1) F, μ $\quad gr(\mu) = gr(supp(\mu))$ is not cyclic.

2) G – amenable group.

Lamplighter groups.

$\mathcal{G}$: a countable group.

$fun(\mathcal{G}, \mathbb{Z}_2)$ – finite support. functions (space of all configurations)

is a group with a pointwise addition.

$$G = \mathcal{L}(\mathcal{G}) = \mathcal{G} \ltimes fun(\mathcal{G}, \mathbb{Z}_2) \quad (\text{amenable})$$

$A, B \quad T: A \to Aut(B)$

$(a, b)(a', b') = (aa', b \cdot T^a(b'))$

"$ax+b$" group $\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right\}$ is an example of a semidirect product.

μ on $G = \mathcal{L}(\mathbb{Z}^d)$

$\{(\underbrace{x}_{\in \mathbb{Z}^d}, \underbrace{\varphi}_{\in \text{config}})\}$ $\quad \left(\begin{array}{l} \{(a, \phi)\} \quad a : \text{a generator of } \mathbb{Z}^d \\ \{(0, \delta_0)\} \end{array} \right.$

generating set of G. • If $(x_{n+1}, \varphi_{n+1}) = (a, \phi)$

$$\left\{ \begin{array}{l} (X_n, \Phi_n) = (x_1, \varphi_1) \cdots (x_n, \varphi_n) \\ (X_{n+1}, \Phi_{n+1}) = (X_n, \Phi_n) \cdot (x_{n+1}, \varphi_{n+1}) \end{array} \right.$$

$X_{n+1} = X_n + a, \ \Phi_{n+1} = \Phi_n$

• If $(x_{n+1}, \varphi_{n+1}) = (0, \delta_0)$

$X_{n+1} = X_n, \ \Phi_{n+1} = \Phi_n + \delta_{X_n}$

If $\underline{d \geq 3}$ $\quad$ cf (Polya)

Non-recurrence of RW on $\mathbb{Z}^d, d \geq 3$.

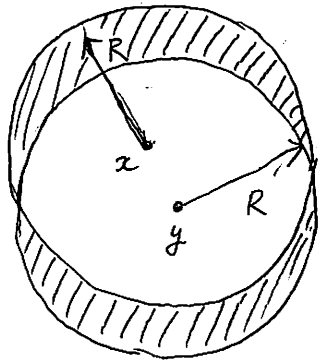
<u>Amenability</u> $\left(\exists m_n \ \ \|m_n - g \cdot m_n\| \to 0 \quad \forall g \in G. \right)$

Triviality of the Poisson boundary.

$\Leftrightarrow$ absense of behavior at ∞.

$\Leftrightarrow$ absense of non-constant bdd harmonic functions.

f : mean value property



$f(x) \approx f(y)$
if $R \gg 1$.

For an arbitrary Markov chain

Poisson boundary (P.b.) $= \{\cdot\}$

$\Longleftrightarrow \quad \| \delta_x P^n - \delta_y P^n \| \to 0$

$$f(x) = \langle f, \delta_x P^n \rangle$$
$$f(y) = \langle f, \delta_y P^n \rangle$$

(G, μ)

$$\underset{\mu^{*n}}{\underset{\sim}{x_n}} = \underset{\mu}{\underset{\sim}{h_1}} \cdots \underset{\mu}{\underset{\sim}{h_n}}$$

$\| x \cdot \mu^{*n} - y \cdot \mu^{*n} \|$
$\to 0 \quad \forall x, \forall y \in G.$

P.b. $(G, \mu) = \{\cdot\} \Longleftrightarrow \mu^{*n}$ converges to a left-invariant mean.

$\Downarrow$

G is amenable.

Converse is not always true as we saw, (tribiality of P.b. depends on μ)

<u>Thm</u> G is amenable

$\left(\Longleftrightarrow \ \exists \mu \in P(G) \ \text{s.t.} \ \text{P.b.}(G, \mu) = \{\cdot\}. \right)$

$\uparrow$
cannot be taken to be finitely supported.
in $G = \mathcal{L}(\mathcal{G})$ case

$\forall K \subseteq G \quad \exists \mu \quad \| g \cdot \mu - \mu \| < \varepsilon.$

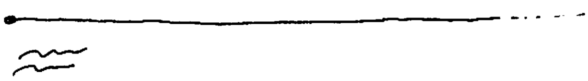
<u>Amenability of actions</u>

1) Invariant mean (approximate invariant sequence) on the action space.

