Reminder

Kennedy 2

$\mathcal{C}$: a category.

I : injective

if $X \overset{\iota}{\hookrightarrow} Y$ embedding and $\varphi : X \to Y$

then $\exists \psi : Y \to I$ and $\psi \circ \iota = \varphi$.

Last time $\mathcal{C} =$ Opsys has sufficiently many injectives.

$=$ G-OpSys does too

Op Sys	G-Op Sys
$\mathbb{B}(H)$	$\ell^\infty(G, \mathbb{B}(H))$

There are lots of injective objects.

For X $\exists$ "good" injective approximation

Suppose X is injective, $X \subset Y$

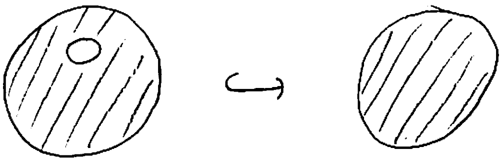
extend $\mathrm{id}|_X$ to a map $\varphi : Y \to X$ (i.e. φ is a projection onto X)

Converse is also true.

Injectivity is preserved by embedding.

We can say $X \sim Y$ if $\exists$ isomorphism $X \xrightarrow{\sim} Y$

complete isometric isomorphism



Fix a category $\mathcal{C}$ of operator systems.

Def 1. Let $X \in \mathcal{C}$. An extension (Y, ι) of X

is $Y \in \mathcal{C}$ plus $\iota : X \hookrightarrow Y$

2. An extension (Y, ι) is essential if whenever

$\varphi : Y \to Z$ and $\varphi|_X$ is an embedding of X,

then φ is an embedding.

So if $\varphi : X \overset{\text{embedding}}{\hookrightarrow} Z$ and (Y, ι) is an essential extension of X, then all extension of φ to Y is an embedding.

X, (Y, ι) : extension means, can think of $\iota(X)$ as a copy of X in Y, so can usually just take ι to be inclusion.

We will show

maximal essential extension $\Longleftrightarrow$ minimal injective extension

Thm Let X be maximal essential (meaning if Y is an essential extension of X, then $X \sim Y$). Then X is injective.

<pf> Zorn's Lemma $\Rightarrow$ every element in C has a maximal essential extension.

$\mathcal{U} \subseteq X_1 \subseteq X_2 \subseteq \cdots \quad X = \bigvee_i X_i$: essential.

Fact Let X, Y be Banach spaces. Then $\mathbb{B}(X, Y^*)$ is a dual space, the weak*-topology is called the point-weak* topology.
We will take X, Y^* to be op systems.
Unit ball is compact by Alaoglu.
Also point weak*-limits preserves the morphisms of the category (Op-Sys, G-Op-Sys.)

$$\varphi_i \xrightarrow{w^*} \varphi \iff \forall a \in X \quad \varphi_i(a) \to \varphi(a) \quad \sigma(Y^*, Y)$$

Suppose $X \subseteq B$, B : injective and w^*-closed.

$X \subseteq B$ [I will find a proj $B \xrightarrow{P} X$]

Use Zorn's Lem to find $Y \subseteq B$ maximal with the property that $\exists$ morphism $\varphi : X + Y \longrightarrow X$ s.t. $\varphi \restriction_X = id_X$.

To find upper bounds to increasing chains.

(Y_i, φ_i) can extend each φ_i to a map $\tilde{\varphi}_i : B \longrightarrow B$.

Take a point-weak* limit pt $\varphi = \lim_i \varphi_i$

Take $Y = \bigvee_i Y_i$ Then (Y, φ) works.

So $\exists$ maximal such (Y, φ)

To finish need to show $X + Y = B$.

<u>Idea</u> If $X \subseteq Y \neq B$, use injectivity of B to extend

$\varphi : Y \to X$ to $\varphi : B \to B$.

Set $X' = \varphi(B)$. This is an essential extension of X.

$X \subset \mathbb{B}(H)$ inj

$\varphi : X \hookrightarrow \mathbb{B}(K)$ emb $\Rightarrow \varphi(X)$ inj

Order all possible extensions φ by

$\varphi \leq \varphi'$ if $\|\varphi_n(x)\| \leq \|\varphi_n'(x)\|$ $\forall x \in M_n(X)$ $\forall n \in \mathbb{N}$

Can Zorn's Lemma

w/ point-weak*

and weak-lower semicontinuity of $\|\cdot\|$ to get minimal

extension φ.

