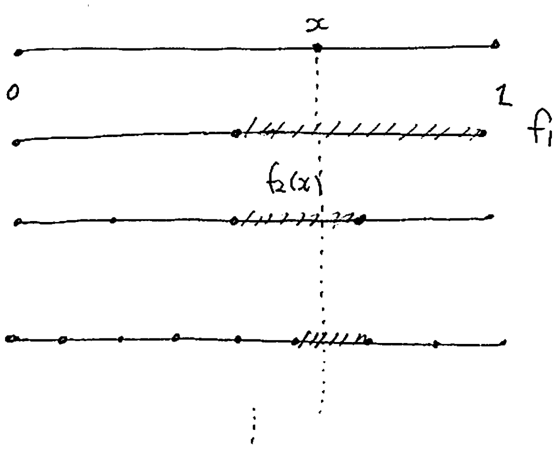


$f \in L^\infty(0,1)$

$x \in (0,1) \quad f_\varepsilon(x) = \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} f(t)dt \quad (\varepsilon > 0)$

Then $f_\varepsilon(x) \xrightarrow{\varepsilon \to 0} f(x)$

for a.e. $x \in (0,1)$.



$f_n(x)$ is the average of f over the order n interval which contains x

$f_n(x) \longrightarrow f(x)$ a.e. x.

$f_0(x) = \text{constant} \left(= \int_0^1 f(t)dt\right)$

f_1 is measurable wrt first order partition.

$\vdots$

f_n is measurable wrt n-th order partition.

$f_n = \mathbb{E}(f \mid \mathcal{O}_n)$ (conditional expectation)

$\mathcal{O}_n$ algebra of sets in n-th order partition

$= \mathbb{E}(f_{n+1} \mid \mathcal{O}_n)$

Convergence of conditional probabilities.

(Doob, Lévy)

<u>Thm</u>

$$\left(\begin{array}{l} \mathcal{O}_1 \subseteq \mathcal{O}_2 \subseteq \cdots \subseteq \mathcal{O}_n \nearrow \mathcal{O}_\infty \\ \mathbb{E}(f \mid \mathcal{O}_n)(x) \xrightarrow{n \to \infty} \mathbb{E}(f \mid \mathcal{O}_\infty)(x) \end{array}\right)$$

$= f(x)$ for a.e. x

<u>Martingale Theorem (Doob)</u>

$$\left(\begin{array}{l} \mathcal{O}_1 \subseteq \mathcal{O}_2 \subseteq \cdots \\ \text{Let } f_n \in L^\infty(\mathcal{O}_n) \text{ s.t. } \mathbb{E}(f_{n+1} \mid \mathcal{O}_n) = f_n \quad \forall n \geq 1. \text{ (Martingale Condition)} \\ \text{Then } f_n \xrightarrow{\text{a.e.}} \exists f_\infty \end{array}\right)$$

μ on F $\quad F \curvearrowright \partial F \quad F \times \partial F \to \partial F$

prob. measures

$\rightsquigarrow (\mu, \lambda) \longmapsto \mu * \lambda$ convolution

<u>claim</u>

$$\left[\begin{array}{l} \exists \lambda \in \mathrm{Prob}(\partial F) \text{ s.t. } \mu * \lambda = \lambda \\ (\text{Such } \lambda \text{ is called } \underline{\mu\text{-stationary}}) \end{array}\right]$$

$$\begin{array}{cccc} T_\mu : & \mathrm{Prob}(\partial F) & \longrightarrow & \mathrm{Prob}(\partial F) \\ & \Psi & & \Psi \\ & \lambda & \longmapsto & \mu * \lambda \end{array}$$

$$\lambda_n := \frac{\mu*\lambda + \mu*\mu*\lambda + \cdots + \overbrace{\mu*\mu*\cdots*\mu}^{n+1}*\lambda}{n}$$

$$\mu*\lambda_n = \frac{\mu*\mu*\lambda + \cdots + \overbrace{\mu*\cdots*\mu}^{n}*\lambda + \overbrace{\mu*\mu*\cdots*\mu}*\lambda}{n}$$

$$\| \lambda_n - \mu*\lambda_n \| \leq \frac{2}{n}$$

∂F cpt $\rightsquigarrow$ $\mathrm{Prob}(\partial F)$ weak*-cpt.