2) (Zimmer) $\lambda_n : X \to P(G)$

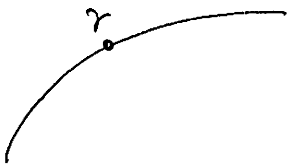
$$\| \lambda_n^{g \cdot x} - g \cdot \lambda_n^{x} \| \to 0$$

Boundary actions $(F \curvearrowright \partial F)$



$x \mapsto \lambda_n^x \quad \|\lambda_n^x - \lambda_n^y\| \to 0$

<u>Thm</u> The action on the Poisson boundary is amenable.



$$\begin{array}{c} (X^{\mathbb{Z}_+}, \mathbb{P}) \\ \downarrow \\ (\Gamma, \nu) \end{array}$$

$\mathbb{P}^\gamma$ – the measure in the path space of the conditional chain

Have trivial <u>Poisson boundary</u>

Doob transform

$$\varphi_\gamma(g) = \frac{d g\nu}{d\nu}(\gamma)$$

$$\frac{\sum_h \varphi_\gamma(gh)\mu(h)}{\varphi_\gamma(g)} = 1$$

<u>Thm</u> G acts on X

$$\left(\begin{array}{c} x \mapsto \varphi_x \quad \text{a minimal harmonic function.} \\ \Downarrow \\ G \curvearrowright X \quad \text{top. amenable} \quad (?) \end{array} \right)$$

<u>Triviality and identifications</u>

$$\begin{tikzcd} (G^{\mathbb{Z}_+}, \mathbb{P}) \arrow[d] \arrow[dr] & \\ & (\Gamma, \nu) \arrow[dl, "=?"] \\ (B, \lambda) & \end{tikzcd}$$

1) G is abelian (nilpotent)

$\Rightarrow (\Gamma, \nu) = \{\circ\}$ for any μ

$\left\{ \begin{array}{l} \text{Choquet – Deny} \\ \text{Blackwell.} \end{array} \right.$

Proof (abelian case) de Finetti 0-1 law

$f \in H^\infty(G,\mu)$

$f(x_n)$ converges for a.e. $\{x_n\}$.

$\parallel$

$f(h_1 + \cdots + h_n)$

$\vdots$ $\quad\quad\quad\quad\quad\quad\quad a \in \mathrm{supp}(\mu)$

$f(h_1 + \cdots + h_n + \underline{\underline{h_{n+1}}})$

μ^{*n} write explicitly.

$G = \mathbb{Z}$ $\quad \mu = \frac{1}{2}(\delta_0 + \delta_1)$

$\mu^{*n}(k) = 2^{-n}\binom{n}{k}$ $\quad k \simeq \frac{n}{2}$ $\quad \binom{n}{k} \sim \binom{n}{k+1}$

Entropy

$a_0, a_1, a_2, \cdots\cdots, a_n, \cdots\cdots$

P_n on A^n $\quad \dfrac{H(P_n)}{n} \xrightarrow{n\to\infty} h$ Shanon entropy

(G, μ) $H(\mu^{*n}) = h_n$ $\quad h_m + h_n \geqslant h_{m+n}$

$(x, y) \longmapsto x \cdot y$

$\mu^{*m} \otimes \mu^{*n} \quad\quad \mu^{*(n+m)}$

$h_\infty = \lim_{n\to\infty} \dfrac{h_n}{n}$ the asymptotic entropy

Thm If $H(\mu) < \infty$, then $h_\infty = 0 \Leftrightarrow (\Gamma, \nu) = \{\cdot\}$

Cor

If $H(\mu) < \infty$, the $P(G, \mu) = \{\cdot\} \Leftrightarrow P(G, \check{\mu}) = \{\cdot\}$

μ^{*n} converges to left invariant

$\Leftrightarrow$ ———— right ————

$\boxed{\text{can fail if } H(\mu) = \infty}$

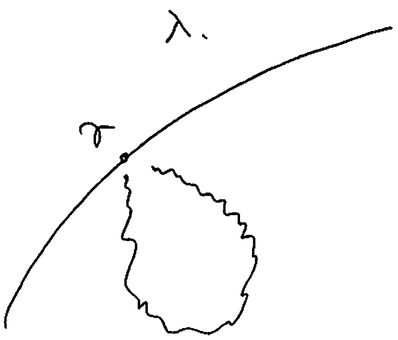
Assume $\operatorname{supp}(\mu) = K \quad |K| < \infty$.

$$\operatorname{supp}(\mu^{*n}) = K^n \Rightarrow h_\infty(G,\mu) \leq \lim_{n\to\infty} \frac{H(\mu^{*n})}{n} \leq \lim_{n\to\infty} \overset{v}{\overset{\shortparallel}{\frac{\log |K^n|}{n}}} \qquad (*)$$

Cor

(If G has subexponential growth, then $\Gamma = \{\cdot\}$ for any finitely supported measure.)

$$\ell(G,\mu) = \lim_{n\to\infty} \frac{|\mu^{*n}|}{n} \qquad \boxed{h \leq \ell \cdot v} \text{ volume growth in } (*)$$



Th $(B,\lambda) \cong (\Gamma, \nu)$

$\Leftrightarrow h(\mathbb{P}^\gamma) = 0$

Thm F, μ with a finite first moment. $\left(\sum_g |g|\,\mu(g) < \infty\right)$

$\Rightarrow \partial F$ is the Poisson boundary.

Lemma $x_n \in F$ then the following are equivalent:

1) $\exists\, \gamma \in \partial F,\ \ell \geq 0 \quad d(x_n, [e,\gamma)(\ell n)) = o(n)$

2) $|x_n| = n\ell + o(n)$

$d(x_n, x_{n+1}) = o(n)$