Kennedy 3 (IV)

$G \curvearrowright X$ is a boundary action (X is a G-boundary)

if $\forall \nu \in P(X)$

$$\overline{G\cdot\nu}^{w^*} \supseteq \{\delta_x \mid x \in X\}$$

(**Thm** X is a G-boundary $\iff$ $C(X)$ is an essential extension of $\mathbb{C}$ in G-OpSys.)

Means:
If $\varphi : C(X) \longrightarrow Z$ is G-equivariant ucp

then φ is completely isometric.

Pf (Sketch) Suppose X is a G-boundary.
Fix $f \in C(X)$, $\varepsilon > 0$, $x \in X$. s.t. $|f(x)| > \|f\| - \varepsilon$
Take $\gamma \in Z^*$. Then $\varphi^* : Z^* \longrightarrow C(X)^*$
state.

$$\varphi^*(\gamma) = \nu \in P(X).$$

$t \in G$ $\langle t\varphi(f), \gamma\rangle = \langle f, t\varphi^*(\gamma)\rangle = \langle f, t\gamma\rangle$

Take t_n s.t. $t_n\gamma \longrightarrow \delta_x$.

$$\langle t_n \varphi(f), \gamma\rangle \longrightarrow f(x)$$

Thus $\|\varphi(f)\| \geq \|f\|_\infty - \varepsilon$. $\square$

Cor Injective envelope of $\mathbb{C}$ in G-OpSys. is $C(\partial_F G)$
where $\partial_F G$ is the universal Furstenberg boundary.

<Pf> Inj envelope is maximal essential extension $\square$

Cor (G is amenable $\iff$ $\partial_F G$ is trivial
$\iff$ No nontrivial boundary actions.)

<pf> G amenable $\Longleftrightarrow$ $\exists\, \nu \in \ell^\infty(G)^*$ positive unital
G-invariant.

$$\nu : \ell^\infty(G) \xrightarrow{G\text{-equivariant, ucp}} \mathbb{C}. \qquad (G \curvearrowright \mathbb{C} \text{ trivial action})$$

Can be viewed as the projection.

$\Longleftrightarrow$ $\exists$ G-equivariant ucp projection onto $\mathbb{C}$.

$\ell^\infty(G, \mathbb{C}) \longrightarrow \mathbb{C}$

ℓ injective
in G-OpSys.

$\Longleftrightarrow$ $\mathbb{C}$ is the injective envelope

i.e. $\mathbb{C} = C(\partial_F G)$. $\square$

Simplicity

Thm Let A be a C^*-algebra. Suppose that $A \subseteq B$, where B is a C^*-algebra and the extension $A \subseteq B$ is essential in Op-Sys. Then if A is simple, so is B.

$(1_A = 1_B \in A)$
A, B : unital!

<pf> Let $I \underset{\neq}{\trianglelefteq} B$, A : simple.

Let $\pi : B \longrightarrow B/I$

Since A is simple, the restriction $\pi \restriction_A$ is an isomorphism,

hence ucp c isometric.

By essentiality, π must be isometric on B.

Hence $I = 0$. $\square$

Recall

Goal determine when $C_r^*(G)$ is simple.

(Hamana)

Thm Let G be discrete. Then $C(X) \rtimes_r G$ is an essential extension of $C_r^*(G)$, whenever X is a G-boundary. (in Op-Sys)

Cor (If $C(X) \rtimes_r G$ is not simple, then $C_r^*(G)$ is not simple.)

Reduced crossed product

Idea $G \curvearrowright A$ a C^*-algebra.

Want to define a C^*-algebra that "encodes" the action

Want

$A \rtimes_r G$. generated by G and A.

Work on $\ell^2(G) \otimes H$ where $A \subseteq \mathbb{B}(H)$

$$A \ni a \mapsto \begin{array}{c} \\ g \\ g' \end{array} \begin{bmatrix} \ddots & & & \\ & g^{-1}a & & \\ & & \ddots & \\ & & & g'^{-1}a \end{bmatrix} = \sum_{g \in G} \delta_g \otimes g^{-1}a$$

can identify A with the image in $\mathbb{B}(\ell^2 G \otimes H)$

Left-regular repr.

$$G \ni g \longmapsto \lambda_g = L_g \otimes 1_H$$

$$\lambda_g \, a \, \lambda_g^* = g \cdot a$$

$$A \rtimes_r G = C^*(\lambda(G), A)$$

Two problems X : a G-boundary

1 When is $C(X) \rtimes_r G$ simple ?
(In terms of dynamics of $G \curvearrowright X$, preferably)

(*) 2 If $C(X) \rtimes_r G$ is simple, can we show $C_r^*(G)$ is simple ?

Def Let $G \curvearrowright X$ is <u>topologically free</u> if for $g \in G \setminus \{e\}$ the fixed point set $\{x \in X \mid g \cdot x = x\}$ has empty interior.

(<u>Thm (Archbold – Spielberg)</u> Let X be a compact minimal G-space. If X is topologically free, then $C(X) \rtimes_r G$ is simple.)