$\rightsquigarrow$ λ_∞ = weak*- limit pt of $(\lambda_n)_{n=1}^{\infty}$.

Then λ_∞ is μ-stationary

$\mu*\lambda = \lambda$. $\hat{f} \in C(\partial F)$

$f(g) = \langle \hat{f}, g\lambda \rangle$

a function on F.

$\mu*\lambda = \lambda$ means

$$\sum_{g\in G} \mu(g)\, g\lambda = \lambda$$

$\Downarrow$

$$\sum_{g\in G} f(g)\mu(g) = f(e) \quad \text{mean value property}$$

f is a μ-harmonic function, $f = P_\mu f$, where

$$P_\mu f = \sum_{h\in G} f(gh)\mu(h)$$

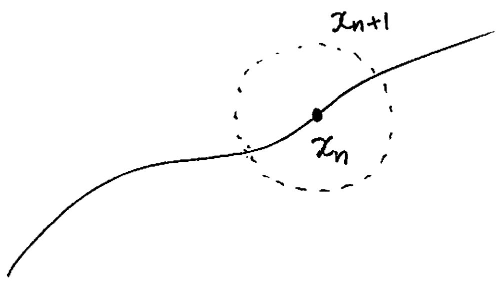
harmonic functions $\Longleftrightarrow$ martingales.

$(X^{\mathbb{Z}_+}, \mathbb{P})$ $\quad \mathcal{a}_n = \sigma(x_0, x_1, \cdots, x_n)$

$F_n : \{x_n\}_{n=1}^{\infty} \longmapsto f(x_n)$

sample path

$\{F_n\}_{n=1}^{\infty}$ is a martingale $\Longleftrightarrow$ f is a μ-harmonic function.



F, μ

$\partial F, \lambda \qquad \mu * \lambda = \lambda.$

$\hat{f} \in C(\partial F) \quad f(g) = \langle \hat{f}, g\lambda \rangle \quad f \in H^\infty(G, \mu)$

bounded, μ-harmonic.

$F_n(\{x_m\}) = f(x_n)$ is martingale

$\Rightarrow F_n(\{x_m\}) = f(x_n)$ converges a.e.

$\langle \hat{f}, x_n\lambda \rangle$

(So far we haven't used free group property
only the cpt-ness of ∂F)

Now $\boxed{\partial F \text{ is separable}}$

$\downarrow$ move quantifiers (N_n : null $\Rightarrow N = \bigcup N_n$: null)

for a.e. $\{x_n\}$ and any $\hat{f} \in C(\partial F)$

$\langle \hat{f}, x_n\lambda \rangle$ converges

For a.e. $\{x_n\}$ $\; x_n\lambda$ weak*-converges.

$\boxed{\text{used}}$

(∂F is compact and separable.)

<u>claim</u> Let λ be a non-atomic measure on ∂F.

(If $g_n \in F \quad g_n \to \infty, \qquad g_n\lambda \longrightarrow \mathcal{H}$

$\Rightarrow g_n \longrightarrow \exists \gamma \in \partial F, \; \mathcal{H} = \delta_\gamma$)

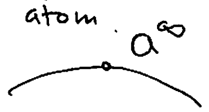
Enough to show :

If $g_n \to \gamma, \; g_n\lambda \to \mathcal{H} \Rightarrow \mathcal{H} = \delta_\gamma$

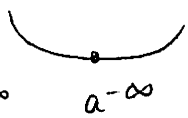
(Then by compactness we are done)

[<u>Not true</u> if λ has an atom

$g_n = a^n$



$\lambda = \frac{1}{2}(\delta_{a^\infty} + \delta_{a^{-\infty}})$



$g_n \cdot \lambda = \lambda \not\to \delta_{a^\infty}$]

$gr(\mu)$ is non-elementary = fixes no finite subset $A \subseteq \partial F$.