Now show $X' = \varphi(B)$ is essential

Let $\psi : X' \longrightarrow B$ be a map that extends $\psi\restriction_X = id_X$

Enough to show such maps are embedding of X'.

$\psi \circ \varphi|_X = id_X$ $\psi \circ \varphi|_Y = 0$

ψ fixes X ($\psi|_X = id_X$), $\cancel{\psi|_Y = 0}$ and $\|\psi_n(z)\| \leq \|\psi_n(\varphi_n(b))\|$
$\leq \|\varphi_n(b)\|$

$z \in M_n(X') \Rightarrow z = \varphi_n(b)$, $b \in \varphi_n(M_n(B))$

By minimality of φ, this is an equality.

Thus ψ is an embedding. $\Rightarrow X'$ is an essential extension of X.

Since X is maximal essential, $X = X'$

hence $\varphi : B \to B$ is a proj onto X $(\varphi = P)$

Hence X is injective.

— comment —

Find Y maximal s.t. $X \subseteq Y$.

$\exists$ proj $Y \longrightarrow X$

Show $Y = B$.

So we know there are lots of injectives.

<u>Cor</u>

Maximal essential extn $\Leftrightarrow$ minimal injective extn

<pf>

If $X \in C$. $X \subseteq I \subseteq B$ inj $\quad X \subseteq E \subseteq B$ ess

$\Rightarrow$ the proj onto I is an embedding of X

hence by essentiality of E, also of E.

So all ess. extns of X embeds into I, in particular a maximal essential one, which is injective by thm. $\square$

Every $X \in \mathcal{C}$ has a minimal injective extension that is max. ess.

<u>Def</u> An injective envelope of X is a minimal extn.

Thm
(Injective envelopes are unique up to $\sim$)

<pf> Let (Y_1, ι_1) (Y_2, ι_2) be inj envelopes of X.

The structure of an inj op-system

Thm
(If X is injective, then $X \sim C^*$-alg.)

<pf> Let $X \subseteq B$. be a "concrete" embedding

Define a product on X. Let $\varphi : B \xrightarrow{ucp} X$ be a projection.

$x_1 \circ x_2 = \varphi(x_1 x_2)$ (Choi - Effros product)

X is closed under this operation, $*$. (φ is ucp)

$(X, \circ)$ Satisfies all properties of C^*- alg.

associativity : difficult to check.

Cor (For any $X \in \mathcal{C}$ any injective envelope is a C^*-algebra.)

Injective C^*- algebras are AW^*- algebras.
Very difficult to work with.

Let $A = C(X)$ X: cpt Hausdorff space.

The inj envelope of A in Op-Sys is a commutative C^*- alg.

$\simeq C(Y)$ The space Y is the Gleason cover or projection cover of X. It is Stonean or extremally disconnected,

$C(Y)$ non-separable, difficult to concretely identify.

Dixmier

$C(Y) \simeq B(X)/\sim$ ← equal on measer sets.

A^{**}
$\cup\!|$
$B(A) / M \cong$ inj envelope (A).

Fix G a discrete group. Then $\mathbb{C}$ is a G-op system w/ trivial action.

What is the injective envelope of $\mathbb{C}$ in G-op system?

A $C(\partial_F G)$, $\partial_F G$ is the universal G-boundary.

Def A compact G-sp X is a boundary if $\forall \mu \in P(X)$

($\overline{G\mu}^{w*}$ contains all point masses $\{\delta_x \mid x \in X\}$)

Thm

(X is a G-boundary iff $C(X)$ is an essential extension of $\mathbb{C}$ in G-op sys.)

This explains why the injective envelope, which is maximal essential is $C(\partial_F G)$

In function theory, there is a notion of Shilov boundary of a set of functions $A \subseteq C(X)$. It's the smallest subset of X s.t. an analogue of max modulus principle holds for A

$C(\hat{S}_A)$ = minimal ess. extn of A that is a C^*-alg.