$gr(\mu)$ is non-amenable

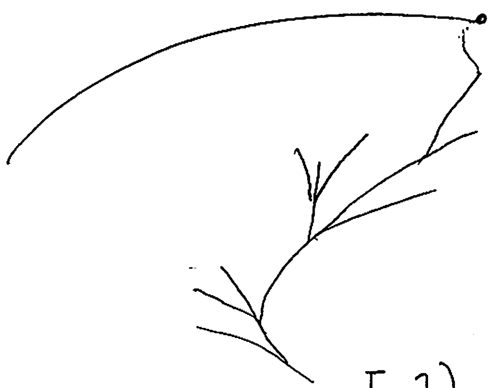
If $gr(\mu)$ is not cyclic then boundary convergence.

G : arbitrary group.

μ : a prob. measure

(G, μ) - can we assign a boundary ?

Poisson boundary



B. bnd : $\{x_n\} \longrightarrow x_\infty \in B$.

$\mathbb{P} \longrightarrow \nu$ on B.

should satisfy

1) measurable

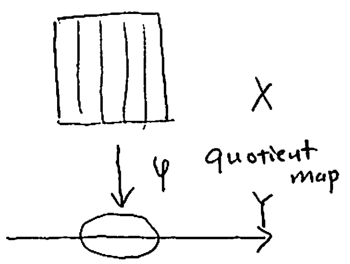
2) If two sample paths are eventually equal (after some time shift)

$$x_{n+i} = x'_{n+i'} \quad n \gg 1$$

Then $\mathrm{bnd}(\{x_n\}) = \mathrm{bnd}(\{x'_n\})$

Lebesgue (- Rokhlin) measure spaces

(e.g. any Polish space with Borel measure.)



preimage sub σ-alg.

$\varphi^{-1}(\mathcal{B}(Y)) \subseteq \mathcal{B}(X)$.

any σ-subalg of $\mathcal{B}(X)$ (complete mod 0 sets) is of this form.

Take the σ-algebra in the path space, which consists of $\sim$ classes.

$\Downarrow$ Rokhlin

certain quotient map. $G^{\mathbb{Z}_+} \xrightarrow{\varphi} B$. Poisson boundary.

$$\begin{array}{ccc} (G^{\mathbb{Z}_+}, \mathbb{P}) & & \\ \downarrow & \searrow & \\ (\Gamma, \nu) & \longrightarrow & \underline{(B, \lambda)} \end{array}$$

Poisson bdry (Γ, ν)

another space satisfying (1) (2).

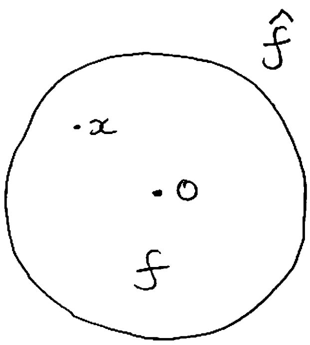
$$L^\infty(\Gamma, \nu) \cong H^\infty(G, \mu)$$ Poisson formula.

$$\hat{f} \longmapsto f(g) = \langle \hat{f}, g\nu \rangle$$

equiv class $\sim$.

$f \in H^\infty(G, \mu)$ $\quad f(x_n) \longrightarrow \hat{f}([\{x_n\}])$

$\rightsquigarrow \hat{f} \in L^\infty(\Gamma, \mu)$



$\mathbb{D} \subseteq \mathbb{C}$.

Poisson kernel

Lebesgue

$$f(x) = \int \hat{f}(\gamma) \underbrace{\Pi(x, \gamma)\, dm(\gamma)}$$

$$= \int \hat{f}(\gamma)\, d\nu_x(\gamma)$$

Brownian motion starting from x.

<u>Question</u>

1) When is the boundary trivial?

2) If the Poisson boundary is non-trivial, can we describe the space?